A warm-up for solving noncompact sigma models:
The Sinh-Gordon model

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Based on

- A. Bytsko, J.T., hep-th/0602093,
- J.T., hep-th/0702122

Goal: Understand string theory on AdS-spaces

Building blocks: nonlin. sigma-models on super-groups.

$$PSL(1,1|2) \leftrightarrow AdS_3 \times S_3$$
$$PSL(2,2|4) \leftrightarrow AdS_5 \times S_5$$

To write out worldsheet-actions, use Poincare coordinates:

$$ds^2 = \frac{1}{z^2}(dz^2 + d\vec{x}^2)\,.$$

Writing $z = e^{\varphi} \Rightarrow$ world-sheet Hamiltonian of the form

$$H = \ldots + e^{2\varphi}(\ldots) + e^{-2\varphi}(\ldots) + \ldots$$

Exponential interactions!

On the other hand: Hope for integrability! (`Bena, Polchinski, Roiban`)

Nice recent story concerning limit (`Hofman, Maldacena`)

$$J \to \infty\,, \qquad E - J = \text{fixed}\,, \qquad y \propto R_{\text{AdS}}^{-1} = \text{fixed}$$

Gauge-fixing: $J \mapsto R = J/\sqrt{y}$, R: world-sheet radius.

$$H_{\text{g.f.}} = \int_0^{2\pi R} d\sigma \Big(\ldots + e^{2\varphi}(\ldots) + e^{-2\varphi}(\ldots) + \ldots \Big)$$

Conjecture: $H_{\text{g.f.}}$ is integrable.

Proposal for factorized scattering theory in infinite volume $R \to \infty$ ($\Leftrightarrow J \to \infty$)

(`Beisert, Staudacher, Hofman, Maldacena ...`) :

▷ Elementary excitations ("magnons") $8_{\text{B}} \oplus 8_{\text{F}}$.

▷ Proposal for $2 \to 2$ scattering matrix (for **all** y !!!)

Proposal has passed highly nontrivial tests!

Important problem: Spektrum for finite $R \Leftrightarrow$ finite J?

Asymptotic Bethe ansatz (`Arutyunov, Frolov, Staudacher`)

- Pretend magnons are free except for magnons crossing (δ-like interactions),

$\Rightarrow$ Ansatz for coord. space wave-fct.: Plane-waves + crossings (S-matrix)

- Periodicity of wave-fct. $\Rightarrow$

$$e^{ip_a R} = \prod_{b \neq a} S(p_a, p_b)\,.$$

Problem: Interactions are not δ-like (vacuum polarization)

$\Rightarrow$ Expect corrections to As. Bethe ansatz!

Classes of integrable models:

(A) **Compact integrable models.**
Models associated to compact groups: WZNW, σ-models, XXX, XXZ,
Ising, Interactions $e^{ib\phi}$: Sine-Gordon ....

(B) **Noncompact integrable models - Real type**
Models associated to non-compact groups: WZNW, σ-models, XXX, XXZ ...
Interactions $e^{b\phi}$: Liouville theory, Sinh-Gordon, (Affine) Toda...

Class (B) is important:

- String theory: Strings on noncompact target spaces (black holes, cosmology),
- Gauge theory - via AdS-CFT correspondence,
- Condensed matter: Integer quantum Hall, electron systems with disorder.

But: Class (B) is very different from (A):

— **Prediction: Bethe ansatz fails in class (B)!**

Consider prototype: Sinh-Gordon model (on circle with circumference R)

$$H_{\mathrm{ShG}} = \int_0^R dx \left\{ 4\pi \, \Pi^2 + \frac{1}{16\pi}(\partial_\sigma \varphi)^2 + 2\mu \cosh(b\varphi) \right\} .$$

- **Infinite volume** $R \to \infty$:
  - **Spectrum:** One massive particle, $E = m\cosh\vartheta$, $p = m\sinh\vartheta$.
  - **Scattering:** S-matrix factorizes into two-particle scattering $S(\vartheta_1 - \vartheta_2)$

$$S(\vartheta) = \frac{\sinh\vartheta - i\sin\vartheta_0}{\sinh\vartheta + i\sin\vartheta_0} .$$

  - **Fields, correlation functions:** ...?!

