$\mathcal{N} = 2$ Superconformal Index and Ruijsenaars-Schneider models

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D. Gaiotto, L. Rastelli, and SR to appear

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Objectives

- The objective : To find an explicit form for the superconformal index for a large class of $\mathcal{N}=2$ SCFTs which one can obtain by compactifying the $(2,0)$ theory on a Riemann surface. Most of these theories are non-Lagrangian and thus direct computations are not possible.

- The strategy : "bottom-up", "experimental math" approach; fully exploit the intuition about the hidden $6d$ origin of the $4d$ theories to generalize directly computable results for Lagrangian theories to non-Lagrangian ones.

- By-product : An AGT-like relation between the superconformal index of the $4d$ theories to $2d$ gauge theories and to integrable systems.

Outline

- The $\mathcal{N} = 2$ generalized quiver theories
- The superconformal index
- The logic of the argument I
- The Hall-Littlewood index as an example
- The logic of the argument II
- RS models and the index
- Summary

$\mathcal{N} = 2$ quiver gauge theories

- $\mathcal{N} = 2$ SCFTs obtained by compactifying the $(2, 0)$ theory on a punctured Riemann surface. (Gaiotto 0904.2715)
- The moduli of the Riemann surface map to gauge couplings of the corresponding $4d$ theory.
- The punctures are associated with flavor symmetries.
- Basic building blocks: theories corresponding to spheres with three punctures (no moduli=no tunable couplings)
  - Free hypermultiplets of $SU(k)$ theories correspond to spheres with two "maximal" punctures and one $U(1)$ puncture.
  - *All* the three-punctured spheres which are not free hypers do not have Lagrangian description.
  - An example of interacting theory corresponding to three-punctured spheres is the $SU(3)$ theory with three maximal punctures is an SCFT with E_6 flavor symmetry.
- "Gluing" three-punctured spheres at the punctures corresponds to gauging an $SU(k)$ flavor symmetry factor.
- Different "pair-of-pants" decompositions correspond to different S-duality frames.

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The superconformal index

- The superconformal index (Kinney-Maldacena-Minwalla-Raju 2006) encodes the information about the protected spectrum of a SCFT that can be obtained from representation theory alone.
- It is evaluated by a trace formula of the schematic form

$$\mathcal{I}(\mu_i) = \mathrm{Tr}(-1)^F \, e^{-\sum_i \mu_i T_i} \, e^{-\beta\,\delta}\,, \qquad \delta = 2\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\} \;(\geq 0)\,,$$

  where $\mathcal{Q}$ is the supercharge "with respect to which" the index is calculated and $\{T_i\}$ a complete set of generators that commute with $\mathcal{Q}$ and with each other.
- The trace is over the states of the theory on S^3 (in the radial quantization). States with $\delta \neq 0$ cancel pairwise, so the index counts states with $\delta = 0$ and it is independent of β.
- For a theory with a Lagrangian description one can compute the index in the free limit of the theory using simple matrix integral techniques.

$\mathcal{N} = 2$ index

- $\mathcal{N} = 2$ SCFTs have 8 supercharges (and eight superconformal counterparts): $\mathcal{Q}_{I\alpha}, \quad \tilde{\mathcal{Q}}_{I\dot{\alpha}}$.
- Here $I = 1, 2$ are $SU(2)_R$ indices and $\alpha = \pm$, $\dot{\alpha} = \pm$ Lorentz indices.
- For concreteness we choose to compute the index with respect to $\tilde{Q}_{1\dot{-}}$ - all other choices are equivalent.
- The elements of the superconformal algbra which commute with $\tilde{Q}_{1\dot{-}}$ are

$$\delta_- \equiv 2\left\{Q_{1-}, (Q_{1-})^\dagger\right\} = E - 2j_1 - 2R - r,$$
$$\delta_+ \equiv 2\left\{Q_{1+}, (Q_{1+})^\dagger\right\} = E + 2j_1 - 2R - r,$$
$$\bar{\delta}'_+ \equiv 2\{\tilde{Q}_{2\dot{+}}, (\tilde{Q}_{2\dot{+}})^\dagger\} = E + 2j_2 + 2R + r\,,$$
$$\bar{\delta}_- \equiv 2\{\tilde{Q}_{1\dot{-}}, (\tilde{Q}_{1\dot{-}})^\dagger\} = E - 2j_2 - 2R + r\,.$$

