(Not quite) Recent Developments in $d = 4, \mathcal{N} = 2$ SCFTs

Yuji Tachikawa (IAS)

in collaboration with

O. Aharony arXiv:0711.4352
and
A. Shapere arXiv:0804.1957

Mar. 2009

Contents

Contents

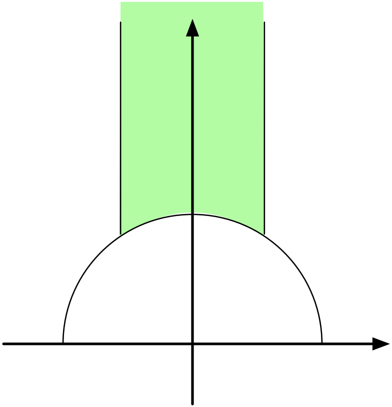
Montonen-Olive S-duality

$\mathcal{N} = 4$ SU(N)

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

$$\tau \to \tau + 1, \qquad \tau \to -\frac{1}{\tau}$$

- Exchanges monopoles W-bosons
- Comes from S-duality of Type IIB



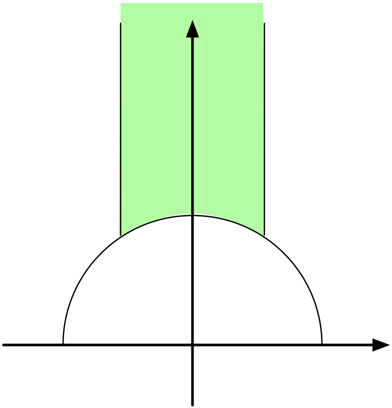
S-duality in $\mathcal{N} = 2$

SU(2) with $N_f = 4$

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$$

$$\tau \to \tau + 1, \qquad \tau \to -\frac{1}{\tau}$$

- Exchanges monopoles and quarks
- Comes from S-duality of Type IIB



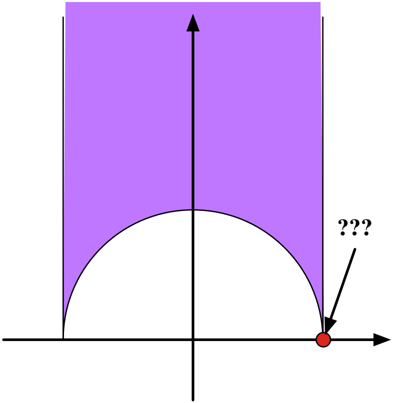
S-duality in $\mathcal{N} = 2$

SU(3) with $N_f = 6$

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$$

$$\tau \to \tau + 2, \qquad \tau \to -\frac{1}{\tau}$$

- Exchanges monopoles and quarks
- Infinitely Strongly coupled at $\tau = 1$



New S-duality [Argyres-Seiberg]

SU(3) + 6 flavors

at coupling τ

$\updownarrow$

SU(2) + 1 flavor + SCFT[E_6]

at coupling $\tau' = \dfrac{1}{1-\tau}$, SU(2) $\subset E_6$ is gauged

Strongly coupled $\mathcal{N} = 2$ SCFT

- Consider SW curve given by

$$y^2 = x^3 + u^4$$

- Electron & Monopole both massless at $u = 0$
  ⟶ conformal theory
  [Argyres-Douglas, Argyres-Plesser-Seiberg-Witten,Eguchi-Hori-Ito-Yang]
- Four-dimensional total space $=\mathbb{C}^2$/tetrahedral
- Known to possess E_6 as the flavor symmetry !
  [Minahan-Nemechansky]
- $\lambda_{SW} = u\dfrac{dx}{y}$ has $\dim = 1$

  ⟶ $\dim(u) = 3$, $\dim(x) = 4$, $\dim(y) = 6$

Argyres-Seiberg: Dimensions

$SU(3) + 6$ flavors

$$\dim(\mathrm{tr}\,\phi^2) = 2, \qquad \dim(\mathrm{tr}\,\phi^3) = 3$$

$\Updownarrow$

$SU(2) + 1$ flavor $+ \mathrm{SCFT}[E_6]$

$$\mathrm{tr}\,\varphi^2 \text{ of } SU(2) : \dim = 2, \qquad u \text{ of } E_6 : \dim = 3$$