- **Finite volume** R:
  - **Spectrum:** Conserved quantities, classification of eigenvectors - today!

First example for QFT from class (B) where spectrum was understood for finite volume!

First step: Construct integrable lattice regularization of the Sinh-Gordon model:

- Hilbert space $\mathcal{H}$
- Hamiltonian H,
- set $\mathcal{Q} = \{\mathsf{T}_0, \mathsf{T}_1, \dots\}$ of commuting conserved charges

1. Definition of $\mathcal{H}$:

Discretize Sinh-Gordon variables as

$$\Pi_n \to \Pi(x)\,\Delta\,, \quad \Phi_n \to \Phi(x)\,, \quad x = n\Delta\,.$$

Quantize:

$$[\,\Pi_n\,,\,\Phi_n\,] \,=\, \frac{1}{i}\delta_{n,m}\,.$$

$\Rightarrow$ Hilbert space $\mathcal{H} \equiv (L^2(\mathbb{R}))^{\otimes \mathrm{N}}$.

2. Construction of $\mathcal{Q}$.

Let

$$\mathsf{M}(u) \equiv \begin{pmatrix} \mathsf{A}(u) & \mathsf{B}(u) \\ \mathsf{C}(u) & \mathsf{D}(u) \end{pmatrix} \equiv L_N(u) \cdot \ldots \cdot L_2(u) \cdot L_1(u)\,,$$

where

$$L(u) \equiv \begin{pmatrix} L_{11}(u) & L_{12}(u) \\ L_{21}(u) & L_{22}(u) \end{pmatrix}, \qquad \left[\begin{array}{ll} L_{11}(u) = e^{+\frac{\beta}{8}(\Pi_n+2s)}\left(1+e^{-\beta(\Phi_n+s)}\right)e^{+\frac{\beta}{8}(\Pi_n+2s)} & \\ L_{12}(u) = \sinh\left(\pi b u + \frac{\beta}{2}\Phi_n\right) & \\ & \beta = b\sqrt{8\pi} \\ L_{21}(u) = \sinh\left(\pi b u - \frac{\beta}{2}\Phi_n\right) & \\ L_{22}(u) = e^{-\frac{\beta}{8}(\Pi_n-2s)}\left(1+e^{+\beta(\Phi_n-s)}\right)e^{-\frac{\beta}{8}(\Pi_n-2s)} & \end{array}\right]$$

Consider the one–parameter family of operators:

$$\mathsf{T}(u) = \mathrm{tr}\left(\mathsf{M}(u)\right) = \mathsf{A}(u) + \mathsf{D}(u)\,.$$

$\Rightarrow$ The operators T_m which are defined by $\mathsf{T}(u) = e^{\pi b N u}\sum_{m=0}^{N}(-e^{-2\pi b u})^m\,\mathsf{T}_m$, are positive self–adjoint and commuting, $[\mathsf{T}_m, \mathsf{T}_n] = 0$.

3. Construction of H

There exists an operator H *which has the following properties.*

(a) *The operator* H *is local:*

$$\mathsf{H} = \sum_{n=1}^{N} H_{n,n+1} \,.$$

(b) H *commutes with the conserved charges* T_k*:*

$$\left[\,\mathsf{H}\,,\,\mathsf{T}_k\,\right] \;=\; 0, \;\; \text{for} \;\; k = 0,\ldots,N.$$

(c) *Classical continuum limit* $(\Pi_n \to \Pi(x)\Delta,\ \Phi_n \to \Phi(x),\ x = n\Delta.)$

$$\begin{Bmatrix} N \to \infty \\ \Delta \to 0 \\ s \to \infty \end{Bmatrix} \;\; \text{with} \;\; \begin{Bmatrix} R = N\Delta/2\pi \\ m = \frac{4}{\Delta} e^{-\pi b s} \end{Bmatrix} \;\; \text{fixed.}$$

$$\mathsf{H} \underset{\hbar\to 0}{\longrightarrow} \sum_n \frac{1}{\Delta} H^{\mathsf{G},\mathrm{cl}}_{n,n+1} \to \int_0^{2\pi R} dx\, \left(\tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\partial_x\Phi)^2 + \tfrac{m^2}{\beta^2}\cosh\beta\Phi\right) + \text{const}$$

