BPS-states and automorphic representations

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Rutgers, September 17, 2013

Talk based on:

Pioline, D.P. [0902.3274]

Bao, Kleinschmidt, Nilsson, D.P., Pioline [0909.4299; 1005.4848]

Alexandrov, D.P, Pioline [1010.5792;1304.0766]

D.P. [1103.1014]

+ work in progress with Fleig, Gustafsson, Kleinschmidt

Experimental fact: BPS-states in string theory are intimately connected with automorphic forms

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BPS-states constitute a discrete subsector which is "stable"

$$\text{BPS-states} \quad \subset \quad \text{physical states}$$

"small" (non-generic) representations of the super-Poincaré algebra

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"small" (non-generic) representations of the super-Poincaré algebra

The (weighted) degeneracies of BPS-states are often captured by the Fourier coefficients of some automorphic form

$$f(\gamma g) = f(g) \qquad \gamma \in G(\mathbb{Z}) \quad g \in G(\mathbb{R})$$

Knowledge of these degeneracies is important for many reasons, some of which are:

→ **black hole entropy calculations**

→ **determining exact effective actions**

→ **wall-crossing phenomena**

→ **mathematical applications** (e.g. topological invariants of Calabi-Yau manifolds, moonshine phenomena etc.)

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- → How do we single out which automorphic form is relevant for different physical situations?
- → Can we obtain explicit formulas for the Fourier coefficients of higher rank Lie groups?
- → Could we use physical reasoning to give new mathematical predictions?

To address these questions we (as physicists) have to learn some seemingly abstract mathematics!

In this talk I will give my perspective on this fascinating story.

Outline

1. Eisenstein Series on $SL(2)$: Math versus Physics

2. Langlands Eisenstein Series & Automorphic Representations

3. BPS-States in N=2 Theories & Special Representations

4. Conclusions and Future Prospects

1. Eisenstein Series on $SL(2)$

- Math versus Physics-

Non-holomorphic Eisenstein series

Consider the sum:

$$E_s(\tau) = \sum_{(c,d)=1} \frac{y^s}{|c\tau + d|^{2s}}$$

non-holomorphic
Eisenstein series
$s \in \mathbb{C}$

→ a function on $\mathbb{H} = \{\tau = x + iy \in \mathbb{C} \,|\, y > 0\}$

→ invariant under $\tau \mapsto \gamma \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

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→ invariant under $\tau \mapsto \gamma \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$

→ converges absolutely for $\Re s > 1$

→ $\Delta_{\mathbb{H}} E_s = s(s-1)E_s$ $\qquad \Delta_{\mathbb{H}} = y^2(\partial_x^2 + \partial_y^2)$

Invariance under $\tau \mapsto \tau + 1$ yields the **Fourier expansion**

$$E_s(\tau) = \underbrace{C(y;s)}_{\substack{\text{constant term}\\ \text{zero mode}}} + \underbrace{\sum_{n\neq 0} F_n(y;s) e^{2\pi i n x}}_{\text{non-zero mode}}$$

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$$\rightarrow \quad C(y;s) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}$$

completed zeta-function:

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

$$\rightarrow \quad F_n(y;s) = \frac{2\sqrt{y}}{\xi(2s)} |n|^{s-1/2} \mu_{1-2s}(n) K_{s-1/2}(2\pi|n|y)$$

divisor sum: $\mu_s(n) = \sum_{d|n} d^s$

For certain values of s this has a **physical interpretation**

[Green, Gutperle]

Perturbative quantum effects (weak-coupling limit $y \to \infty$)

$$C(y;3/2) = y^{3/2} + \frac{\xi(2)}{\xi(3)} y^{-1/2}$$

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Perturbative quantum effects (weak-coupling limit $y \to \infty$)

$$C(y;3/2) = y^{3/2} + \frac{\xi(2)}{\xi(3)} y^{-1/2}$$

tree-level

one-loop

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- Supersymmetry: Laplacian eigenfunction (with fixed eigenvalue)

[Green, Gutperle][Green, Sethi]

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[Green, Gutperle][Green, Sethi]

But this is just the tip of the iceberg!

The process of "compactification" leads to enhanced symmetries called **U-duality**

$$SL(2,\mathbb{Z}) \subset G(\mathbb{Z})$$

discrete Lie group (U-duality group)

Physical observables should be invariant under U-duality

This implies that **automorphic forms on higher rank Lie groups** occur naturally in string theory

For example, in the case of **maximal supersymmetry**, we have the following list of U-duality groups occurring in different spacetime dimensions D

[Cremmer, Julia][Hull, Townsend][Witten]

D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3)\times\mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z})\times\mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$(\mathrm{Spin}(5)\times\mathrm{Spin}(5))/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

Automorphic forms have been extensively studied in this context:

[Kiritsis, Pioline][Obers, Pioline][Basu][Green, Russo, Vanhove][Green, Miller, Russo, Vanhove][Green, Miller, Vanhove][Pioline][Fleig, Kleinschmidt][Bao, Carbone]

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But much more remains to be done! Explicit Fourier coefficients for exceptional groups, non-BPS protected quantities, non-maximally supersymmetric compactifications...

