Is There an *F*-Theorem?

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Based on:

- 1011.5487 with C. Herzog, I. Klebanov, and T. Tesileanu
- 1103.1181 with D. Jafferis, I. Klebanov, and B. Safdi
- 1105.2817 and 1106.5484 with D. Gulotta and C. Herzog
- 1105.4598 with I. Klebanov and B. Safdi
- 1111.6290 and 1112.5342 with I. Klebanov, S. Sachdev, and B. Safdi

Rutgers, March 20, 2012

Introduction

Question: What is a good measure of the number of degrees of freedom in CFT_3 on $\mathbb{R}^{2,1}$? (i.e. that decreases under RG flow and is stationary at RG fixed points)

- Same question was asked in other spacetime dimensions. An answer: conformal anomaly coefficients c (in 2d) and a (in 4d) [Zamolodchikov '86; Cardy '88; Komargodski, Schwimmer '11]

$$\langle T^\mu_\mu \rangle_{2\text{d}} = -\frac{c}{12} R \,, \quad \langle T^\mu_\mu \rangle_{4\text{d}} = -\frac{a}{16\pi^2} \text{Euler density} + \frac{c}{16\pi^2} \text{Weyl}^2 \,.$$

- But $\langle T^\mu_\mu \rangle_{3\text{d}} = 0$, so no obvious candidate in 3d.
- Conjecture ("F-Theorem"): F (to be defined shortly) decreases along RG flow and is stationary at RG fixed points.

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- But $\langle T^\mu_\mu \rangle_{3d} = 0$, so no obvious candidate in 3d.

- Conjecture ("F-Theorem"): F (to be defined shortly) decreases along RG flow and is stationary at RG fixed points.

"*F*" is for "free energy"

Two equivalent definitions for F:

- In Euclidean signature, use a Weyl rescaling to map the CFT from $\mathbb{R}^3$ to S^3 (of radius R)

$$ds^2_{S^3} = \frac{4R^2}{\left(1 + x_1^2 + x_2^2 + x_3^2\right)^2} \left[(dx_1)^2 + (dx_2)^2 + (dx_3)^2\right] .$$

- Compute the partition function on S^3, and define

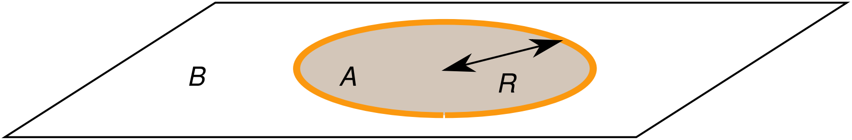
$$\log Z_{S^3} = a_3 \frac{R^3}{\epsilon^3} + a_1 \frac{R}{\epsilon} - F + O(\epsilon/R) ,$$

where ϵ is a UV cutoff.

  - This definition requires subtraction of cubic and linear UV divergences.

"F" is also entanglement entropy

- F is the finite part of the vacuum entanglement entropy between a disk and the complement of the disk.



$$S(R) = \alpha \frac{R}{\epsilon} - F + O(\epsilon/R)\,,$$

where $S(R) = -\operatorname{tr}(\rho_A \log \rho_A)$ with $\rho_A \equiv \operatorname{tr}_B |0\rangle\langle 0|$.

  - This definition requires subtraction of a linear divergence.

- [Casini, Huerta, Myers '11] showed the equivalence of these two definitions.

The Story of F as a free energy

- Starts with [Drukker, Marino, Putrov '10; Herzog, Klebanov, SSP, Tesileanu '10] .
  - Field theory computation of F for 3d field theories with gravity duals. Strong test of AdS/CFT.
  - Used [Kapustin, Willett, Yaakov '09] . Only $\mathcal{N} \geq 3$ SUSY.
- [Jafferis '10] : in $\mathcal{N} = 2$ theories one should extremize F in order to find the correct IR R-charges.
  - Analog of a-maximization in 4d.
- [Jafferis, Klebanov, SSP, Safdi '11; Martelli, Sparks '11, Cheon *et al.* '11, . . .] : Tests of F-extremization in field theories with gravity duals.
  - The extremum of F is always a *maximum*. Consequently, F decreases under SUSY RG flow.
- [Klebanov, SSP, Safdi '11] : Tests of the "F-theorem" in non-SUSY flows.

