Bounds on 4D Conformal and Superconformal Field Theories

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(with David Simmons-Duffin [arXiv:1009.2087])

Motivation

- Near-conformal dynamics could play a role in BSM physics!
  - Walking/Conformal Technicolor [Many people...]
  - Conformal Sequestering [Luty, Sundrum '01; Schmaltz, Sundrum '06]
  - Solution to $\mu/B\mu$ problem [Roy, Schmaltz '07; Murayama, Nomura, DP '07]
  - Flavor Hierarchies [Georgi, Nelson, Manohar '83; Nelson, Strassler '00]
  - ...

- Many ideas make assumptions about operator dimensions that are difficult to check
- Hard to calculate *anything* in non-SUSY theories! Lattice studies may be only hope.
- In $\mathcal{N} = 1$ SCFTs, we actually know lots about chiral operators, but not much about non-chiral operators...

Example: Nelson-Strassler Flavor Models ['00]

- Idea: Matter fields T_i have large anomalous dimensions under some CFT, flavor hierarchies generated dynamically!

$$W = T_1\mathcal{O}_1 + T_2\mathcal{O}_2 + y^{ij}T_iT_jH + \dots$$

- Interactions of matter T_i with CFT operators $\mathcal{O}_i$ are marginal
- Yukawa couplings y^{ij} flow to zero at rate controlled by $\dim T_i$
- Since T_i are chiral, $\dim T_i = \frac{3}{2}R_{T_i}$ (superconformal $U(1)_R$)
- Can write down lots of concrete models and then *calculate* dimensions using a-maximization! [DP, Simmons-Duffin '09]

Example: Nelson-Strassler Flavor Models ['00]

- Soft-mass operators $K \sim \frac{1}{M_{pl}^2} X^\dagger X T_i^\dagger T_j$ also flow to zero
  - Rate controlled by $\dim T_i^\dagger T_j$
- Maybe can solve SUSY flavor problem???
  - No way to calculate dimensions...
- Similar issue arises in Conformal Sequestering, $\mu/B\mu$ solution
- Can we say *anything* about $\dim T^\dagger T$, given $\dim T$?

- Recently, the papers:

  Rattazzi, Rychkov, Tonni, Vichi [arXiv:0807.0004]

  Rychkov, Vichi [arXiv:0905.2211]

  addressed a similar question in non-SUSY CFTs, deriving *bounds* on $\dim \phi^2$ as a function of $\dim \phi$...

Outline

CFT Review: Primary Operators

- In addition to Poincaré generators P^a and M^{ab}, CFTs have dilatations D and special conformal generators K^a

$$[K^a, P^b] = 2\eta^{ab}D - 2M^{ab}$$

- *Primary* operators $\mathcal{O}^I(0)$ are defined by

$$[K^a, \mathcal{O}^I(0)] = 0$$

(descendants obtained by acting with P^a)

CFT Review: Primary Operators

- Primary 2-pt and 3-pt functions fixed by conformal symmetry in terms of dimensions and spins, up to overall coefficients $\lambda_{\mathcal{O}}$

$$\langle \mathcal{O}^{a_1\dots a_l}(x_1)\mathcal{O}^{b_1\dots b_l}(x_2)\rangle = \frac{I^{a_1b_1}\dots I^{a_lb_l}}{x_{12}^{2\Delta}}$$

$$\langle \phi(x_1)\phi(x_2)\mathcal{O}^{a_1\dots a_l}(x_3)\rangle = \frac{\lambda_{\mathcal{O}}}{x_{12}^{2d-\Delta+l}x_{23}^{\Delta-l}x_{13}^{\Delta-l}}Z^{a_1}\dots Z^{a_l}$$

$$\left(I^{ab} = \eta^{ab} - 2\frac{x_{12}^a x_{12}^b}{x_{12}^2} \quad , \quad Z^a = \frac{x_{31}^a}{x_{31}^2} - \frac{x_{32}^a}{x_{32}^2}\right)$$

- Higher n-pt functions *not* fixed by conformal symmetry alone, but are determined once spectrum and $\lambda_{\mathcal{O}}$'s are known...