To this end we must understand the mathematics better

$SL(2,\mathbb{Z})$-Eisenstein series revisited

Using the isomorphism

$$\mathbb{H} \cong SL(2,\mathbb{R})/SO(2) \qquad\qquad SO(2) = \mathrm{Stab}(i)$$

we can think of the Eisenstein series as a function on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ via

$$E_s(g) = E_s(nak) = E_s\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k\right) = E_s(x+iy)$$

where we used the Iwasawa decomposition

$$g = nak \in SL(2,\mathbb{R}) = NAK$$

Borel subgroup $\qquad B = NA = \left\{\begin{pmatrix} \star & \star \\ & \star \end{pmatrix}\right\}$

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The function $E_s(g)$ is then a (spherical) **automorphic form** on $SL(2,\mathbb{R})$

$$E_s(\gamma g k) = E_s(g)$$

$$\gamma \in SL(2,\mathbb{Z}) \qquad k \in SO(2)$$

We can now recast the Fourier expansion more group-theoretically:

$$E_s(g) = E_s^{\mathrm{const}}(g) + \sum_{\psi \text{ generic}} W_\psi(g)$$

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$$\psi \,:\, N(\mathbb{Z})\backslash N(\mathbb{R}) \longrightarrow U(1)$$

unitary character on $N(\mathbb{R})$

(trivial on $N(\mathbb{Z})$)

$$\psi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(e^{xE_\alpha}) = e^{2\pi i m x}$$

$$x \in \mathbb{R} \qquad m \in \mathbb{Z}$$

ψ generic $\longleftrightarrow$ $m \neq 0$

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$$E_s^{\text{const}}(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E_s(ng)dn \; = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}$$

Non-constant term

$$W_\psi(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E_s(ng)\overline{\psi(n)}dn$$

$$= \frac{2y^{1/2}}{\xi(2s)} |m|^{s-1/2} \mu_{1-2s}(m) K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}$$

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This is an example of a (spherical) **Whittaker vector:**

$$W_\psi(ngk) = \psi(n)W_\psi(g)$$

The Whittaker vector is determined by its restriction to A:

$$W_\psi(g) = W_\psi(nak) = \psi(n)W_\psi(a)$$

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These formulas now have a **natural generalization** to higher rank Lie groups!

Euler products

Before we proceed with the higher rank case we mention some further properties of the Fourier expansion, namely that it decomposes into Euler products

$$E_s^{\mathrm{const}}(g) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}$$

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$$\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\prod_{p<\infty}\frac{1-p^{-2s}}{1-p^{-2s+1}}$$

- One can incorporate the prefactor as the $p=\infty$ part of the Euler product
- These observations form the basis of Langlands' general constant term formula

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"real Whittaker vector": $W_\infty(g) = \frac{2\pi^s}{\Gamma(s)} y|m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}$

$$g = nak = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k \in SL(2,\mathbb{R})$$

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"p -adic Whittaker vector"

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$1 \in SL(2, \mathbb{Q}_p)$

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This implies that the instanton measure in string theory is completely determined by a p -adic Whittaker vector!

[Kazhdan, Pioline, Waldron][Neitzke, Gunaydin, Pioline, Waldron][Pioline][Pioline, D.P.]

2. Langlands Eisenstein Series and Automorphic Representations

Eisenstein series on semi-simple Lie groups

$G(\mathbb{R}) = B(\mathbb{R})K(\mathbb{R})$ semi-simple Lie group in its split real form

(quasi-)character $\chi\ :\ B(\mathbb{Z})\backslash B(\mathbb{R}) \to \mathbb{C}^\times$

defined by

$$\chi(b) = \chi(na) = \chi(a) = e^{\langle \lambda + \rho | H(a) \rangle}$$

$$H\ :\ A(\mathbb{R}) \to \mathfrak{h} = \mathrm{Lie}\, A(\mathbb{R})$$

$$H(a) = H\left(e^{\sum_{\alpha\in\Pi} y_\alpha H_\alpha}\right) = \sum_{\alpha\in\Pi} y_\alpha H_\alpha$$

$$\lambda \in \mathfrak{h}^\star \otimes \mathbb{C} \qquad\qquad \rho = \frac{1}{2}\sum_{\alpha>0}\alpha$$

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Extend to the whole group by: $\chi(g) = \chi(nak) = \chi(na)$

Eisenstein series on semi-simple Lie groups

Given this data the **Langlands Eisenstein series** is defined by:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

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- Converges absolutely on a subspace of $\mathfrak{h}^\star \otimes \mathbb{C}$

  Godement's domain
  $\{\lambda | \langle \lambda, \alpha \rangle > 1, \forall \alpha \in \Pi\}$

- Can be continued to a meromorphic function on all of $\mathfrak{h}^\star \otimes \mathbb{C}$