- E is the conformal dimension, (j_1, j_2) the Cartan generators of the $SU(2)_1 \otimes SU(2)_2$ isometry group, and $(R\,, r)$, the Cartan generators of the $SU(2)_R \otimes U(1)_r$ R-symmetry group.
- The index we will compute is

$$\boxed{\mathcal{I}(p, q, t, \dots) = \mathrm{Tr}(-1)^F\, p^{\frac{1}{2}\delta_+}\, q^{\frac{1}{2}\delta_-}\, t^{R+r}\, e^{-\beta\,\bar{\delta}_-}\ \dots.}$$

TQFT structure

- The superconformal index does not depend on the tunable parameters/coupling of the theory.
- For Gaiotto theories this means that the index does not depend on the moduli of the underlying Riemann surface.
- Thus, it is expected that the index will be given by a $2d$ TQFT computation.
- The structure constants of this TQFT are the indices of the three-punctured spheres,

$$\mathcal{I}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$$

where $\mathbf{x}_i$ are fugacities of the Cartan subgroup of the flavor symmetry.

- A basic property of a TQFT is that the different pair-of-pants decompositions of the riemann surface give the same result - the algebra defined by the structure constants is associative:

$$\oint \prod_{i=1}^{k-1} \frac{dx^i}{2\pi i x_i} \Delta(\mathbf{x})\, \mathcal{I}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x})\, \mathcal{I}_V(\mathbf{x})\, \mathcal{I}(\mathbf{x}^{-1}, \mathbf{x}_3, \mathbf{x}_4)\,.$$

The associativity implies that this index is invariant under permutations of $\mathbf{x}_i$.

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Our strategy I - Look for a nice "basis"

- We want to obtain the superconformal index for all the $\mathcal{N} = 2$ generalized quivers.
- Our strategy in solving the problem is to rewrite the index of the Lagrangian theories in such a way that the Riemann surface underlying the theory will be clearly visible in the expressions. Thus, allowing for generalizations to arbitrary rank and Riemann surface.
- Choose a basis for symmetric functions (in case of $SU(n)$ gauge group) $f^\lambda(a_1, \ldots, a_n)$ orthonormal with respect to a measure $\hat{\Delta}(a_1, \ldots, a_n)$.
- Define structure constants

$$\mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \mathcal{K}(\mathbf{a}_1)\mathcal{K}(\mathbf{a}_2)\mathcal{K}(\mathbf{a}_3) \sum_{\mu,\nu,\lambda} C_{\mu\nu\lambda}\, f^\mu(\mathbf{a}_1)\, f^\nu(\mathbf{a}_2)\, f^\lambda(\mathbf{a}_3),$$

such that

$$\mathcal{I}_V(\mathbf{a})\, (\mathcal{K}(\mathbf{a}))^2\, \Delta(\mathbf{a}) = \hat{\Delta}(\mathbf{a})\,,$$

with Δ being the Haar measure and $\mathcal{I}_V(\mathbf{a})$ is the index of the vector multiplet.
- Gluing two spheres is then just multiplying the structure constants

$$\oint \prod_{i=1}^{k-1} \frac{da^i}{2\pi i a_i}\, \Delta(\mathbf{a})\, \mathcal{I}_V(\mathbf{a}) \mathcal{I}(\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2) \mathcal{I}(\mathbf{a}^{-1}, \mathbf{a}_3, \mathbf{a}_4) =$$

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- S-duality implies that the structure constants are associative: $C_{\alpha\beta}{}^{\gamma} C_{\gamma\delta\rho} = C_{\alpha\delta}{}^{\gamma} C_{\gamma\beta\rho}$.
- "Diagonalize" the bais such that the only non-zero structure constants will be $C_{\alpha\alpha\alpha}$.
- We will see that this diagonal basis representation of the index of Lagrangian three-punctured spheres is naturally generalizable to arbitrary rank and punctures.

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Hall-Littlewood index

- We take the limit $p,\, q \to 0$ of the full index

$$\mathcal{I} = \text{Tr}(-1)^F\, p^{\frac{1}{2}\delta_{1+}}\, q^{\frac{1}{2}\delta_{1-}}\, t^{R+r}\, e^{-\frac{1}{2}\beta\,\bar{\delta}_{-}}\,.$$

- Alternatively can state that it is given by

$$\mathcal{I} = \text{Tr}(-1)^F\, t^{E-R}\,,$$

evaluated on states satisfying $j_1 = 0$ and $E - 2R - r = 0$.