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The Story of F as entanglement entropy

- [Myers, Sinha '10] defined a function a_d^* which one can show decreases under holographic RG flow in d boundary dimensions.
- $a_2^* \propto c$ and $a_4^* \propto a$.
- For a CFT with a holographic dual, a_d^* is the universal part of the entanglement entropy between a disk of radius R and its complement.
- [Casini, Huerta '10; Casini, Huerta, Myers '11] : a_d^* can be computed by evaluating the free energy on S^d or on $S^1 \times \mathbb{H}^{d-1}$.
- [Liu and Mezei '12] propose that

$$\mathcal{F}(R) = RS'(R) - S(R)$$

interpolates monotonically between F_{UV} $(R = 0)$ and F_{IR} $(R = \infty)$.

- Proof that $\mathcal{F}'(R) = RS''(R) < 0$ in [Casini, Huerta '12] . (?)

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Spin-offs

- Thorough tests of AdS/CFT and of the Seiberg-like dualities of [Giveon, Kutasov '08] .
- Better understanding of 7d tri-Sasakian geometry [Herzog, Klebanov, SSP, Tesileanu '10; Gulotta, Herzog, SSP '11; Gulotta, Ang, Herzog '11; Gulotta, Herzog, Nishioka '11] .
- Computation of *all* Rényi entropies in simple 3d field theories [Klebanov, SSP, Sachdev, Safdi '11] . (Only CS was known before.)

Outline

F can be computed in many CFT_3's, with or without supersymmetry.

I'll talk about:

- $\mathcal{N} \geq 3$ field theories with gravity duals.
- An $\mathcal{N} = 2$ example and F-maximization.
- $1/N$ expansions.
- RG flows in non-SUSY theories.

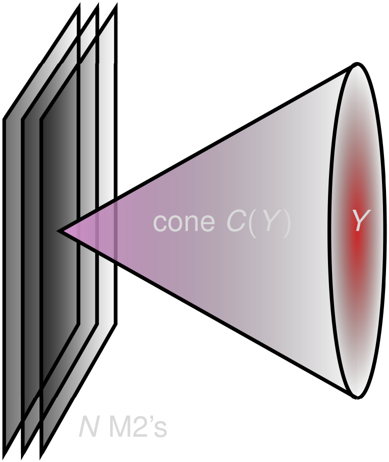
M-theory compactifications

Why use field theory to compute F in CFTs with gravity duals? Don't we already know the answer?

- Consider $AdS_4 \times Y$ compactifications of M-theory (11-d SUGRA).
- Take a stack of N coincident M2-branes sitting at the tip of the Calabi-Yau cone over Y.
- Close to the M2-branes, the metric is

$$ds_{11}^2 = ds_{AdS_4}^2 + 4L^2 ds_Y^2 \,,$$

where L is the radius of AdS_4.



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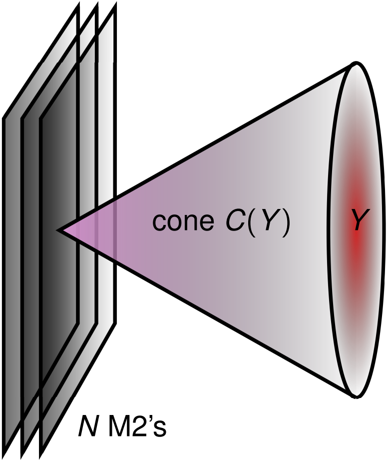

A Puzzle

- SUGRA predicts:

$$F = N^{3/2}\sqrt{\frac{2\pi^6}{27\mathrm{Vol}(Y)}}\,. \qquad (\star)$$

  - The same $N^{3/2}$ scaling was observed in [Klebanov, Tseytlin '96] for the thermal free energy.