Argyres-Seiberg: Flavor symmetry

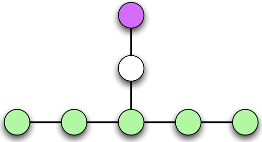
SU(3) + 6 flavors

- Flavor symmetry: $U(6) = \mathbf{U(1)} \times \mathbf{SU(6)}$

↕

SU(2) + 1 flavor + SCFT[E_6]

- **SO(2)** acts on 1 flavor = 2 half-hyper of **SU(2)** doublet
- **SU(2)** $\subset E_6$ is gauged
- **SU(2)** $\times$ **SU(6)** $\subset E_6$ is a maximal regular subalgebra



Current Algebra Central Charge

Normalize s.t. a free hyper in the fund. of $\mathbf{SU}(N)$ contributes $\mathbf{2}$ to k_G

$$J^a_\mu(x)J^b_\nu(0) = \frac{3}{4\pi^2} k_G \delta^{ab} \frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{x^8} + \cdots$$

A bifundamental hyper under $\mathbf{SU}(N) \times \mathbf{SU}(M)$

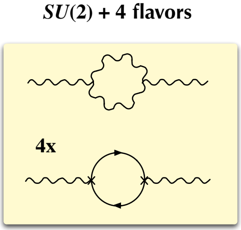
$$\longrightarrow \quad k_{\mathsf{SU}(N)} = 2M, \qquad k_{\mathsf{SU}(M)} = 2N$$

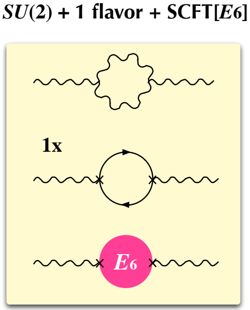
SU(3) + 6 flavors

$$k_{\mathsf{SU}(6)} = 6$$

k for SCFT[E_6]

SU(2) $\subset E_6$ central charge:

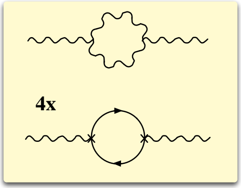




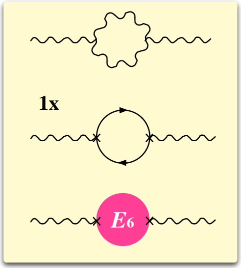
k for SCFT[E_6]

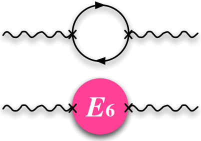
SU(2) $\subset E_6$ central charge:

$SU(2)$ + 4 flavors



$SU(2)$ + 1 flavor + SCFT[E_6]





$= \langle j_\mu j_\nu \rangle_{\text{free hyper}}$

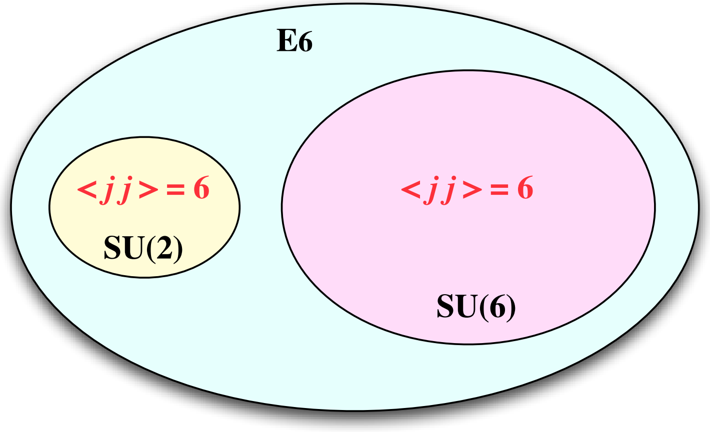
$= \langle j_\mu j_\nu \rangle_{\text{SCFT}[E6]}$