CFT Review: Operator Product Expansion

Let ϕ be a scalar primary of dimension d in a 4D CFT:

$$\phi(x)\phi(0) \quad = \quad \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}} C_I(x,P)\,\mathcal{O}^I(0) \qquad \text{(OPE)}$$

- Sum runs over *primary* $\mathcal{O}$'s
- $C_I(x,P)$ fixed by conformal symmetry [Dolan, Osborn '00]
- $\mathcal{O}^I = \mathcal{O}^{a_1\dots a_l}$ can be any spin-l Lorentz representation (traceless symmetric tensor) with $l = 0, 2, \dots$
- Unitarity tells us that $\Delta_{\mathcal{O}} \geq l + 2 - \delta_{l,0}$ and that $\lambda_{\mathcal{O}}$ is real

CFT Review: Conformal Block Decomposition

Use OPE to evaluate 4-point function

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \sum_{\mathcal{O}\in\phi\times\phi}\lambda_{\mathcal{O}}^2 C_I(x_{12},\partial_2)C_J(x_{34},\partial_4)\langle\mathcal{O}^I(x_2)\mathcal{O}^J(x_4)\rangle \\ &\equiv \frac{1}{x_{12}^{2d}x_{34}^{2d}}\sum_{\mathcal{O}\in\phi\times\phi}\lambda_{\mathcal{O}}^2\, g_{\Delta,l}(u,v)\end{aligned}$$

- $u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}$, $v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$ conformally-invariant cross ratios.
- $g_{\Delta,l}(u,v)$ conformal block ($\Delta=\dim\mathcal{O}$ and $l=$ spin of $\mathcal{O}$)

CFT Review: Conformal Blocks

Explicit formula [Dolan, Osborn '00]

$$
\begin{aligned}
g_{\Delta,l}(u,v) &= \frac{(-1)^l}{2^l}\frac{z\bar{z}}{z-\bar{z}}[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - z \leftrightarrow \bar{z}] \\
k_\beta(x) &= x^{\beta/2}{}_2F_1(\beta/2,\beta/2,\beta;x),
\end{aligned}
$$

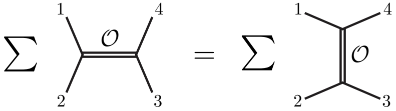
where $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$.

- Similar expressions in other even dimensions, recursion relations known in odd dimensions
- Alternatively can be viewed as eigenfunctions of the quadratic casimir of the conformal group [Dolan, Osborn '03]

CFT Review: Crossing Relations

- Four-point function $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ is clearly symmetric under permutations of x_i
- After OPE, symmetry is non-manifest!
- Switching $x_1 \leftrightarrow x_3$ gives the "crossing relation":

$$\sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 g_{\Delta,l}(u,v) \quad = \quad \left(\frac{u}{v}\right)^d \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 g_{\Delta,l}(v,u)$$



- Other permutations give no new information

Outline

Review: Method of Rattazzi et. al. [arXiv:0807.0004]

- Let's study the OPE coefficient of a particular $\mathcal{O}_0 \in \phi \times \phi$
- We can rewrite crossing relation as

$$\underbrace{\lambda^2_{\mathcal{O}_0} F_{\Delta_0,l_0}(u,v)}_{\mathcal{O}_0} \quad = \quad \underbrace{1}_{\text{unit op.}} \quad - \quad \underbrace{\sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda^2_{\mathcal{O}} F_{\Delta,l}(u,v)}_{\text{everything else}},$$

where

$$F_{\Delta,l}(u,v) \quad \equiv \quad \frac{v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u)}{u^d - v^d}.$$

Review: Method of Rattazzi et. al. [arXiv:0807.0004]

Idea: Find a linear functional α such that

$$\begin{aligned} \alpha(F_{\Delta_0,l_0}) &= 1, \quad \text{and} \\ \alpha(F_{\Delta,l}) &\geq 0, \quad \text{for all other } \mathcal{O} \in \phi \times \phi. \end{aligned}$$

Applying to both sides:

$$\begin{aligned} \alpha\left(\lambda^2_{\mathcal{O}_0} F_{\Delta_0,l_0}\right) &= \alpha(1 - \sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda^2_{\mathcal{O}} F_{\Delta,l}) \\ \lambda^2_{\mathcal{O}_0} &= \alpha(1) - \sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda^2_{\mathcal{O}} \alpha(F_{\Delta,l}) \leq \alpha(1) \end{aligned}$$

since $\lambda^2_{\mathcal{O}} \geq 0$ by unitarity.