Good news: We do have integrable lattice regularization

$\Rightarrow$ **Strategy: Diagonalize** $\mathsf{T}(u) \Rightarrow \cdots \Rightarrow$ **Diagonalization of** H

Bad news: The Bethe ansatz fails:

– one would need pseudo-vacuum Ω annihilated by $\mathsf{B}(u)$

– Since interactions involve **real exponentials** $e^{b\varphi}$:

$\mathsf{B}(u)\Omega = 0$ leads to equations like $e^{\phi}\Omega = 0$

– **no normalizable** solution $\Omega \in \mathcal{H}$!!!

Reasons:

– exponential interactions e^{ϕ},

– non-compactness of target space!

However, there is something better than Bethe ansatz:

Q-operators and separation of variables

Assume we have an **operator** $\mathsf{Q}(u)$ related to $\mathsf{T}(u)$ via the **Baxter equation**

$$\mathsf{Q}(u)\mathsf{T}(u) = \big(a(u)\big)^{\mathrm{N}}\mathsf{Q}(u-ib) + \big(d(u)\big)^{\mathrm{N}}\mathsf{Q}(u+ib),$$

which furthermore satisfies

$$\left[\begin{array}{ll} \text{(a)} & \mathsf{Q}(u) \text{ is normal}, \quad \mathsf{Q}(u)\mathsf{Q}^*(v) = \mathsf{Q}^*(v)\mathsf{Q}(u), \\ \text{(b)} & \mathsf{Q}(u)\,\mathsf{Q}(v) = \mathsf{Q}(v)\,\mathsf{Q}(u)\,, \\ \text{(c)} & \mathsf{Q}(u)\,\mathsf{T}(v) = \mathsf{T}(v)\,\mathsf{Q}(u)\,, \end{array}\right]$$

- $\mathsf{Q}(u)$ is repackaging of conserved quantities
- Diagonalization of $\mathsf{Q}(u) \Rightarrow$ diagonalization of $\mathsf{T}(u)$.
- Eigenvalues $q(u)$ of $\mathsf{Q}(u)$ must satisfy the Baxter equation.
- Explicit construction of $\mathsf{Q}(u)$ (`Bytsko, J.T.`) $\Rightarrow$ analytic and asymptotic properties of $q(u)$: **Quantization conditions!**

Necessary conditions on the spectrum:

A function $t(u)$ *can be an eigenvalue of the transfer–matrix* $\mathsf{T}(u)$ **only if** *there exists a function* $q_t(u)$ *which satisfies*

$$\left[\begin{array}{ll}
\text{(i)} & t(u)\, q_t(u) \;=\; \big(a(u)\big)^{\mathrm{N}} q_t(u-ib) + \big(d(u)\big)^{\mathrm{N}} q_t(u+ib), \\
 & \text{where } \; d(u) = a(-u) = 1 + (\tfrac{m\Delta}{4})^2\, e^{-\pi b(2u+ib)}\,, \\
\text{(ii)} & q_t(u) \;\text{ is meromorphic in }\; \mathbb{C}, \;\text{ with poles in }\; \pm\,\Upsilon_{-s}, \\
\text{(iii)} & q_t(u) \sim \begin{cases} e^{+i\pi \mathrm{N}\,\sigma\, u - i\frac{\pi}{2}\mathrm{N}u^2} \text{ for } |u| \to \infty, \;\; |\arg(u)| < \frac{\pi}{2}, \\ e^{-i\pi \mathrm{N}\,\sigma\, u - i\frac{\pi}{2}\mathrm{N}u^2} \text{ for } |u| \to \infty, \;\; |\arg(u)| > \frac{\pi}{2}. \end{cases}
\end{array}\right]$$

Separation of Variables

Main idea (Sklyanin): Diagonalize $\mathsf{B}(u)$, parametrize eigenvalues $b(u)$ as

$$b(u) \sim \prod \sinh 2\pi b(u - y_k) \,.$$

$\Rightarrow$ wave-functions $\Psi(y_1 \dots y_\mathrm{N})$.