$\Rightarrow$

$$\langle j_\mu j_\nu \rangle_{\text{SCFT}[E6]} = 3 \langle j_\mu j_\nu \rangle_{\text{free hyper}}$$

$$= 6$$

k for SCFT[E_6]



Central charges a and c of conformal algebra

$$\langle T^\mu_\mu \rangle = a \times \text{Euler} + c \times \text{Weyl}^2$$

- 1 free hyper : $a = 1/24$, $c = 1/12$
- 1 free vector : $a = 5/24$, $c = 1/6$
- **SU(3) + 6 flavors**: $a = 29/12$, $c = 17/6$
- **SU(2) + 1 flavor**: $a = 17/24$, $c = 2/3$

SCFT[E_6]

$$a = \frac{29}{12} - \frac{17}{24} = \frac{41}{24}, \qquad c = \frac{17}{6} - \frac{2}{3} = \frac{13}{6}$$

Summary

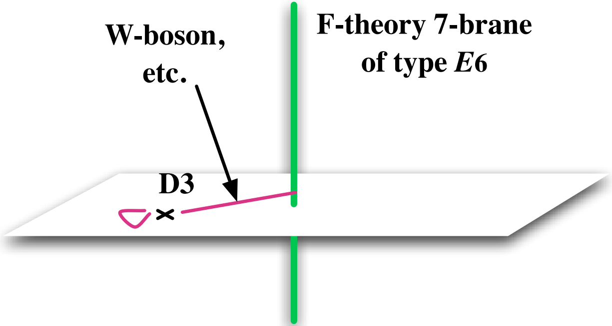
G	D_4	E_6	E_7	E_8
k_G	4	6	8	???
$24a$	23	41	59	???
$6c$	7	13	19	???

- D_4 : SU(2) + 4 flavors
- E_6 : SU(3) + 6 flavors $\leftrightarrow$ SU(2) + 1 flavor + SCFT[E_6]
- E_7 : USp(4) + 6 flavors $\leftrightarrow$ SU(2) + SCFT[E_7]

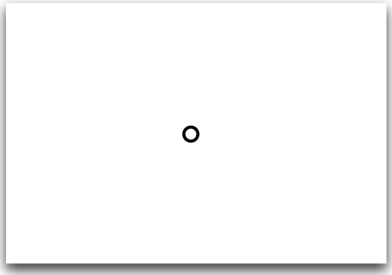
- What would be k_{E_8}, a_{E_8} and c_{E_8}?

Contents

$\mathcal{N} = 2$ SCFT from F-theory

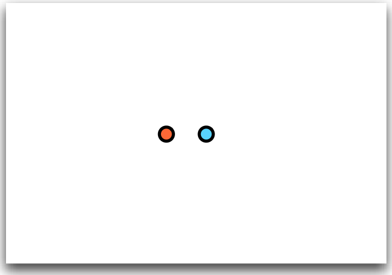


$\mathcal{N} = 2$ SU(2) from orientifolds



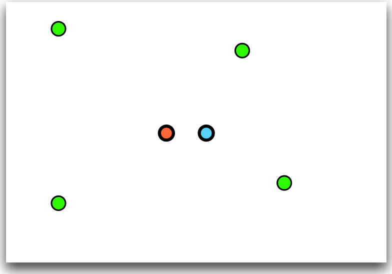
- Enhanced **SU(2)** symmetry at the origin $u = 0$ $\longrightarrow$

$\mathcal{N} = 2$ SU(2) from orientifolds



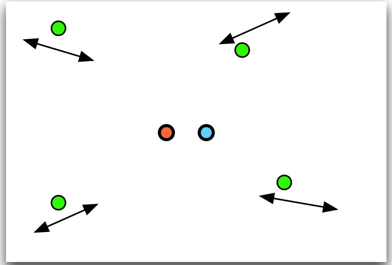
- Enhanced **SU(2)** symmetry at the origin $u = 0$ ⟶
- Monopole point $u = \Lambda^2$
- Dyon point $u = -\Lambda^2$
- O7 splits into 7-branes $A + B$