Review: Method of Rattazzi et. al. [arXiv:0807.0004]

- To make the bound $\lambda^2_{\mathcal{O}_0} \leq \alpha(1)$ as strong as possible: *Minimize* $\alpha(1)$ subject to the constraints $\alpha(F_{\Delta_0,l_0}) = 1$ and $\alpha(F_{\Delta,l}) \geq 0$ $(\mathcal{O} \neq \mathcal{O}_0)$.
- This is an infinite dimensional *linear programming problem*... to use known algorithms (e.g., simplex) we must make it finite
- Can take α to be linear combinations of derivatives at some point in $z, \overline{z}$ space

$$\alpha : F(z, \overline{z}) \mapsto \sum_{m+n \leq 2k} a_{mn} \partial_z^m \partial_{\overline{z}}^n F(1/2, 1/2)$$

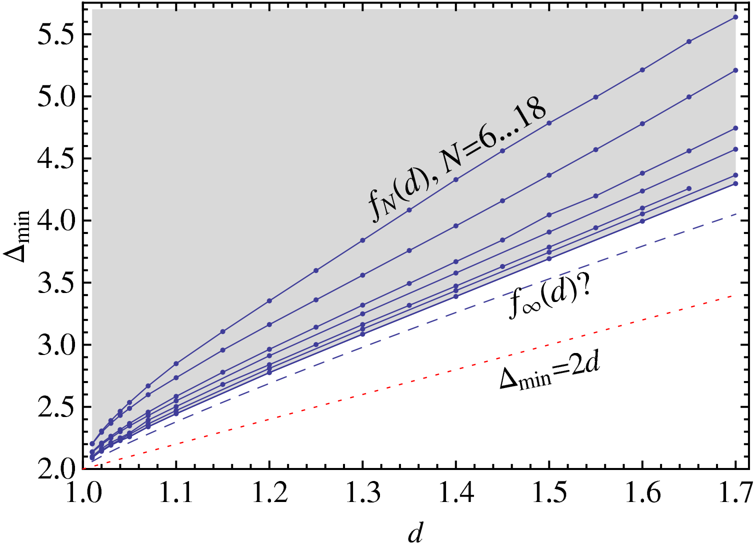
- Discretize constraints to $\alpha(F_{\Delta_i,l_i}) \geq 0$ for $D = \{(\Delta_i, l_i)\}$
- Take $k, D \to \infty$ to recover "optimal" bound

Review: Method of Rattazzi et. al. [arXiv:0807.0004]

- Can make any assumptions about the spectrum that we want!
- E.g., can assume that all scalars appearing in the OPE $\phi \times \phi$ have dimension larger than some $\Delta_{\min} = \dim \mathcal{O}_0$
- If $\lambda^2_{\mathcal{O}_0} \leq \alpha(1) < 0$, there is a contradiction with unitarity and the assumed spectrum can be ruled out

By scanning over different $\Delta_{\min}$, one can obtain bounds on $\dim \phi^2$ as a function of $d = \dim \phi$

Bounds on $\dim \phi^2$ (taken from arXiv:0905.2211)



Limitations

- Only looked at single real ϕ, can't distinguish between $\mathcal{O}$'s in different global symmetry reps
- E.g., chiral Φ in an $\mathcal{N}=1$ SCFT:
  $\mathrm{Re}[\Phi] \times \mathrm{Re}[\Phi]$ contains $\mathcal{O}$'s from both $\Phi \times \Phi$ and $\Phi^\dagger \times \Phi$
- $\Phi \times \Phi \subset \Phi^2 + \dots$, with $\dim \Phi^2 = 2 \dim \Phi$:
  Φ^2 satisfies bound and we learn nothing about $\Phi^\dagger \Phi$...
- Supersymmetry also relates different conformal primaries, so we should additionally take this information into account

Let's try to generalize the method to deal with this case!