Key observations:

- (Sklyanin) $\mathsf{T}(u)\Psi_t = t(u)\Psi_t(u) \Leftrightarrow$

$$t(y_k)\Psi(\dots y_k \dots) = (a(y_k))^{\mathrm{N}}\Psi(\dots y_k - ib \dots) + (d(y_k))^{\mathrm{N}}\Psi(\dots y_k + ib \dots)$$

- (Bytsko, J.T.) Properties (i)-(iii) $\Rightarrow$ Ansatz

$$\Psi_t = \prod_{k=1}^{N} q_t(y_k)$$

yields **normalizable** eigenstates of $\mathsf{T}(u)$.

$\overset{\text{Bytsko,J.T.}}{\Rightarrow}$ **Full characterization of the spectrum:**

A function $t(u)$ *is eigenvalue of the transfer–matrix* $\mathsf{T}(u)$ **if and only if** *there exists a function* $q_t(u)$ *which satisfies*

$$\left[\begin{array}{rl} \text{(i)} & t(u)\, q_t(u) \;=\; \big(a(u)\big)^{\mathrm{N}} q_t(u-ib) + \big(d(u)\big)^{\mathrm{N}} q_t(u+ib), \\ & \text{where } \; d(u) = a(-u) = 1 + (\tfrac{m\Delta}{4})^2\, e^{-\pi b(2u+ib)}\,, \\ \text{(ii)} & q_t(u) \; \text{is meromorphic in } \mathbb{C}, \; \text{with poles in } \; \pm\,\Upsilon_{-s}, \\ \text{(iii)} & q_t(u) \sim \begin{cases} e^{+i\pi \mathrm{N}\,\sigma\, u - i\frac{\pi}{2}\mathrm{N}u^2} \text{ for } |u| \to \infty, \;\; |\arg(u)| < \frac{\pi}{2}, \\ e^{-i\pi \mathrm{N}\,\sigma\, u - i\frac{\pi}{2}\mathrm{N}u^2} \text{ for } |u| \to \infty, \;\; |\arg(u)| > \frac{\pi}{2}. \end{cases} \end{array}\right]$$

Task: Classify set $\mathcal{Q}$ of solutions to the Baxter equation (i)

Let $\mathcal{Y}_M$ be the set of all functions $Y(\vartheta)$ which satisfy the integral equation

$$(\mathrm{I})_{\mathrm{L}} \qquad \begin{aligned} \log Y(\vartheta) = & \int_{\mathcal{C}} \frac{d\vartheta'}{4\pi}\, \sigma(\vartheta-\vartheta')\log(W(\vartheta')+Y(\vartheta')) \\ & - N\arctan\left(\frac{\cosh(\vartheta+i\tau)}{\sinh\sigma}\right) - N\arctan\left(\frac{\cosh(\vartheta-i\tau)}{\sinh\sigma}\right) \\ & + \sum_{a=1}^{M}\log S(\vartheta-\vartheta_a-i\tfrac{\pi}{2}) + \frac{1}{2}\sum_{a=M'+1}^{M}\log S(\vartheta-\vartheta_a-i\tfrac{\pi}{2})\,. \end{aligned}$$

and which have the properties

$$\left[\begin{array}{ll} (\mathrm{Y}_1)_{\mathrm{L}} & \log Y(\vartheta) \sim -i\delta \mathrm{N}((\vartheta\mp\sigma\mp i\tfrac{\pi}{2})^2-\tau^2) \text{ for } |u|\to\infty,\ |\arg(\pm u)|<\tfrac{\pi}{2}\,, \\ (\mathrm{Y}_2)_{\mathrm{L}} & Y(\vartheta) \text{ is meromorphic with poles of maximal order N in } \pm\Upsilon_{-s\pm i\tau}\,, \\ (\mathrm{Y}_3)_{\mathrm{L}} & \text{There exist complex numbers } \vartheta_a\in\mathbb{S},\ a=1,\ldots,M, \text{ such that} \\ & W(\vartheta)+Y(\vartheta)=0 \text{ if } \vartheta=\vartheta_a\pm i\tfrac{\pi}{2}\,. \end{array}\right]$$

