$\mathcal{N} = 1$ SQCD and the Transverse Field Ising Model

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(with David Poland [arXiv:1104.1425])

Motivation

4D CFTs and SCFTs are interesting for many reasons:

- Basic building blocks of QFTs,
- Dual to theories of quantum gravity in AdS,
- Could play a role in BSM physics!
  - Walking/Conformal Technicolor [Many people...]
  - Conformal Sequestering [Luty, Sundrum '01; Schmaltz, Sundrum '06]
  - Solution to $\mu/B\mu$ problem [Roy, Schmaltz '07; Murayama, Nomura, Poland '07]
  - Flavor Hierarchies [Georgi, Nelson, Manohar '83; Nelson, Strassler '00; Poland, DSD '09; Craig '10] ...

Unfortunately,

- Phenomenological applications often involve statements about operator dimensions that are difficult to check.
- In $\mathcal{N} = 1$ SCFTs, we know lots about chiral operators, but not much about non-chiral operators. Gravity duals useful at very strong coupling $\lambda \gg 1$.

Example: $\mu/B\mu$ problem

- X gets SUSY-breaking vev in the IR,

$$\langle X \rangle = M + \theta^2 F$$

$$\int d^4\theta X^\dagger H_u H_d \to \mu, \qquad \int d^4\theta X^\dagger X H_u H_d \to B\mu$$

- $B\mu$ often too big for natural EWSB, but can be smaller if X participates in strong dynamics above $\Lambda_{\mathrm{SUSY-breaking}}$ and $\dim X^\dagger X > 2 \dim X$.
- In specific examples, can calculate $\dim X$ ($\Delta = \frac{3}{2} R$ for chiral operators).
- Can't say much about $\dim X^\dagger X$ (though some bounds exist [Rattazzi, Rychkov, Tonni, Vichi '08; Poland, DSD '10]).

Progress at $\lambda \sim O(1)$

Hard to say things at $\lambda \sim O(1)$. But one major exception is in (planar) $\mathcal{N}=4$ SYM (and related theories).

- *Integrability* lets one calculate operator dimensions at $\lambda \sim O(1)$ and match onto both weak-coupling Feynman diagram expansion and strong coupling calculations in AdS.

$$
\begin{aligned}
f(\lambda) &= 8\lambda - \frac{8\pi^2}{3}\lambda^2 + \frac{88\pi^4}{45}\lambda^3 + \ldots \\
&= 4\lambda^{1/2} - \frac{3\log 2}{\pi} - \frac{K}{4\pi^2}\lambda^{-1/2} + \ldots
\end{aligned}
$$

[Beisert, Alday, Staudacher, Minahan, Zarembo, many others...]

- Some of the first evidence: Minahan, Zarembo '02 showed that the 1-loop dilatation operator acting on scalars $\mathrm{Tr}(X_i \ldots X_j)$ is equivalent to an exactly solvable Heisenberg spin chain.

Why $\mathcal{N}=1$ SQCD?

- Only a narrow range of theories studied so far ($\mathcal{N}=4$, orbifolds, β-deformation, recently $\mathcal{N}=2$ [Gadde, Pomoni, Rastelli '10])
- None of them phenomenologically relevant.
- Success suggests interesting structures might lie hidden in a wider range of theories (not necessarily full integrability).

We need more data! This talk: mimic Minahan and Zarembo for (planar) $\mathcal{N}=1$ SQCD

- Simplest nontrivial $\mathcal{N}=1$ superconformal theory.
- *Two* weak coupling (Banks-Zaks) regimes. For now, we'll focus on the weakly-coupled electric description. Eventually, we might hope to learn more about Seiberg duality.