$\mathcal{N} = 2$ SU(2) from orientifolds



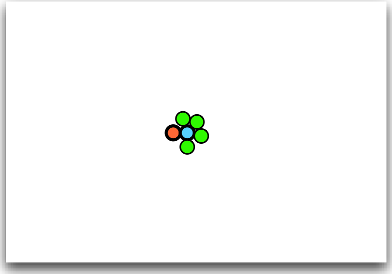
- Enhanced **SU(2)** symmetry at the origin $u = 0$ →
- Monopole point $u = \Lambda^2$
- Dyon point $u = -\Lambda^2$
- O7 splits into 7-branes $A + B$
- 4 additional D7-branes $u \sim m_i^2$

$\mathcal{N} = 2$ SU(2) from orientifolds



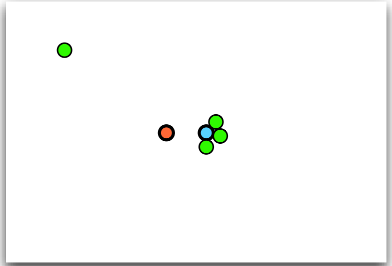
- Enhanced **SU(2)** symmetry at the origin $u = 0$ ⟶
- Monopole point $u = \Lambda^2$
- Dyon point $u = -\Lambda^2$
- O7 splits into 7-branes $A + B$
- 4 additional D7-branes $u \sim m_i^2$

D_4 singularity



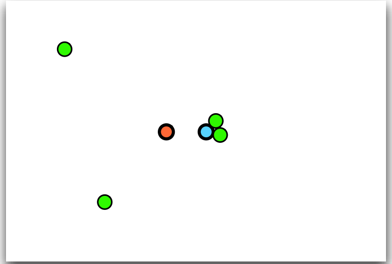
- Can put all 7-branes together
- O7+4 D7, dilaton tadpole=0
- $D_4 = \mathbf{SO(8)}$ symmetry on the 7-branes
- flavor symmetry from the D3 pov

Argyres-Douglas points



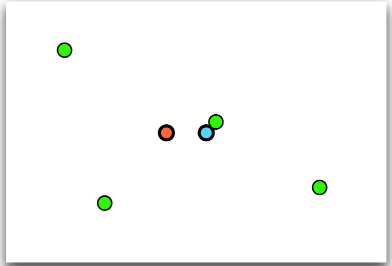
- Can move one D7 away
- Orientifold split again
- $A_2 = \mathbf{SU(3)}$ symmetry

Argyres-Douglas points



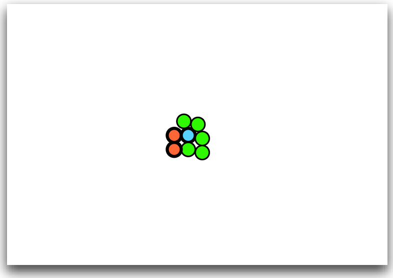
- Can move one D7 away
- Orientifold split again
- $A_2 = \mathbf{SU(3)}$ symmetry
- $A_1 = \mathbf{SU(2)}$ symmetry

Argyres-Douglas points



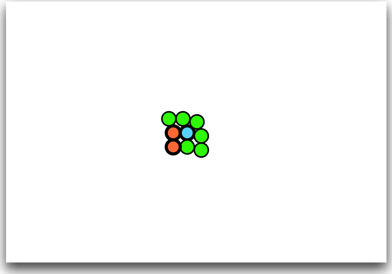
- Can move one D7 away
- Orientifold split again
- $A_2 = \mathbf{SU(3)}$ symmetry
- $A_1 = \mathbf{SU(2)}$ symmetry
- A_0 no symmetry

E_n singularities



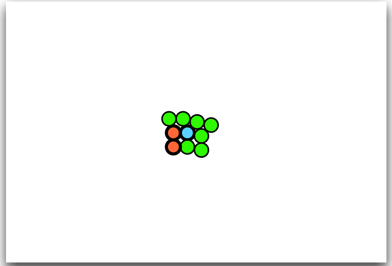
- Can add one A brane, and then another C brane
- E_6 symmetry