- Puzzle: In CFT_3 one expects a field theory written in terms of $N \times N$ matrices. Naively, number of degrees of freedom is N^2. So how can it be $N^{3/2}$?

- Resolution: The field theory intuition is correct only in the 't Hooft limit where N/k is kept fixed.

- A non-'t Hooft limit of the CFT_3 reproduces $N^{3/2}$ (details to follow).

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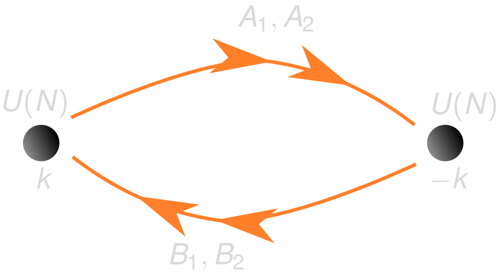
What can we learn?

- There are many proposals for field theories dual to $AdS_4 \times Y$ for various Y.
- Gravity predicts # of d.o.f.'s is $\sim N^{3/2}/\sqrt{\text{Vol}(Y)}$. Can we match this with a field theory computation?
- If we can, is this an easier way of computing Vol(Y)?

Localization in ABJM Theory

- Let's start with the simplest example: $Y = S^7/\mathbb{Z}_k$.
- The dual field theory is ABJM theory [Aharony *et al.* '08] .



- Field content: 1 $\mathcal{N} = 2$ vector multiplet for each gauge group (1 gauge field, 1 gaugino, 1 real scalar σ); 4 bifundamental chiral fields (each contains 1 complex scalar and 1 fermion).

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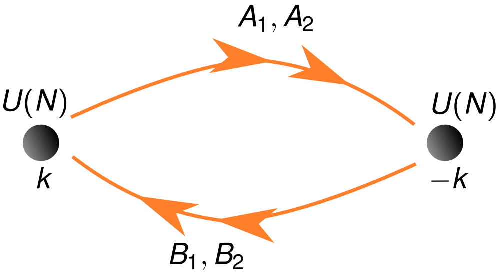

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- Due to [Kapustin, Willett, Yaakov '10] inspired by [Pestun '07] .
- How do we compute $Z = \int [DX] \exp[-S]$?
- Clever trick: change the theory by considering $S_t = S + t\{Q, \mathcal{V}\}$ where Q is a supercharge and $\mathcal{V}$ is such that $\{Q, \mathcal{V}\}$ is positive definite.
- One can show that $Z = \int [DX] \exp[-S_t]$ is independent of t.
- Then take t to be large. The integral localizes where $\{Q, \mathcal{V}\} = 0$ and $Z = \exp[-S_{t,\text{classical}}] \times$ one-loop determinant.
- Complicated calculation gives (λ_i, $\tilde{\lambda}_i$ are e'values of σ, $\tilde{\sigma}$)

$$Z = \frac{1}{(N!)^2} \int \prod_{i=1}^{N} \frac{d\lambda_i \, d\tilde{\lambda}_i}{(2\pi)^2} \frac{\prod_{i<j} \left(4 \sinh \frac{\lambda_i - \lambda_j}{2} \sinh \frac{\tilde{\lambda}_i - \tilde{\lambda}_j}{2}\right)^2}{\prod_{i,j} \left(2 \cosh \frac{\lambda_i - \tilde{\lambda}_j}{2}\right)^2} \times \exp\left(\frac{ik}{4\pi} \sum (\lambda_i^2 - \tilde{\lambda}_i^2)\right)$$

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Saddle point (large N) approximation

- The following techniques were developed in [Herzog, Klebanov, SSP, Tesileanu '10] .
- We focus on large N where we can use the saddle point approximation:

$$Z = \int d\lambda_i d\tilde{\lambda}_j \, e^{-F(\lambda_i, \tilde{\lambda}_j)} \approx e^{-F_{\text{critical}}} \,,$$

where to compute F_{critical} we require

$$\frac{\partial F}{\partial \lambda_i} = \frac{\partial F}{\partial \tilde{\lambda}_j} = 0 \,. \tag{1}$$