- The states contributing to this index are annihilated by three supercharges, two chiral and one anti-chiral.
- For Lagrangian theories the only "letters" contributing to this index are a scalar q $(t^{\frac{1}{2}})$ from the hypermultiplet and a gaugino $\bar{\lambda}_{1\dot{+}}$ $(-t)$ from the vector multiplet.

HL index - $SU(2)$ quivers

- The quiver theories with $\mathcal{N} = 2$ supersymmetry are the simplest: all the relevant theories have Lagrangian description.
- The basic building block corresponding to a sphere with three punctures is a free hypermultiplet.
- The HL index of the free hyper-multiplet is given by

$$\mathcal{I}(a_1, a_2, a_3) = \frac{1}{\prod_{\pm 1}(1 - t^{\frac{1}{2}}\ a_1^{\pm 1}\ a_2^{\pm 1}\ a_3^{\pm 1})}\,.$$

- The index of the free hyper-multiplet can be written as

$$\begin{aligned}\mathcal{I}(a_1, a_2, a_3) &= \frac{1+t^2}{1-t^2} \prod_{i=1}^{3} \frac{1}{(1-ta_i^2)\,(1-t/a_i^2)} \sum_{\lambda=0}^{\infty} \frac{1}{P_\lambda^{HL}(t^{\frac{1}{2}}, t^{-\frac{1}{2}}|\, t)} \prod_{i=1}^{3} P_\lambda^{HL}(a_i, a_i^{-1}|\, t) \\ &= \mathcal{N}(t) \prod_{i=1}^{3} \mathcal{K}(a_i) \sum_{\lambda=0}^{\infty} C_{\lambda\lambda\lambda} \prod_{i=1}^{3} f^\lambda(a_i)\,.\end{aligned}$$

where

$$P_\lambda^{HL}(a, a^{-1}|t) = \mathcal{N}_\lambda(\mathrm{t})\,(\chi_\lambda(a) - t\,\chi_{\lambda-2}(a))$$

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HL index - $SU(2)$ quivers (cont.)

- The Hall-Littlewood polynomials are orthonormal under the following measure

$$\Delta_{HL}(z) = \frac{1}{2}\,\frac{(1-\,z^2)\,(1-z^{-2})}{(1-t\,z^2)\,(1-t\,z^{-2})} \qquad (\,=\Delta(a)\,\mathcal{K}(a)\,)\,.$$

- The index of the vector multiplet is

$$\mathcal{I}_V(z) = (1-t)\,(1-t\,z^2)\,(1-t\,z^{-2}) \quad (\,=(1-t)\mathcal{K}^{-1}(a)\,)\,.$$

- Using the orthogonality of the HL polynomials we can immediately write the index of any $SU(2)$ quiver

$$\mathcal{I}_{\mathfrak{g},s}(a_i) = (1-t)^{\mathfrak{g}-1}\,(1+t)^{2\mathfrak{g}-2+s}\prod_{i=1}^{s}\mathcal{K}(a_i)\sum_{\lambda=0}^{\infty}\frac{\prod_{i=1}^{s}P_\lambda^{HL}(a_i,a_i^{-1}|\,\mathfrak{t})}{\left[P_\lambda^{HL}(t^{\frac{1}{2}},t^{-\frac{1}{2}}|\,\mathfrak{t})\right]^{2\mathfrak{g}-2+s}}\,.$$

HL index - higher rank generalization

- The expression for the $SU(2)$ index is tightly tied to the underlying Riemann surface and to the rank of the group. We thus can conjecture a simple generalizion to higher ranks.
- The HL polynomials can be defined for $U(k)$ groups

$$P_\lambda^{HL}(x_1,\dots,x_k|\,t)=\mathcal{N}_\lambda(t)\sum_{\sigma\in S_k}\sigma\left(x_1^{\lambda_1}\dots x_k^{\lambda_k}\prod_{i<j}\frac{x_i-t\,x_j}{x_i-x_j}\right).$$

and thus for higher rank building blocks, the T_k theories, the HL index is given by

$$\boxed{\mathcal{I}(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)=\mathcal{N}_k(t)\prod_{l=1}^{3}\mathcal{K}(\mathbf{a}_l)\sum_\lambda\frac{1}{P_\lambda^{HL}(t^{\frac{k-1}{2}},\dots,t^{\frac{1-k}{2}})}\prod_{l=1}^{3}P_\lambda^{HL}(\mathbf{a}_l)}.$$