(see Rattazzi, Rychkov, Vichi [arXiv:1009.5985] for more on global symmetries)

Outline

$\mathcal{N} = 1$ Superconformal Algebra

$$
\begin{array}{cccccc}
\text{dim} & & & & & \\
+1 & & & P_a & & \\
+1/2 & & Q_\alpha & & \overline{Q}_{\dot\alpha} & \\
0 & M_{\alpha\beta} & & D, R & & M_{\dot\alpha\dot\beta} \\
-1/2 & & S_\alpha & & \overline{S}_{\dot\alpha} & \\
-1 & & & K_a, & &
\end{array}
$$

$$\{Q, \overline{Q}\} = P \qquad\qquad \{S, \overline{S}\} = K$$

- Superconformal primary means $[S, \mathcal{O}(0)] = [\overline{S}, \mathcal{O}(0)] = 0$
- Descendants obtained by acting with $P, Q, \overline{Q}$
- Chiral means $[\overline{Q}, \phi(0)] = 0$

Superconformal Block Decomposition

ϕ: scalar chiral superconformal primary of dimension d in an SCFT (lowest component of chiral superfield Φ)

$$\langle \phi(x_1)\phi^\dagger(x_2)\phi(x_3)\phi^\dagger(x_4)\rangle \;=\; \frac{1}{x_{12}^{2d}x_{34}^{2d}} \sum_{\mathcal{O}\in\Phi\times\Phi^\dagger} |\lambda_{\mathcal{O}}|^2(-1)^l \mathcal{G}_{\Delta,l}(u,v)$$

- Sum over superconformal primaries $\mathcal{O}^I$ with zero R-charge
- $\lambda_{\mathcal{O}}$ real for even spin $\mathcal{O}^I$, imaginary for odd spin $\mathcal{O}^I$
- $x_1 \leftrightarrow x_3$ gives crossing relation only involving $\mathcal{O}^I \in \Phi\times\Phi^\dagger$
- Must organize superconformal descendants into reps of the conformal subalgebra...

Superconformal Block Derivation

Multiplet built from $\mathcal{O}$ (generically) contains four conformal primaries with vanishing R-charge and definite spin:

name	operator	dim	spin
$\mathcal{O}$	$\mathcal{O}$	Δ	l
J, N	$Q\overline{Q}\mathcal{O} + \#P\mathcal{O}$	$\Delta + 1$	$l+1, l-1$
D	$Q^2\overline{Q}^2\mathcal{O} + \#PQ\overline{Q}\mathcal{O} + \#PP\mathcal{O}$	$\Delta + 2$	l

- Superconformal symmetry fixes coefficients of $\langle \phi\phi^\dagger J\rangle, \langle \phi\phi^\dagger N\rangle, \langle \phi\phi^\dagger D\rangle$ in terms of $\langle \phi\phi^\dagger \mathcal{O}\rangle$
- Must also normalize J, N, D to have canonical 2-pt functions
- Superconformal block is then a sum of $g_{\Delta,l}$'s for $\mathcal{O}, J, N, D$

Superconformal Block Derivation

We find,[1]

$$\mathcal{G}_{\Delta,l} = g_{\Delta,l} - \frac{(\Delta+l)}{2(\Delta+l+1)} g_{\Delta+1,l+1} - \frac{(\Delta-l-2)}{8(\Delta-l-1)} g_{\Delta+1,l-1} + \frac{(\Delta+l)(\Delta-l-2)}{16(\Delta+l+1)(\Delta-l-1)} g_{\Delta+2,l}.$$

- When unitarity bound $\Delta \geq l+2$ is saturated, multiplet is shortened.
- $\mathcal{G}_{\Delta,l}$ can also be determined from consistency with $\mathcal{N}=2$ superconformal blocks computed by Dolan and Osborn ['01].