- Original range of integration was the real line, but saddle point can be anywhere in the complex plane!
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Solution: 1. Numerics

- Think of λ_i and $\tilde{\lambda}_j$ as coordinates of particles in $\mathbb{C} = \mathbb{R}^2$ acted on by forces $\partial F / \partial \lambda_i$ and $\partial F / \partial \tilde{\lambda}_j$.
- Add “viscosities’ τ_λ and $\tau_{\tilde{\lambda}}$ and a time direction and use relaxation method:

$$\tau_\lambda \frac{d\lambda_i}{dt} = \frac{\partial F}{\partial \lambda_i}, \qquad \tau_{\tilde{\lambda}} \frac{d\tilde{\lambda}_j}{dt} = \frac{\partial F}{\partial \tilde{\lambda}_j}.$$

- Note that τ_λ and $\tau_{\tilde{\lambda}}$ need not be real!

This is for $k = 1$ and $N = 20$—compare to $N = 40$ on next slide.

Movie 1

Imaginary parts stay of order 1, while real parts grow as $\sqrt{N}$.

Movie 2

Solution: 2. Analytical formulas

Saddle point equations are:

$$\frac{ik}{2\pi}\lambda_i = \sum_{j\neq i} \coth\frac{\lambda_j-\lambda_i}{2} - \sum_j \tanh\frac{\tilde\lambda_j-\lambda_i}{2}\,,$$

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- Assume $\lambda_i = N^\alpha x_i + i y_i$ and $\tilde\lambda_j = N^\alpha \tilde x_j + i\tilde y_j$ as $N\to\infty$.
- Key insight:

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with *exponentially small corrections*.

- LHS is $O(N^\alpha)$ at large $N \implies$ RHS must also be $O(N^\alpha)$.
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Why $N^{3/2}$?

- Introduce $\rho(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i)$.
- y_j and $\tilde{y}_j$ approach continuous functions $y(x)$ and $\tilde{y}(x)$.
- In the continuum limit $F(\lambda_i, \tilde{\lambda}_i)$ becomes a *local* functional!

$$F = \frac{k}{2\pi} N^{1+\alpha} \int dx \rho(x) x \left[y(x) - \tilde{y}(x)\right] + N^{2-\alpha} \int dx \rho(x)^2 \left[\pi^2 - (\tilde{y}(x) - y(x))^2\right] + \dots .$$

- To balance out these terms we need $\alpha = 1/2$. Then $F \sim N^{3/2}$!
- Need to minimize F under the constraints $\int dx \rho(x) = 1$ and $\rho(x) \geq 0$ almost everywhere.

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- y_j and $\tilde{y}_j$ approach continuous functions $y(x)$ and $\tilde{y}(x)$.
- In the continuum limit $F(\lambda_i, \tilde{\lambda}_i)$ becomes a *local* functional!

$$F = \frac{k}{2\pi}N^{1+\alpha}\int dx\,\rho(x)x\left[y(x) - \tilde{y}(x)\right] + N^{2-\alpha}\int dx\,\rho(x)^2\left[\pi^2 - (\tilde{y}(x) - y(x))^2\right] + \dots .$$

- To balance out these terms we need $\alpha = 1/2$. Then $F \sim N^{3/2}$!
- Need to minimize F under the constraints $\int dx\,\rho(x) = 1$ and $\rho(x) \geq 0$ almost everywhere.

- Solution is

$$\rho(x) = \frac{1}{2x_*}\,, \qquad y(x) = -\tilde{y}(x) = \frac{\pi}{2}\frac{x}{x_*} \qquad \text{for } x \in [-x_*, x_*]\,,$$

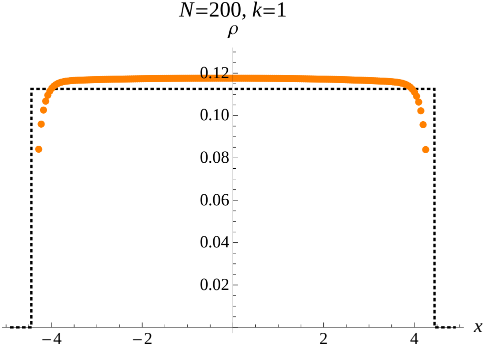
with $x_* = \pi\sqrt{2/k}$.