- The conjecture for the index with arbitrary punctures

$$\boxed{\mathcal{I}_{\Lambda_1,\Lambda_2,\Lambda_3}(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)=\mathcal{N}_k(t)\prod_{l=1}^{3}\mathcal{K}_{\Lambda_l}(\mathbf{a}_l)\sum_\lambda\frac{1}{P_\lambda^{HL}(t^{\frac{k-1}{2}},\dots,t^{\frac{1-k}{2}})}\prod_{l=1}^{3}P_\lambda^{HL}(\mathbf{a}_l(\Lambda_l))}.$$

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$$\mathcal{I}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3})=\frac{(1+t)(1+t+t)}{(1-t)^2}\prod_{i=1}^{3}\mathcal{K}(\mathbf{a}_i)\sum_{\lambda_1,\lambda_2}\frac{1}{P^{HL}_{\lambda_1,\lambda_2}(t,t^{-1},1|\,t)}\prod_{i=1}^{3}P^{HL}_{\lambda_1,\lambda_2}(\mathbf{a_i}|\,t)\,.$$

- This expression agrees with the one obtained from Argyres-Seiberg duality (Gadde-Rastelli-SR-Yan 1003.4244) and thus in particular is consistent with this duality.
- For T_3 theory the flavor symmetry is known to enhance: $SU(3)^3\rightarrow E_6$.
- The above expression can be shown (order by order in t) to be equal to

$$\mathcal{I}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3})=\sum_{k=0}^{\infty}[0,k,0,0,0,0]_{\mathbf{z}}\,t^k\,,$$

  where $\mathbf{z}$ is an E_6 fugacity and $[0,k,0,0,0,0]_{\mathbf{z}}$ are the characters of the irreducible representation of E_6 with Dynkin labels $[0,k,0,0,0,0]$.
- The E_6 covariant expression was conjectured in Benvenuti-Hanany-Mekareeya 1005.3026 (see also Gaiotto-Neitzke-Tachikawa 0810.4541)

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Summary and Comments

- We can "guess" the HL index for any $\mathcal{N} = 2$ generalized quiver for arbitrary rank and types of punctures.
- This guess can be subjected to numerous checks.
- An immediate generalization is to index with more superconformal fugacities turned on. It so happens that with $p = 0$ and q, t generic all one has to do is to to exchange HL polynomials with Macdonald polynomials.
- In another simle specialization of parameters, $t = q$, the relevant functions are Schur polynomials and the index is directly related to 2d qYM.
- Where do these special polynomials come from?
- Macdonald polynomials are simultaneous eigenfunctions of a set commuting "Hamiltonians" defining an integrable quantum mechanics: the trigonometric Ruijsenaars-Schneider (RS) model .
- This model is a one parameter generalization of the Calogero-Moser-Sutherland systems (the limit $p = 0$ and $q, t = q^g \to 1$ which gives Jack polynomials).
- In particular it admits an elliptic version with three parameters direcly analogous to our p, q, and t.
- In what follows we will see hoiw these "Hamiltonians" emerge from the index.

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Summary and Comments

- We can "guess" the HL index for any $\mathcal{N} = 2$ generalized quiver for arbitrary rank and types of punctures.
- This guess can be subjected to numerous checks.
- An immediate generalization is to index with more superconformal fugacities turned on. It so happens that with $p = 0$ and q, t generic all one has to do is to to exchange HL polynomials with Macdonald polynomials.
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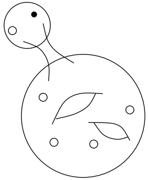
Strategy II - More explicit derivation of the expressions

- The index has many poles in flavor fugacities.
- The index of the free hyper is

$$\mathcal{I}_{hyp.}(b,c;a)=\prod_{i,j=1}^{N}\prod_{m,n\geq 0}\frac{1-p^{n+1}q^{m+1}t^{-\frac{1}{2}}(ab_ic_j)^{-1}}{1-p^nq^mt^{\frac{1}{2}}ab_ic_j}\frac{1-p^{n+1}q^{m+1}t^{-\frac{1}{2}}ab_ic_j}{1-p^nq^mt^{\frac{1}{2}}(ab_ic_j)^{-1}}\,.$$

- A natural question is what are the residues?