$$\left(\sigma(\vartheta) \equiv \frac{4\sin\vartheta_0\cosh\vartheta}{\cosh 2\vartheta-\cos 2\vartheta_0}\,, \qquad \vartheta_0 \equiv \frac{\pi}{2}\frac{b}{\delta}\,, \qquad \sigma \equiv \frac{\pi}{2}\frac{s}{\delta}\,, \qquad \tau \equiv \frac{\pi}{2}\frac{\delta'}{\delta}\,.\right)$$

Theorem 1. *There is a one-to-one correspondence between the solutions* $Y(\vartheta) \in \mathcal{Y}_M$ *of the integral equations* $(\mathrm{I})_{\mathrm{L}}$ *and the elements* $Q \in \mathcal{Q}_M$.

- *Given* $Q \in \mathcal{Q}_M$, *get* $Y(\vartheta)$:

$$W(\vartheta) + Y(\vartheta) \;=\; Q(\vartheta + i\tfrac{\pi}{2})Q(\vartheta - i\tfrac{\pi}{2})\,.$$

$\mathbb{Z} = \{\vartheta_1, \ldots, \vartheta_M\}$: *set of zeros of* $Q(\vartheta)$ *within* $\mathbb{S}$.

- *Given* $Y(\vartheta) \in \mathcal{Y}_M$ *(zeros:* $\vartheta_a \in \mathbb{S} \setminus \partial\mathbb{S}$, $\vartheta'_b \in \partial\mathbb{S}$*) get* $Q \in \mathcal{Q}_M$ *as*

$$(\mathrm{Q})_{\mathrm{L}} \qquad \begin{aligned} \log Q(\vartheta) \;=\; & \int_{\mathcal{C}} \frac{d\vartheta'}{4\pi}\, \frac{\log(W(\vartheta') + Y(\vartheta'))}{\cosh(\vartheta - \vartheta')} - N \arctan\left(\frac{\cosh\vartheta}{\sinh\sigma}\right) \\ & + \sum_{a=1}^{M'} \int_{\mathcal{C}_a}^{\vartheta} d\vartheta' \frac{1}{\sinh(\vartheta' - \vartheta_a)} + \frac{1}{2}\sum_{b=1}^{M-M'} \int_{\mathcal{C}_a}^{\vartheta} d\vartheta' \frac{1}{\sinh(\vartheta' - \vartheta'_b)}\,. \end{aligned}$$

Continuum limit: Easy!

Continuum limit:

$\mathrm{N} \to \infty$, $s \to \infty$ such that

$$\frac{mR}{2\sin \pi \frac{b}{2\delta}} \equiv 2\mathrm{N} \exp\left(-\frac{\pi}{2\delta} s\right)$$

is kept constant.

$$\Rightarrow \cdots \Rightarrow \cdots \Rightarrow$$

Main claim:

Main claim:

The Hilbert space of the Sinh-Gordon model contains (is equal to !?) $\mathcal{H}_{\mathrm{TBA}} = \bigoplus_{M=0}^{\infty} \mathcal{H}_M$. The spaces $\mathcal{H}_M$ have ONB $e_{\mathbf{k}}$ labelled by $\mathbf{k} = (k_1, \dots, k_M) \in \mathbb{Z}^M$,, $k_1 > \cdots > k_M$, $e_{\mathbf{k}}$: eigenvect. to the conserved quantities of the model.