[1] after plenty of algebra

Outline

Bounds on Dimension of $\Phi^\dagger\Phi$

Isolating the lowest dimension scalar $\Phi^\dagger\Phi \in \Phi \times \Phi^\dagger$, we have

$$|\lambda_{\Phi^\dagger\Phi}|^2 \mathcal{F}_{\Delta_{\min},0} \;\; = \;\; 1 - \sum_{\mathcal{O}\neq\Phi^\dagger\Phi} |\lambda_{\mathcal{O}}|^2 \mathcal{F}_{\Delta,l},$$

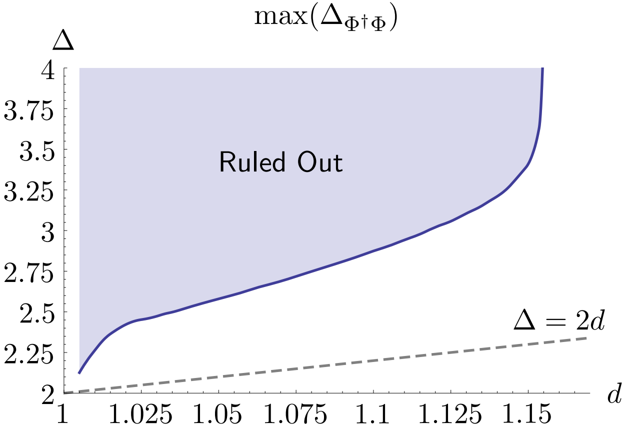
where $\Delta_{\min} = \dim \Phi^\dagger\Phi$, and $\mathcal{F}_{\Delta,l}$ is $F_{\Delta,l}$ with $g_{\Delta,l} \to (-1)^l \mathcal{G}_{\Delta,l}$.

Now minimize $\alpha(1)$ subject to

- $\alpha(\mathcal{F}_{\Delta,0}) \geq 0$ for all $\Delta \geq \Delta_{\min}$,
- $\alpha(\mathcal{F}_{\Delta,l}) \geq 0$ for all $\Delta \geq l+2$ and $l \geq 1$,
- $\alpha(\mathcal{F}_{\Delta_{\min},0}) = 1$

If $\alpha(1) < 0$, we get $|\lambda_{\Phi^\dagger\Phi}|^2 < 0 \implies \Phi^\dagger\Phi$ can't have dim $\Delta_{\min}$

Upper Bound on Dimension of $\Phi^\dagger\Phi$



- Scanning over $\Delta_{\min}$, minimizing $\alpha(1)$ over 21 dimensional space of derivatives

Flavor Currents

- If ϕ transforms under flavor symmetry with charges T^I, conserved currents J^I appear in the $\phi \times \phi^\dagger$ OPE:

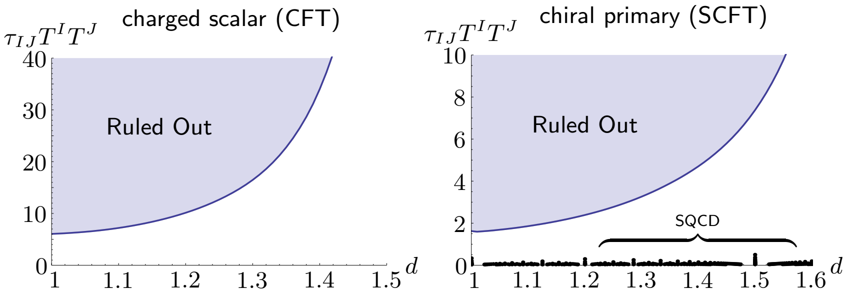
$$\langle \phi\phi^\dagger J^I \rangle \quad \sim \quad -\frac{i}{2\pi^2} T^I \qquad \text{(Ward id.)}$$