E_n singularities



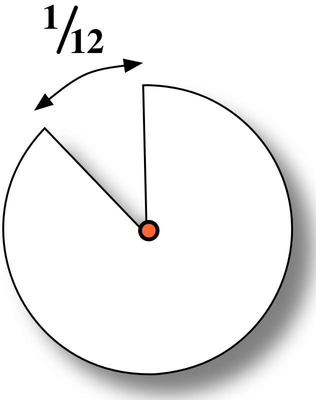
- Can add one A brane, and then another C brane
- E_6 symmetry
- E_7 symmetry

E_n singularities



- Can add one A brane, and then another C brane
- E_6 symmetry
- E_7 symmetry
- E_8 symmetry

Deficit angle



- Codimension two object
- Deficit angle $2\pi/12$ per one 7-brane
- angular periodicity

$$2\pi \rightarrow \frac{2\pi}{\Delta}$$

where

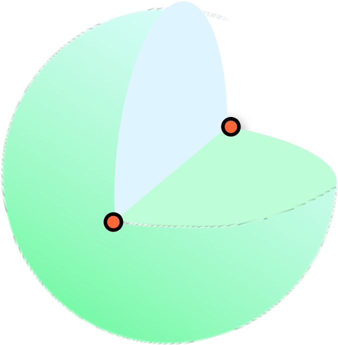
$$\Delta = \frac{12}{12 - n_7}$$

Summary

G	H_0	H_1	H_2	D_4	E_6	E_7	E_8
n_7	2	3	4	6	8	9	10
Δ	6/5	4/3	3/2	2	3	4	6
τ	ω	i	ω	arb.	ω	i	ω

- Probe by one D3 $\longrightarrow$ isolated SCFT with flavor symmetry G
- Probe by N D3s $\longrightarrow$ rank N version, with flavor symmetry G
- Can take near-horizon limit when $N \gg 1$

Near horizon limit



- $AdS_5 \times S^5/\Delta$
- $|x|^2 + |y|^2 + |z|^2 = 1$,
  $z \sim z \exp(2\pi i/\Delta)$
- G-type 7-brane at $z = 0$,
  wrapping S^3
- $F_5 = N\Delta(\mathrm{vol}_{S^5} + \mathrm{vol}_{AdS_5})$

[Fayyazuddin-Spałinski,
Aharony-Fayyazuddin-Maldacena]

Central charges from AdS/CFT

- SUSY relates k_G, a and c to the triangle anomalies

$$k_G \delta^{ab} = -3\,\mathrm{tr}(R_{\mathcal{N}=1} T^a T^b)$$
$$a = \frac{3}{32}\left[3\,\mathrm{tr}\,R^3_{\mathcal{N}=1} - \mathrm{tr}\,R_{\mathcal{N}=1}\right]$$
$$c = \frac{1}{32}\left[9\,\mathrm{tr}\,R^3_{\mathcal{N}=1} - 5\,\mathrm{tr}\,R_{\mathcal{N}=1}\right]$$

- AdS/CFT relates triangle anomalies to the Chern-Simons term

$$\mathrm{tr}(R_{\mathcal{N}=1} T^a T^b) \longrightarrow A_R \wedge \mathrm{tr}\,F_G \wedge F_G$$
$$\mathrm{tr}(R^3_{\mathcal{N}=1}) \longrightarrow A_R \wedge F_R \wedge F_R$$
$$\mathrm{tr}(R_{\mathcal{N}=1}) \longrightarrow A_R \wedge \mathrm{tr}\,R \wedge R$$

Chern-Simons terms : $O(N^2)$

- A_R enters in 10d bulk fields:

$$F_5 = N\Delta(\mathrm{vol}_{S^5} + \mathrm{vol}_{AdS_5}) + N\Delta\, dA_R \wedge \omega$$
$$ds^2_{10} = ds^2_{AdS_5} + ds^2_{S^5} + (k^R A_R)^2$$