- Comparison: analytical formula (dashed) and numerics (orange):
- Free energy is

$$F = \frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2}$$

in agreement with the gravity computation.

- Also in agreement with the large N/k limit of [Drukker *et al.* '10] .



More general theories with $\mathcal{N} = 3$ SUSY

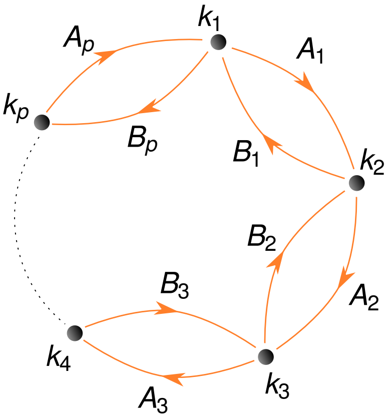
- We can consider more general "necklace" quiver gauge theories with $\mathcal{N} = 3$ SUSY [Jafferis, Tomasiello '08] .

- The $\mathcal{N} = 3$ superpotential is

$$W = -\sum_{a=1}^{p} \frac{2\pi}{k_a} (B_{a-1} A_{a-1} - A_a B_a)^2$$

- Take the gauge groups to be all $U(N)$.

- Require $\sum_{a=1}^{p} k_a = 0$.



- Dual SUGRA background is $AdS_4 \times Y$ where Y is a tri-Sasakian space. (=an Einstein manifold which is the base of a hyperKähler cone.)
- From [Kapustin, Willet, Yaakov '10] :

$$Z \propto \int \left(\prod_{a,i} \frac{d\lambda_{a,i}}{2\pi} \right) \prod_{a=1}^{p} \left(\frac{\prod_{i<j} \left(2 \sinh \frac{\lambda_{a,i} - \lambda_{a,j}}{2} \right)^2}{\prod_{i,j} 2 \cosh \frac{\lambda_{a,i} - \lambda_{a+1,j}}{2}} \exp \left[\frac{i}{4\pi} \sum_i k_a \lambda_{a,i}^2 \right] \right)$$

Numerics for 3 nodes:

Movie 3

4 nodes:

Movie 4

- The long-range forces between the eigenvalues cancel iff

$$\lambda_{a,i} = N^{1/2} x_i + i y_{a,i} \,,$$

i.e. to leading order in N the real parts of $\lambda_{a,i}$ are the same.
- Then

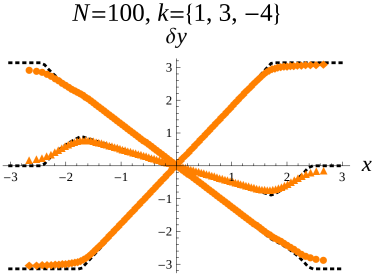
$$F[\rho, \delta y_a] = \frac{N^{3/2}}{2\pi} \int dx\, x \rho(x) \sum_{a=1}^{p} q_a \delta y_a(x) + \frac{N^{3/2}}{2} \int dx\, \rho(x)^2 \sum_{a=1}^{p} f(\delta y_a(x)) + o(N^{3/2}) \,,$$

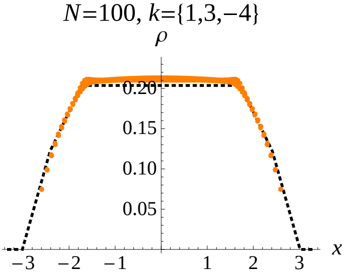
where $k_a = q_a - q_{a-1}$,$\delta y_a = y_{a-1} - y_a$, and as before