Outline

Outline

1. $\mathcal{N}=1$ SQCD
2. 1-loop Dilatation Operator
3. Spin Chain Solution
4. Outlook

$\mathcal{N}=1$ SQCD

- Matter content:

	$\mathrm{SU}(N_c)$	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$	$U(1)_B$	$U(1)_R$
Q_{ai}	$\square$	$\square$	1	1	$1-\frac{N_c}{N_f}$
$\widetilde{Q}^{\tilde{i}a}$	$\overline{\square}$	1	$\overline{\square}$	-1	$1-\frac{N_c}{N_f}$

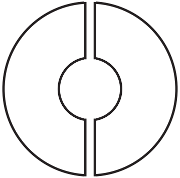
$$i,\tilde{\imath} = 1,\ldots,N_f$$
$$a = 1,\ldots,N_c$$

- Conformal window: $\frac{3}{2} < \frac{N_f}{N_c} < 3$
- Veneziano limit: $N_f, N_c \to \infty$, with $\frac{N_f}{N_c}$ fixed. Electric weak-coupling regime: $\epsilon = \frac{3N_c}{N_f} - 1 \ll 1$,

$$\lambda \equiv \frac{g^2 N_c}{8\pi^2} = \epsilon + \frac{\epsilon^2}{2} + \frac{9}{4}(1+2\zeta_3)\epsilon^3 \ldots$$

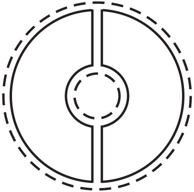
Large-N Counting

- 't Hooft limit: Double lines for adjoints; Single lines for fundamentals



$$\sim \; N_c^{2-2h-b}$$

- Veneziano limit: Double (gauge+flavor) lines for fundamentals



$$\sim \; N_c^{2-2h-b} N_f^b \quad \sim \quad N^{2-2h}$$

Generalized Single Trace Operators

Basic objects are "generalized single-trace" (GST) operators. Using only scalars, we have: flavor singlets,

$$Q_{ai}\, Q^{\dagger ib}\, \widetilde{Q}^{\dagger}_{b\tilde{\jmath}} \widetilde{Q}^{\tilde{\jmath}c} \ldots Q^{ka} \quad = \quad \mathrm{Tr}(XY\ldots X) \quad \text{“closed”}$$

where,

$$X \equiv (QQ^{\dagger})^{a}_{b}, \quad Y \equiv (\widetilde{Q}^{\dagger}\widetilde{Q})^{a}_{b}$$

and flavor adjoint+bifundamentals,

$$\left.\begin{array}{c} Q^{\dagger}XY\ldots XQ \\ \widetilde{Q}XY\ldots XQ \\ Q^{\dagger}XY\ldots X\widetilde{Q}^{\dagger} \\ \widetilde{Q}XY\ldots X\widetilde{Q}^{\dagger} \end{array}\right\} \quad \text{“open”}$$

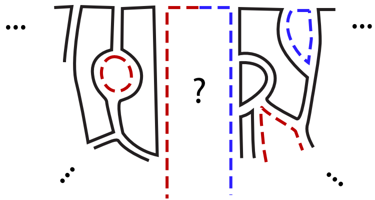
Factorization

Dimensions and correlators of generalized *multi*-trace operators factorize at large N:

$$\begin{aligned} \dim(\mathcal{O}_1\mathcal{O}_2) &= \dim\mathcal{O}_1 + \dim\mathcal{O}_2 + O(1/N^2) \\ \langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle &= \sum_{\text{pairings}}\prod_{\text{pairs}}\langle\mathcal{O}_i\mathcal{O}_j\rangle + O(1/N) \end{aligned}$$

Note: operators with left-right flavor indices contracted are GMT

$$\text{e.g.:}\qquad \mathrm{Tr}(MM\ldots M) \;=\; Q_{ai}\,\widetilde{Q}^{ib}\,Q_{bj}\,\widetilde{Q}^{jc}\,\ldots\,\widetilde{Q}^{ka}$$



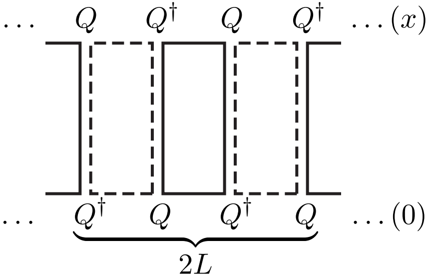
Outline

Tree-Level

To compute anomalous dimensions, we must study 2-pt functions

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \quad \sim \quad x^{-2(\Delta_0+\lambda\Delta_1+\dots)} = x^{-2\Delta_0}(1-\lambda\Delta_1\log(x^2)+\dots)$$