- Consider a general quiver associated to Riemann surface $\mathcal{C}$ with index $\mathcal{I}^{\mathcal{C}}$ and couple a free hyper-multiplet to it.
- It is possible to compute the residue of the full theory $\mathcal{I}$ at a pole of the $U(1)$ fugacity without explicitly knowing the index of the theory associated to the Riemann surface $\mathcal{C}$, $\mathcal{I}^{\mathcal{C}}$.
- The residue can be presented as a difference operator acting on the $SU(N)$ flavor fugacity living "on the tube" $\mathcal{I}^{\mathcal{C}}$.
- This operator is one of the RS "Hamiltonians" !!
- Using S-duality we can argue that the index is diagonal in the basis of eigen-functions of these operators.



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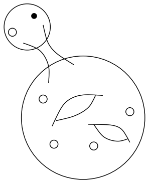

The appearance of the RS "hamiltonians"

- After some work one can show that the index of the theory with a free hyper coupled to it has poles at

$$a = t^{\frac{1}{2}}\, q^{\frac{1}{N}r}\, p^{\frac{1}{N}r'}\,, \qquad r, r' \in \mathbb{N}\,.$$

- The contour integrals involved in gluing the sphere to the Riemann surface $\mathcal{C}$ are "pinched" at these values of a and that is why the poles appear.

- The residue at $a = t^{\frac{1}{2}}$ is given simply by $\mathcal{I}_{\mathcal{C}}$, i.e. by the index on the Riemann surface. That is computing this resideue simply amounts to removing the $U(1)$ puncture.

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- This operator, up to trivial manipulations, IS the basic elliptic RS difference operator.
- Higher RS operators are obtained from other residues. In particular the $N-1$ independent Hamiltonians are encoded inside $\mathfrak{S}_{(r,0)}$.

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Comments on the RS hamiltonians

- Thus, the residues of the index are obtained by acting with difference operators on it.
- Although the operators act on a given flavor fugacity, any choice of the flavor fugacity will give the same result due to S-duality,

$$\mathfrak{S}_{(1,0)}(a)\,\mathcal{I}_{\mathcal{C}}(a,b,\cdots) = \mathfrak{S}_{(1,0)}(b)\,\mathcal{I}_{\mathcal{C}}(a,b,\cdots)\,.$$

- All these operators are commuting: physically the reason is again S-duality. (*We can choose the operators to act on different $U(1)$ punctures*)

$$\left[\mathfrak{S}_{(r,r')},\mathfrak{S}_{(s,s')}\right] = 0\,.$$

- The operators are self-adjoint under a natural measure constructed from the Haar measure and the index of the vector multiplet.
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A construction of the general index

- S-duality is very constraining!! We can exploite it to write the index of the generic quivers.
- Defining the eigenfunctions of the RS difference operators by ψ^λ and also defining the eigenvalues as

$$\mathfrak{S}_{(1,0)} \cdot \psi^\lambda = E_\lambda \, \psi^\lambda \,,$$

we obtain

$$\begin{aligned}\mathfrak{S}_{(1,0)}\, \mathcal{I}_{0,3} &= \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\, E_\alpha\, \psi^\alpha(a)\psi^\beta(b)\psi^\gamma(c) = \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\, E_\beta\, \psi^\alpha(a)\psi^\beta(b)\psi^\gamma(c) \\ &= \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\, E_\gamma\, \psi^\alpha(a)\psi^\beta(b)\psi^\gamma(c)\,.\end{aligned}$$

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Summary and Comments

- We have obtained explicit expression for the (two parameter) superconformal index of all Gaiotto's theories.
- The expressions for the index are manifestly S-duality invariant and have a uniform form for all types of punctures.
- The basic trick of the argument I was to write the index in a convenient discrete basis.
- This basis is related to a very generic family of symmetric functions: Macdonald polynomials and their elliptic generalizations.
- These functions are eigenfunctions of RS difference operators. We have seen how these operators are encoded in the index through residue computations.

Summary and Comments cont.

- Although looking on residues seems ad hoc they actually have physical meaning!!
- One can argue that the residues of the index of the type we discussed today give the index of a theory in presence of surface defects.
- The expressions we get for the index are suggestive of a 2d YM interpretation analogous to the AGT conjecture.
- 2d gauge theories are related to Calogero-Moser-Sutherland type of models (Gorsky-Nekrasov,...) and it will be interesting to understand these relations further.
- Another interesting question for further research is whether there is a direct physical derivation of our results. E.g. whether starting from the $(2,0)$ 6d theory and compactifying on $S^3 \times S^1$ one can obtain the 2d gauge theory and/or the integrable quantum mechanical systems
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