- $O(N^2)$ contribution $\sim \int_{S^5/\Delta} F_5^2 \sim N^2\Delta$

Chern-Simons terms : $O(N)$

- Coupling on the 7-brane stack

$$n_7 \int C_4 \wedge [\mathrm{tr}\, R_T \wedge R_T - \mathrm{tr}\, R_N \wedge R_N] + \int C_4 \wedge \mathrm{tr}\, F_G \wedge F_G$$

- $O(N)$ contribution to $a, c \sim \int_{S^3} n_7 C_4 \sim n_7 N \Delta$
- $O(N)$ contribution to $k_G \sim \int_{S^3} C_4 \sim N \Delta$

Chern-Simons terms : $O(1)$

For $\mathcal{N} = 4$ SU(N),

- central charge $\sim N^2 - 1$
- Supergravity analysis gives N^2
- -1 comes from the decoupling of the center-of-mass motion of N D3's

In our case,

- motion transverse to 7-brane : coupled
- motion parallel to 7-brane : decoupled
- subtract the contribution from a free hyper:

$$\delta a = -1/24, \qquad \delta c = -1/12$$

Summary

We get

$$\begin{aligned} k_G &= 2N\Delta \\ a &= \frac{1}{4}N^2\Delta + \frac{1}{2}N(\Delta - 1) - \frac{1}{24} \\ c &= \frac{1}{4}N^2\Delta + \frac{3}{4}N(\Delta - 1) - \frac{1}{12} \end{aligned}$$

Summary

We get

$$k_G = 2N\Delta$$
$$a = \frac{1}{4}N^2\Delta + \frac{1}{2}N(\Delta - 1) - \frac{1}{24}$$
$$c = \frac{1}{4}N^2\Delta + \frac{3}{4}N(\Delta - 1) - \frac{1}{12}$$

Let's put $N = 1$ ...

Summary

G	H_0	H_1	H_2	D_4	E_6	E_7	E_8
k_G	12/5	8/3	3	4	6	8	12
$24a$	43/5	11	14	23	41	59	95
$6c$	11/5	3	4	7	13	19	31

- Results for D_4, E_6, E_7 perfectly match with field theoretical calculation !
- Results for E_8 were predictions.

Summary

G	H_0	H_1	H_2	D_4	E_6	E_7	E_8
k_G	12/5	8/3	3	4	6	8	12
$24a$	43/5	11	14	23	41	59	95
$6c$	11/5	3	4	7	13	19	31

- Results for D_4, E_6, E_7 perfectly match with field theoretical calculation !
- Results for E_8 were predictions.
- Later confirmed by a field-theoretical construction [Argyres-Wittig]

USp(6) with one 14-dim rep. $\oplus$ 11 fundamentals

$\leftrightarrow$ **SO(5)** with **SCFT[E_8]**

Contents

Argyres-Douglas points

How about them ?

G	H_0	H_1	H_2	D_4	E_6	E_7	E_8
k_G	12/5	8/3	3	4	6	8	12
$24a$	43/5	11	14	23	41	59	95
$6c$	11/5	3	4	7	13	19	31

- Simpler field theory realization : just take **SU(2)** with $N_f = 1, 2, 3$, set the mass appropriately.
- a, c linear combination of 't Hooft anomalies of R symmetry
- R symmetry known, but it's accidental in the IR
- couldn't calculate a and c field theoreticaly...