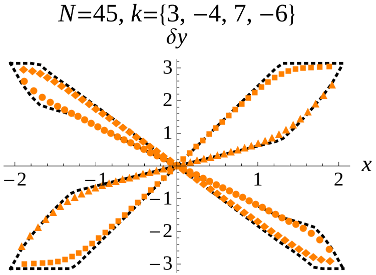
$$f(\delta y_a) = \pi^2 - (\delta y_a)^2 \qquad \text{if } |\delta y_a| < \pi \,.$$

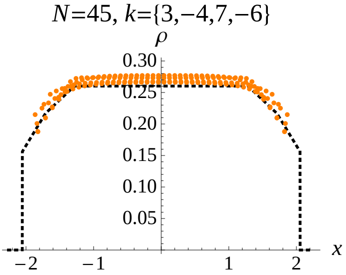
Features of the solution:

- $\rho(x)$ and $\rho(x)\delta y_a(x)$ are piecewise linear functions.
- The δy_a saturate at $\pm\pi$: $\delta y_a(x) = \pm\pi$ for $|x| > x_{a*}$ for some x_{a*}.






N=45, k={3, −4, 7, −6}
δy
3
2
1
−1
−2
−3
−2
−1
1
2
x


N=45, k={3,−4,7,−6}
ρ
0.30
0.25
0.20
0.15
0.10
0.05
−2
−1
1
2
x

What are we learning from all this?

- In a large class of $\mathcal{N} = 3$ theories, we can show that the free energy on S^3 grows as $N^{3/2}$ at large N as expected from the dual gravity side.

- Using AdS/CFT, we can compute the volumes of the internal spaces Y:

$$F = N^{3/2}\sqrt{\frac{2\pi^6}{27\mathrm{Vol}(Y)}} . \qquad (\star)$$

- For necklace quivers the volume agrees with the geometrical computation of [Lee, Yee '06, Yee '07] .

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Volume formula

- For $p = 3$ nodes, one obtains

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{|k_1||k_2| + |k_1||k_3| + |k_2||k_3|}{(|k_1| + |k_2|)(|k_2| + |k_3|)(|k_1| + |k_3|)} .$$

- Tree formula: The volume of the tri-Sasakian spaces can be written as a sum over tree diagrams [Herzog, Klebanov, SSP, Tesileanu '10] . Proof in [Gulotta, Herzog, SSP '11] .

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{\sum_{(V,E)\in\mathcal{T}} \prod_{(a,b)\in E} |q_a - q_b|}{\prod_{a=1}^{p} \left[\sum_{b=1}^{p} |q_a - q_b|\right]} ,$$

where $k_a = q_a - q_{a-1}$, and $\mathcal{T}$ is the set of all trees with nodes $V = \{1, 2, \ldots, p\}$ and edges E. (# of edges is $p - 1$ for any tree with p nodes!)

$\mathcal{N} = 2$ theories

- Correct R-symmetry in the IR extremizes e^{-F} [Jafferis '10] .
- For each bifundamental $X_{a,b}$, replace

$$\frac{1}{\sqrt{\cosh\frac{\delta\lambda}{2}}} \to \exp\left[\ell\left(1 - R[X_{a,b}] + i\frac{\delta\lambda}{2\pi}\right)\right]$$

where

$$\ell(z) = -z\ln\left(1 - e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2\left(e^{2\pi i z}\right)\right) - \frac{i\pi}{12}\,.$$

- Extremize over $R[X_{a,b}]$ under the constraint that the superpotential should have R-charge $R[W] = 2$.

- The superpotential $W = \mathrm{tr}[\epsilon^{ij}\epsilon^{kl} A_i B_k A_j B_l]$ has $R[W] = 2$, so

$$R[A_1] + R[A_2] + R[B_1] + R[B_2] = 2\,.$$

- The free energy is

$$F(R[A_i], R[B_i]) = \frac{4\sqrt{2}\pi k^{1/2} N^{3/2}}{3} \sqrt{R[A_1]R[A_2]R[B_1]R[B_2]}\,.$$

- F is maximized when $R[A_i] = R[B_i] = 1/2$.
- When $k = 1$, one can trigger an RG flow by adding A_1^2 to the superpotential. Then $R[A_1] = 1$ and F is maximized when $R[A_2] = R[B_1] = R[B_2] = 1/3$.
- Holographic RG flow to $U(1)_R \times SU(3)$-invariant fixed point of $\mathcal{N} = 8$ gauged SUGRA [Warner '83; Corrado, Pilch, Warner '01; Benna, Klebanov, Klose, Smedback '08] . A_2, B_1, and B_2 are expected to form an $SU(3)$ triplet.