At tree level,



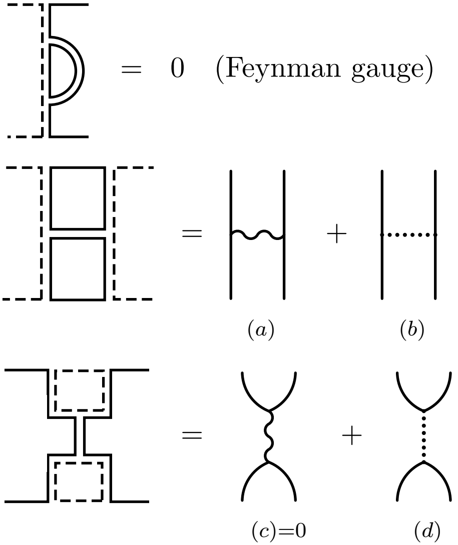
$$= \quad N_c^L N_f^L x^{-2(2L)}$$

One Loop Contributions

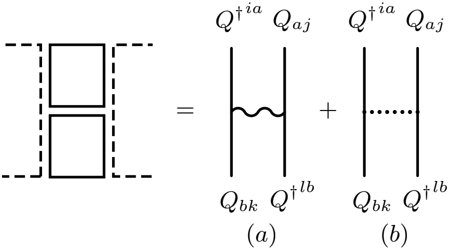
wavefunction:

color-loop:

flavor-loop:



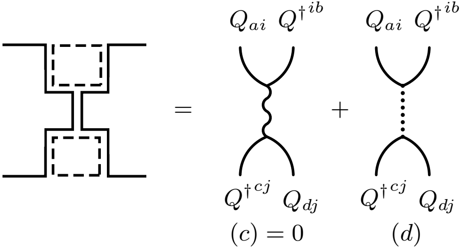
Color Loops



Graph (b) comes from the D-term potential: $(Q^\dagger T^A Q - \widetilde{Q} T^A \widetilde{Q}^\dagger)^2$

$$(a)+(b) \;=\; \left(\begin{array}{c|cccc} & Q^\dagger Q & Q^\dagger \widetilde{Q}^\dagger & \widetilde{Q} Q & \widetilde{Q}\widetilde{Q}^\dagger \\ \hline QQ^\dagger & A+B & & & \\ \widetilde{Q} Q & & A-B & & \\ Q^\dagger \widetilde{Q}^\dagger & & & A-B & \\ \widetilde{Q}\widetilde{Q}^\dagger & & & & A+B \end{array}\right) \times \lambda\, \mathbb{I}_{\text{flavor}}$$

Flavor Loops



Graph (d) is a 4-scalar contact interaction, so equals (b) up to overall factors

$$(c) + (d) \;=\; \left(\begin{array}{c|cc} & QQ^{\dagger} & \widetilde{Q}^{\dagger}\widetilde{Q} \\ \hline Q^{\dagger}Q & B & -B \\ \widetilde{Q}\widetilde{Q}^{\dagger} & -B & B \end{array}\right) \times \frac{N_f}{N_c}\lambda \times 2T^A \otimes T^A$$

In the planar limit, $2T^A \otimes T^A$ is equivalent to $\mathbb{I}_{\text{color}}$.
Exception: 2-field operators, where $\mathrm{Tr}(T^A) \otimes \mathrm{Tr}(T^A) = 0$.

Fixing A, B from Consistency

$$
\gamma_{\text{two-field}} = \left(\begin{array}{c|cccc}
 & Q^\dagger Q & Q^\dagger \widetilde{Q}^\dagger & \widetilde{Q} Q & \widetilde{Q}\widetilde{Q}^\dagger \\
\hline
Q Q^\dagger & A+B & & & \\
\widetilde{Q} Q & & A-B & & \\
Q^\dagger \widetilde{Q}^\dagger & & & A-B & \\
\widetilde{Q}\widetilde{Q}^\dagger & & & & A+B
\end{array}\right)\lambda
$$

No need to evaluate any diagrams:

$$
\begin{aligned}
\gamma(\widetilde{Q}Q) &= (A-B)\lambda = -\lambda && \text{(chiral: } \Delta = 3R/2) \\
\gamma(Q^\dagger Q - \widetilde{Q}\widetilde{Q}^\dagger) &= (A+B)\lambda = 0 && \text{(conserved current)}
\end{aligned}
$$

(remember $\lambda = \frac{3N_c}{N_f} - 1$ at one loop)

So $A = -B = -\frac{1}{2}$.