The corresponding function $Q_{\mathbf{k}}(\vartheta)$ can be represented as

$$\text{(Q)} \quad \log Q_{\mathbf{k}}(\vartheta) = \int_{\mathbb{R}} \frac{d\vartheta'}{2\pi} \, \frac{\log(W(\vartheta') + Y_{\mathbb{T}_{\mathbf{k}}}(\vartheta'))}{\cosh(\vartheta - \vartheta')} - mR\frac{\cosh\vartheta}{2\sin\vartheta_0} + \sum_{a=1}^{M} \int_{\mathcal{C}_a}^{\vartheta} d\vartheta' \frac{1}{\sinh(\vartheta' - \vartheta_a)} \,.$$

where $\mathbb{T}_{\mathbf{k}} = [\vartheta_1, \dots, \vartheta_M]$ is the unique solution of

$$\text{(B)} \qquad 2\pi k_a + mR\sinh\vartheta_a + \sum_{\substack{b=1 \\ b\neq a}}^{M} \arg S(\vartheta_a - \vartheta_b) + i\int_{\mathbb{R}} \frac{d\vartheta}{2\pi} \, \sigma(\vartheta_a - \vartheta + i\tfrac{\pi}{2}) \log(1 + Y_{\mathbb{T}_{\mathbf{k}}}(\vartheta)) \; = 0 \,,$$

with $Y_{\mathbb{T}}(u)$ being defined as the unique solution to

$$\text{(A)} \quad \log Y_{\mathbb{T}}(\vartheta) - \int_{\mathbb{R}} \frac{d\vartheta'}{2\pi} \, \sigma(\vartheta - \vartheta') \log(1 + Y_{\mathbb{T}}(\vartheta')) + mR\cosh\vartheta + \sum_{a=1}^{M} \log S(\vartheta - \vartheta_a - i\tfrac{\pi}{2}) = 0 \,.$$

The energies are calculated from $Y_{\mathbb{T}}(\vartheta)$ as follows:

$$E_{\mathbf{k}} = \sum_{a=1}^{M} m \cosh \vartheta_a - m \int_{\mathbb{R}} \frac{d\vartheta}{2\pi} \cosh \vartheta \, \log(1 + Y_{\mathbb{T}}(\vartheta))$$

Excited state TBA !

(Generalizes work of Al.B. Zamolodchikov, S. Lukyanov for the ground state)

The IR limit $R \to \infty$

Considering IR limit $R \to \infty$, notice that $Y_{\mathbb{T}}(\vartheta) = \mathcal{O}(e^{-mR})$.

$\Rightarrow$

Particle picture:

- M: number of particles,
- ϑ_a: Rapidity of particle a,

$$E_a = m \cosh \vartheta_a \,, \qquad p_a = m \sinh \vartheta_a \,,$$

- ϑ_a quantized by "asymptotic Bethe ansatz equations".

$$e^{-iRp_a} = \prod_{\substack{b=1 \\ b \neq a}}^{M} S(\vartheta_a - \vartheta_b)$$

For finite R:

$$e^{-iR(p_a+\Phi_a)} = \prod_{\substack{b=1 \\ b\neq a}}^{M} S(\vartheta_a - \vartheta_b)\,,$$

where

$$\Phi_a \equiv \int_{\mathbb{R}} \frac{d\vartheta}{2\pi R}\,\sigma(\vartheta_a - \vartheta + i\tfrac{\pi}{2})\log(1 + Y_{\mathbb{T}}(\vartheta) = \mathcal{O}(e^{-mR})\,.$$

Φ_a: Effects of vacuum polarization!

The UV limit $R \to 0$:

— Connection with Liouville theory!

Compare with “String Bethe Ansatz”:

- Valid for $J \to \infty$, J: angular momentum on S_5

  $\sim$ Radius of world-sheet cylinder in gauge-fixed action.

- For finite J: corrections $\mathcal{O}\big(e^{-2\pi J/\sqrt{\lambda}}\big)$ to String Bethe Ansatz

  (Schäfer-Nameki, Zamaklar, Zarembo)

  $\equiv$ counterparts of e^{-mR}-corrections in Sinh-Gordon

- Gauge theory side: “Wrapping interactions”:
  (see Kotikov, Lipatov, Rej, Staudacher, Velizhanin)

Prediction: Bethe ansatz will fail for AdS_5 sigma model, finite J.

But don’t forget:

There is sthg. better than Bethe ansatz!