Twisting and a and c

- $\boldsymbol{a}$ and $\boldsymbol{c}$ measures the response of the CFT to the external gravity

$$\langle T^{\mu}_{\mu} \rangle = a \times \text{Euler} + c \times \text{Weyl}^2$$

- the best way to couple $\mathcal{N} = 2$ supersymmetric theory to gravity = topological twisting.
- Are $\boldsymbol{a}$ and $\boldsymbol{c}$ encoded in the topological theory ?
  Yes ! in the so-called $A^{\chi}B^{\sigma}$ term which is known for 10 yrs by [Witten, Moore, Mariño, Losev, Nekrasov, Shatashvili]

Topological twisting

$$\delta\phi = \epsilon^{\alpha}\psi_{\alpha}$$

- On a curved bkg, constant ϵ_{α} not possible $\longrightarrow$ no global susy

Topological twisting

$$\delta\phi = \epsilon_{i}^{\alpha}\psi_{\alpha}^{i}$$

- On a curved bkg, constant ϵ_{α}^{i} not possible $\longrightarrow$ no global susy
- They are $\mathsf{SU(2)}_R$ doublet

Topological twisting

$$\delta\phi = \epsilon_i^\alpha \psi_\alpha^i$$

- On a curved bkg, constant ϵ_α^i not possible ⟶ no global susy
- They are $\mathrm{SU}(2)_R$ doublet
- Introduce external $\mathrm{SU}(2)_R$ gauge field ($a = 1, 2, 3$)

$$F^a_{\mu\nu,R} = R_{\mu\nu\rho\sigma}\Omega^{\rho\sigma,a}$$

i.e. self-dual part of metric connection.

- ϵ_i^α : $2 \times 2 = 1 + 3$
- One global susy preserved !

$A^\chi B^\sigma$ term

Just as nontrivial $\tau(u) F\tilde{F}$ is generated, on a curved manifold

$$S_{\text{curved}} = [\log A(u)] R\tilde{\tilde{R}} + [\log B(u)] R\tilde{R} + \cdots$$

are generated. Then we have

$$\langle O_1 O_2 \cdots \rangle = \int [du] e^{-S} A(u)^\chi B(u)^\sigma O_1 O_2 \cdots$$

$$\text{with} \quad \chi = \frac{1}{32\pi^2} \int d^4x \sqrt{g} R_{abcd} \tilde{\tilde{R}}_{abcd}, \qquad \sigma = \frac{1}{48\pi^2} \int d^4x \sqrt{g} R_{abcd} \tilde{R}_{abcd}.$$

$A^\chi B^\sigma$ and R anomaly

$$\langle O_1 O_2 \cdots \rangle = \int [du] A(u)^\chi B(u)^\sigma O_1 O_2 \cdots$$

means, on a curved manifold, $\langle O_1 O_2 \cdots \rangle$ nonzero only if

$$R(O_1) + R(O_2) + \cdots = -\chi R(A) - \sigma R(B) - R([du])$$

i.e. the vacuum has the R-anomaly

$$\Delta R = \chi R(A) + \sigma R(B) + \frac{\chi + \sigma}{2} r + \frac{\sigma}{4} h$$

where r, h the number of free vectors / hypers

R-anomaly in physical/twisted theories

$$a = \frac{3}{32}\left[3\,\mathrm{tr}\,R^3_{\mathcal{N}=1} - \mathrm{tr}\,R_{\mathcal{N}=1}\right], \quad c = \frac{1}{32}\left[9\,\mathrm{tr}\,R^3_{\mathcal{N}=1} - 5\,\mathrm{tr}\,R_{\mathcal{N}=1}\right]$$

can also be represented as

$$\partial_\mu R^\mu_{\mathcal{N}=1} = \frac{c-a}{24\pi^2} R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \frac{5a-3c}{9\pi^2} F^{\mathcal{N}=1}_{\mu\nu}\tilde{F}^{\mu\nu}_{\mathcal{N}=1}$$

Using $R_{\mathcal{N}=1} = R_{\mathcal{N}=2}/3 + 4I_3/3$ etc., we have

$$\partial_\mu R^\mu_{\mathcal{N}=2} = \frac{c-a}{8\pi^2} R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \frac{2a-c}{8\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a$$