Spin Chain Notation

- We have gauge-adjoint dimers $X = QQ^\dagger, Y = \widetilde{Q}^\dagger \widetilde{Q}$.
- Useful notation: $XYX\ldots \longrightarrow |\uparrow\downarrow\uparrow\ldots\rangle$ (spin chain state). $\mathrm{Tr}(XY\ldots X)$ corresponds to a shift-invariant state.

$$\begin{aligned}\text{color-loop} &= \lambda(A\,\mathbb{I}_i \otimes \mathbb{I}_{i+1} + B\,\sigma_i^z \otimes \sigma_{i+1}^z) \\ \text{flavor-loop} &= \frac{\lambda N_f}{N_c}(B\mathbb{I}_i - B\sigma_i^x)\end{aligned}$$

$$\gamma_{\text{closed}} = \frac{N_f - N_c}{2N_c}\lambda L + \frac{\lambda}{2}\underbrace{\sum_{i=0}^{L-1}(\sigma_i^z\sigma_{i+1}^z - \frac{N_f}{N_c}\sigma^x)}_{H}$$

- H is Hamiltonian for Transverse Field Ising Model ($h = \frac{N_f}{N_c} \approx 3$): exactly solvable (Pfeuty '70)

Outline

Jordan-Wigner Transformation

2 states/site $\implies$ fermions?

$$\begin{aligned} \sigma_i^{\pm} &\equiv \frac{1}{2}(\sigma_i^y \pm i\sigma_i^z) \\ 0 &= [\sigma_i^{\pm}, \sigma_j^{\pm}] = [\sigma_i^{\pm}, \sigma_j^{\mp}] \qquad (i \neq j) \end{aligned}$$

Jordan-Wigner transformation ('28), discrete version of fermionization:

$$c_n^\dagger = \left(\prod_{m=0}^{n-1} \sigma_m^x\right) \sigma_n^+ \qquad c_n = \left(\prod_{m=0}^{n-1} \sigma_m^x\right) \sigma_n^-$$

$$\{c_n^\dagger, c_m\} = \delta_{nm}, \qquad \{c_n, c_m\} = \{c_n^\dagger, c_m^\dagger\} = 0$$

σ^x =parity flip ($Q \leftrightarrow \widetilde{Q}^\dagger$). Each site is sensitive to the overall parity of sites to the left.

Jordan-Wigner Transformation

Key fact: Hamiltonian quadratic in $c, c^\dagger$,

$$\begin{aligned} H &= \sum_{n=0}^{L-1} \sigma_n^z \sigma_{n+1}^z - h\sigma_n^x \\ &= \sum_{n=0}^{L-1} (c_n^\dagger + c_n)(c_{n+1}^\dagger - c_{n+1}) - h(2c_n^\dagger c_n - 1). \end{aligned}$$

Correct boundary condition for c's depends on overall parity,

$$P \equiv \prod_{n=0}^{L-1} \sigma_n^x = \begin{cases} -1: & \text{periodic} \\ +1: & \text{anti-periodic} \end{cases}$$

Diagonalizing the Fermion System

$$
\begin{aligned}
H &= \sum_{n=0}^{L-1}(c_n^\dagger + c_n)(c_{n+1}^\dagger - c_{n+1}) - h(2c_n^\dagger c_n - 1) \\
&= \sum_k \left[-(2\cos k + 2h)c_k^\dagger c_k - i\sin k(c_{-k}^\dagger c_k^\dagger + c_{-k}c_k)\right] + Lh \\
&= \sum_k \epsilon(k)\left(b_k^\dagger b_k - \frac{1}{2}\right) \qquad \text{(Bogoliubov)}
\end{aligned}
$$

$\Longrightarrow$ A system of free fermions, with dispersion relation

$$
\epsilon(k) = 2\sqrt{h^2 + 1 + 2h\cos k} \qquad \left(h = \frac{N_f}{N_c} = 3 + O(\lambda)\right)
$$