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$1/N$ expansion

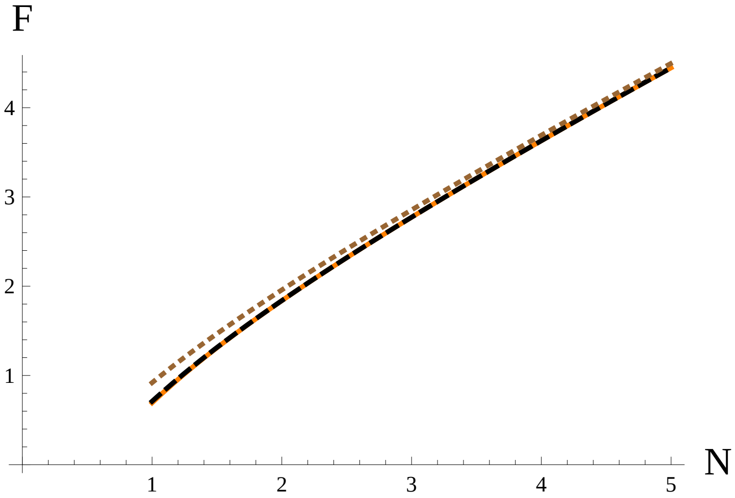
- Localization and F-maximization allow us to compute F in all $\mathcal{N} \geq 2$ supersymmetric theories.
- In $\mathcal{N} = 4$ $U(1)$ gauge theory with N hypers, the partition function is

$$Z = e^{-F} = \frac{1}{2^N} \int_{-\infty}^{\infty} \frac{d\lambda}{\cosh^N(\pi\lambda)} = \frac{2^{-N}\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{N+1}{2}\right)} .$$

- $1/N$ expansion is quite powerful:

$$F = -\log Z = N \log 2 + \frac{1}{2} \log\left(\frac{N\pi}{2}\right) - \frac{1}{4N} + \frac{1}{24N^3} + \dots .$$

Exact result in orange, large N approximation in black and brown.



- Consider $\mathcal{N}=2$ CS theory with level k and N pairs of fundamental and anti-fundamental flavors with no superpotential.
- Assume the flavor fields have R-charge

$$\Delta = \frac{1}{2} + \frac{\Delta_1}{N} + \dots$$

- Denoting $\kappa = 2k/(N\pi)$, the free energy is

$$F(\Delta) = N\log 2 + \frac{1}{2}\log\left(\frac{N\pi}{2}\sqrt{1+\kappa^2}\right) - \left(\frac{\pi^2\Delta_1^2}{2} + \frac{2\Delta_1}{1+\kappa^2} + \frac{1-\kappa^2}{4(1+\kappa^2)^2}\right)\frac{1}{N} + \dots .$$

- Extremizing over Δ_1 one finds

$$\Delta_1 = -\frac{2}{\pi^2(1+\kappa^2)} .$$

- Add superpotential $\sum(Q\tilde{Q})^2 \Longrightarrow$ flow to IR where $\Delta_1 = 0$. $F_{\text{UV}} > F_{\text{IR}}$ because of F-maximization.