- Flavor current conformal blocks are then determined by current 2-pt functions

$$\begin{aligned}
\langle J^I J^J \rangle &\sim \frac{3}{4\pi^4} \tau^{IJ} & \\
\langle \phi\phi^\dagger \phi\phi^\dagger \rangle &\sim -\frac{1}{3} \tau_{IJ} T^I T^J \, g_{3,1} & \text{(general CFTs)}, \\
\langle \phi\phi^\dagger \phi\phi^\dagger \rangle &\sim \quad \tau_{IJ} T^I T^J \, \mathcal{G}_{2,0} & \text{(SCFTs)},
\end{aligned}$$

where $\tau_{IJ} = (\tau^{IJ})^{-1}$ (in SCFTs, $\tau^{IJ} = -3\mathrm{Tr}(RT^I T^J)$).

Upper Bounds on $\tau_{IJ}T^I T^J$



- Example: SUSY QCD with $\frac{3}{2}N_c < N_f < 3N_c$, consider $\langle M M^\dagger M M^\dagger \rangle$: $d = 3 - \frac{3N_c}{N_f}$ and $\tau_{IJ}T^I T^J = \frac{2}{3}\frac{N_f - 1}{N_c^2}$
- In dual AdS$_5$, $(8\pi^2 L)\tau_{IJ} = g^2_{IJ}$. Gauge coupling can't be too strong in presence of charged scalar.

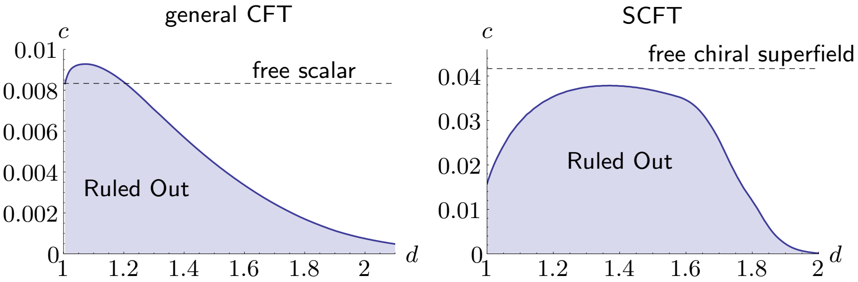
The Stress Tensor

- Ward identity ensures $T^{ab} \in \phi \times \phi$
- $\langle TT \rangle$ is proportional to the central charge c (trace anomaly $16\pi^2 \langle T^a_a \rangle = c(\mathrm{Weyl})^2 - a(\mathrm{Euler})$)
- In an SCFT, T lives in the supercurrent multiplet $\mathcal{J}^a = J^a_R + \theta\sigma_b\bar{\theta}T^{ab} + \ldots$, and c is determined in terms of $U(1)_R$ anomalies
- Conformal block contributions are

$$\langle \phi\phi\phi\phi \rangle \sim \frac{d^2}{90c}\, g_{4,2} \qquad \text{(general CFTs)}$$

$$\langle \phi\phi^\dagger\phi\phi^\dagger \rangle \sim -\frac{d^2}{36c}\, \mathcal{G}_{3,1} \qquad \text{(SCFTs)}$$

Lower Bound on c in General CFT



- Scalar with $d \sim 1$ contributes $\gtrsim 1$ degree of freedom
- In dual AdS$_5$, $c \sim \pi^2 L^3 M_P^3$. Gravity can't be too strong in presence of bulk scalar.
- See also Rattazzi, Rychkov, Vichi [arXiv:1009.2725]

Outline

Outlook

We calculated:

- Superconformal blocks
- Bound $\dim \Phi^\dagger\Phi < f_{\Phi^\dagger\Phi}(d)$
- Bound $\tau_{IJ}T^I T^J \leq f_\tau(d)$ in CFT, SCFT
- Bound $c \geq f_c(d)$ in CFT, SCFT

In the future, we'd like:

- Stronger bounds to make contact with BSM motivation! Improved numerics and better algorithms.
- SUSY theories that come close to saturating bounds on τ, c.
- Bounds in other numbers of dimensions.
- Understand bounds from bulk dual perspective.