R-anomaly in physical/twisted theories

Twisting sets

$$F^a_{\mu\nu} = \text{anti-self-dual part of } R_{\mu\nu\rho\sigma}$$

so

$$\partial_\mu R^\mu = \frac{c-a}{8\pi^2} R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \frac{2a-c}{8\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a$$

becomes

$$\partial_\mu R^\mu = \frac{2a-c}{16\pi^2} R_{\mu\nu\rho\sigma}\tilde{\tilde{R}}_{\mu\nu\rho\sigma} + \frac{c}{16\pi^2} R_{\mu\nu\rho\sigma}\tilde{R}_{\mu\nu\rho\sigma}.$$

Therefore

$$\Delta R = 2(2a-c)\chi + 3c\,\sigma$$

$A^\chi B^\sigma$ and a, c

Comparing

$$\Delta R = 2(2a - c)\chi + 3c\,\sigma$$

and

$$\Delta R = \chi R(A) + \sigma R(B) + \frac{\chi + \sigma}{2} r + \frac{\sigma}{4} h$$

we have

$$a = \frac{1}{4} R(A) + \frac{1}{6} R(B) + \frac{5}{24} r + \frac{1}{24} h,$$
$$c = \frac{1}{3} R(B) + \frac{1}{6} r + \frac{1}{12} h.$$

$A^\chi B^\sigma$

A and B have been calculated
[Witten,Moore,Mariño,Nekrasov,Losev,Shatashivili]

$$A(u)^2 = \det \frac{\partial u_i}{\partial a_I}, \qquad B(u)^8 = D$$

- $u_i = \mathrm{tr}\,\phi^i$: gauge-invariant coordinates
- a^I: special coordinates
- D: the discriminant of the SW curve

⟶ taking their R-charges, we get a and c.

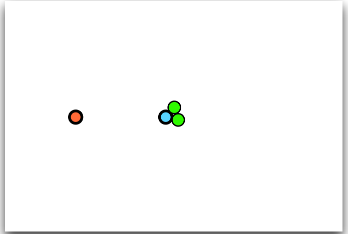
a and c of Argyres-Douglas points

How about them ?

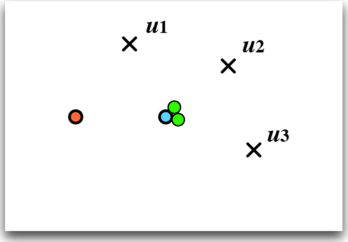
G	H_0	H_1	H_2	D_4	E_6	E_7	E_8
k_G	12/5	8/3	3	4	6	8	12
$24a$	43/5	11	14	23	41	59	95
$6c$	11/5	3	4	7	13	19	31

- It was calculated using AdS/CFT, with $N = 1$.
- Nicely reproduced from $A^{\chi}B^{\sigma}$ terms.

a and c of Argyres-Douglas points

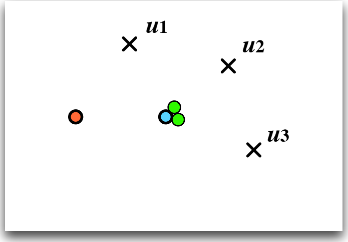


a and c of Argyres-Douglas points



- **USp(2N)** + N_f flavors + 1 antisymmetric

a and c of Argyres-Douglas points



- **USp(2N)** + N_f flavors + 1 antisymmetric

$$A^2 = \det \frac{\partial s_k}{\partial a_I}, \qquad B^8 = D = \prod_{i>j} (u_i - u_j)^6 \prod_i u_i^{1+Nf}$$

- $s_k = \langle \mathrm{tr}\, \phi^{2k} \rangle = k$-th sym. product of u_i

a and c of Argyres-Douglas points

We get

$$k_G = 2N\Delta$$
$$a = \frac{1}{4}N^2\Delta + \frac{1}{2}N(\Delta - 1) - \frac{1}{24}$$
$$c = \frac{1}{4}N^2\Delta + \frac{3}{4}N(\Delta - 1) - \frac{1}{12}$$

which completely agrees with the holographic calculation !

Contents

Summary

- Field theoretical realization of strange $\mathcal{N} = 2$ SCFTs
- central charges :
  pure field theory and AdS/CFT gave the same answer !
- new technique to calculate central charges,
  applicable to generic $\mathcal{N} = 2$ SCFT.