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Non-supersymmetric CFTs

One can compute F in the following theories with no SUSY by evaluating functional determinants [Klebanov, SSP, Safdi '11; Klebanov, SSP, Sachdev, Safdi '11] :

- Free complex scalar and Dirac fermion:

$$F_S = \frac{\log 2}{4} - \frac{3\zeta(3)}{8\pi^2} \approx 0.128\,, \qquad F_D = \frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2} \approx 0.219\,.$$

- CS theory at level k with N_f fermions and N_b bosons with electric charges q_f and q_b:

$$F = \frac{\log 2}{4}(N_f + N_b) + \frac{3\zeta(3)}{8\pi^2}(N_f - N_b) + \frac{1}{2}\log\left[\pi\sqrt{\left(\frac{N_f q_f^2 + N_b q_b^2}{8}\right)^2 + \left(\frac{k}{\pi}\right)^2}\right] + \cdots\,.$$

“Close” RG fixed points

- CFT perturbed by $\mathcal{O}$ w/ dimension $3-\epsilon$. Perturbative IR fixed point at coupling $g_* \propto \epsilon$ with $F_{\text{IR}} - F_{\text{UV}} \propto -\epsilon^3$.
- Double trace deformation in a large N QFT: $\delta\mathcal{L} = \frac{\lambda_0}{2}\Phi^2$ where Φ is a single trace operator with dimension $\Delta < 3/2$.

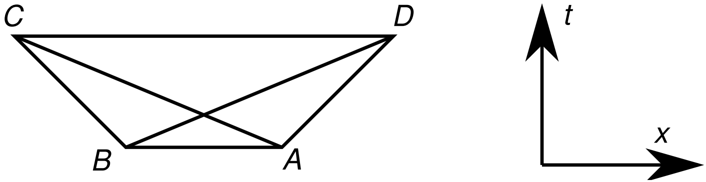
$$F_{\text{IR}} - F_{\text{UV}} = -\frac{\pi}{6}\int_{\Delta}^{3/2} dx (x-1)(x-\frac{3}{2})(x-2)\cot(\pi x) < 0\,.$$

- Fermionic double trace deformation: $\delta\mathcal{L} = \lambda_0 \bar{\chi}\chi$.

$$F_{\text{IR}} - F_{\text{UV}} = -\frac{2\pi}{3}\int_{\Delta}^{3/2} dx \left(x-\frac{1}{2}\right)\left(x-\frac{3}{2}\right)\left(x-\frac{5}{2}\right)\tan(\pi x) < 0\,.$$

Entanglement entropy proof of c-theorem

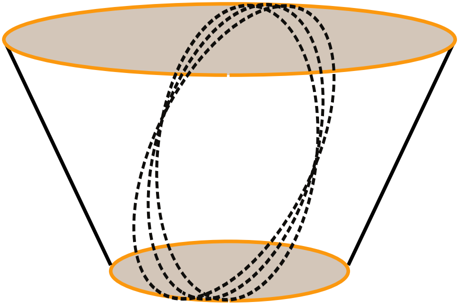
- For a CFT on $\mathbb{R}^{1,1}$, $S(r) = \frac{c}{3} \log r$, so $rS'(r) = c/3$.



- Take $|AB| = r$, $|CD| = R$, $|AD| = |BC| = 0 \Longrightarrow$ $|AC| = |BD| = \sqrt{Rr}$.
- Strong subadditivity: $S(AB \cup BC) + S(AB \cup AD) \geq S(AB) + S(AB \cup BC \cup AD)$.
- But $S(AB \cup BC) = S(AB \cup AD) = S(\sqrt{Rr})$; $S(AB) = S(r)$; $S(AB \cup BC \cup AD) = S(R)$.
- Consequently $2S(\sqrt{Rr}) \geq S(R) + S(r) \Longrightarrow (rS'(r))' \leq 0$.

Entanglement entropy proof of F-theorem

- The 3d proof involves a large number of boosted circles.



- [Casini, Huerta '12] show that $S''(r) < 0$.
- Recall that $\mathcal{F}(r) = rS'(r) - S(r)$, so $\mathcal{F}'(r) = rS''(r)$.

Conclusions

- In all examples, F decreases under RG flow.
- F is calculable in many field theories; if there's $\mathcal{N} \geq 2$ SUSY, F can be expressed as the log of a finite-dimensional integral using localization.
- For field theories on M2-branes, a field theory computation reproduces the $N^{3/2}$ scaling of the number of degrees of freedom.
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