Quasimomenta Quantization

$$
\begin{aligned}
H &= \sum_k \epsilon(k)\left(b_k^\dagger b_k - \frac{1}{2}\right) \\
P &= \begin{cases} -1: & k = \frac{2\pi m}{L} \\ +1: & k = \frac{(2m+1)\pi}{L} \end{cases}
\end{aligned}
$$

We can also think of operators with open flavor indices as states in an open chain with "Dirichlet" boundary conditions,

$$
Q^\dagger XY \ldots YQ \qquad \Longrightarrow \qquad |\uparrow \uparrow\downarrow \ldots \downarrow \uparrow\rangle
$$

This system is still solvable [Douçot, Feigel'man, Ioffe, Ioselevich, cond-mat/0403712]. Same H as above, with an interesting quantization condition,

$$
\frac{\sin(k(L+2))}{\sin(k(L+1))} = -h \qquad \text{(open chains)}.
$$

Example: Closed 4-field Operators

4-field flavor singlet operators: $\mathrm{Tr}(X^2)$, $\mathrm{Tr}(Y^2)$, $\mathrm{Tr}(XY)$.

- Odd parity

$$b_\pi^\dagger|0\rangle \qquad \underbrace{b_0^\dagger|0\rangle}_{\sum k=0} \longleftrightarrow \mathrm{Tr}(X^2-Y^2)$$

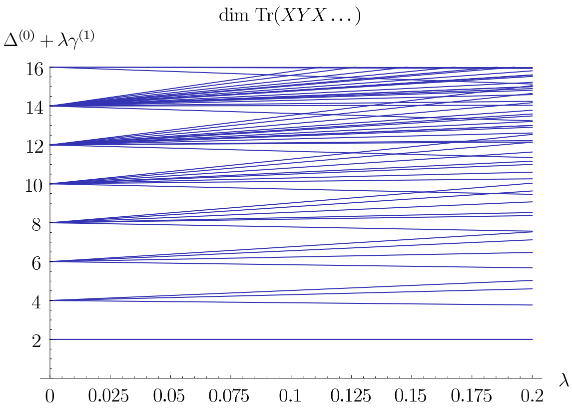
$$\gamma = \left(2+\frac{1}{2}\sqrt{10+6\cos 0}-\frac{1}{2}\sqrt{10+6\cos\pi}\right)\lambda = 3\lambda$$

- Even parity

$$\underbrace{|0\rangle, \quad b_{\frac{\pi}{2}}^\dagger b_{\frac{3\pi}{2}}^\dagger|0\rangle}_{\sum k=0} \qquad \longleftrightarrow \qquad \mathrm{Tr}(X^2+Y^2), \quad \mathrm{Tr}(XY)$$

$$\gamma = \left(2\pm\frac{1}{2}\sqrt{10+6\cos\frac{\pi}{2}}\pm\frac{1}{2}\sqrt{10+6\cos\frac{3\pi}{2}}\right)\lambda = (2\pm\sqrt{10})\lambda$$

Asymptotics



$$
\begin{aligned}
\gamma_\pm &= \lambda L \pm \frac{\lambda L}{4\pi} \int_0^{2\pi} dk \sqrt{10 + 6\cos k} \qquad \text{(Large } L\text{)} \\
&\approx (1 \pm 1.54)\lambda L
\end{aligned}
$$

Outline

Outlook

New features (cf. $\mathcal{N} = 4$ SYM)

- $|0\rangle$ is not BPS.
- fundamental excitations are dimer solitons (non-local).

Possible directions

- Other operators at 1-loop? At 2-loops, mixing with fermions+gauge fields, but still nearest neighbor in X, Y. Nontrivial S-matrix for $b^\dagger, b$?
- Magnetic dual? Building blocks aren't so simple: $q(MM^\dagger)^k q^\dagger$, $\widetilde{q}^\dagger(M^\dagger M)^k \widetilde{q}$, $q(MM^\dagger)^k M\widetilde{q}$, $\widetilde{q}^\dagger(M^\dagger M)^k M^\dagger q^\dagger$. Noncompact spin chain?
- Speculation: TFIM has quantum phase transition at $h = 1$ (naively $\frac{N_f}{N_c} = 1$). Could this be relevant in SQCD?

Outlook

Thanks!