- Automorphic form: $E(\lambda, \gamma g k) = E(\lambda, g)$

  $\gamma \in G(\mathbb{Z})$ $\quad$ $k \in K(\mathbb{R})$

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- Automorphic form: $E(\lambda, \gamma g k) = E(\lambda, g)$

- Satisfies a functional equation in λ

- Eigenfunction of the Laplacian: $\Delta_{G/K} E(\lambda, g) = \frac{1}{2}(\langle \lambda | \lambda \rangle - \langle \rho | \rho \rangle) E(\lambda, g)$

Automorphic representations

Eisenstein series are attached to the (non-unitary) **principal series:**

$$I(\lambda) = \mathrm{Ind}_B^G \chi = \{f : G \to \mathbb{C} \mid f(bg) = \chi(b) f(g), b \in B\}$$

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The theory of Eisenstein series then defines a map

$$E \, : \, I(\lambda) \to \mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$$

from the principal series to **the space of automorphic forms on** $G(\mathbb{R})$

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G acts on $\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ by **right-translation:**

$$[\rho(h)f](g) = f(gh)$$

The irreducible constituents in the decomposition of $\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ under this action are called **automorphic representations**

[Gelfand, Graev, Piatetski-Shapiro][Langlands]...

Automorphic representations

There is an important notion of "size" of an automorphic representation, called the **Gelfand-Kirillov dimension**.

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$$\mathrm{GKdim}(I(\lambda)) = \dim_{\mathbb{R}} B\backslash G = \dim_{\mathbb{R}} N$$

This is important for physics, since we have the rough correspondence:

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This is important for physics, since we have the rough correspondence:

number of independent **physical charges** (e.g. electric, magnetic) ⟷ **Gelfand-Kirillilov dimension** of the associated automorphic representation

Automorphic representations

For example, consider again the **non-holomorphic Eisenstein series**

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$$\lambda+\rho = 2s\Lambda$$ (fundamental weight: $\Lambda = \alpha/2$)

$$H(a) = H(e^{yH_\alpha}) = yH_\alpha$$ $$\langle \Lambda | H_\alpha \rangle = 1$$

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This is attached to the representation $I(s) = \mathrm{Ind}_B^{SL(2,\mathbb{R})} e^{2s\langle \Lambda | H\rangle}$

$$\mathrm{GKdim} I(s) = \dim_{\mathbb{R}} B\backslash SL(2,\mathbb{R}) = 1$$

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$$\mathrm{GKdim} I(s) = \dim_{\mathbb{R}} B\backslash SL(2,\mathbb{R}) = 1$$

This equals the number of summation variables in the Fourier expansion

$$E(s,g) = \sum_{\psi : N(\mathbb{Z})\backslash N(\mathbb{R}) \to U(1)} W_\psi(g) = \sum_{m\in\mathbb{Z}} W_m(g)$$

For $s = 3/2$ this is also the number of instanton charges in string theory

Non-abelian Fourier expansions

Beyond $SL(2,\mathbb{R})$ the unipotent radical N is no longer abelian

$$E(\lambda, g) = E_{\rm ab}(\lambda, g) + E_{nab}(\lambda, g)$$

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Abelian term:

$$E_{\rm ab}(\lambda, g) = \int_{N'(\mathbb{Z})\backslash N'(\mathbb{R})} E(\lambda, n'g)dn'$$

This is the constant term with respect to the derived subgroup

$$N' = [N, N]$$

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Abelian term:

$$E_{\rm ab}(\lambda,g) = \int_{N'(\mathbb{Z})\backslash N'(\mathbb{R})} E(\lambda,n'g)dn'$$

$$= E^{\rm const}(a) + \sum_{\psi\neq 1} W_\psi(g)$$

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Abelian term:

$$E_{\mathrm{ab}}(\lambda, g) = E^{\mathrm{const}}(a) + \sum_{\psi \neq 1} W_{\psi}(g)$$

ψ is trivial on N' and so restricts to a character on the abelianization:

$$N_{\mathrm{ab}} = N' \backslash N$$

Non-abelian Fourier expansions

Beyond $SL(2,\mathbb{R})$ the unipotent radical N is no longer abelian

$$E(\lambda, g) = E_{\mathrm{ab}}(\lambda, g) + E_{nab}(\lambda, g)$$

Abelian term:

$$E_{\mathrm{ab}}(\lambda, g) = E^{\mathrm{const}}(a) + \sum_{\psi \neq 1} W_{\psi}(g)$$

Therefore the Whittaker vector is given by the same formula as before:

$$W_{\psi}(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)\overline{\psi(n)}dn$$

This can be evaluated locally at each prime!

Non-abelian Fourier expansions

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$$E(\lambda, g) = E_{\rm ab}(\lambda, g) + E_{nab}(\lambda, g)$$

Abelian term:

$$E_{\rm ab}(\lambda, g) = E^{\rm const}(a) + \sum_{\psi\neq 1} W_\psi(g)$$

The **non-abelian term** is more tricky and will be discussed in more detail later for specific examples.

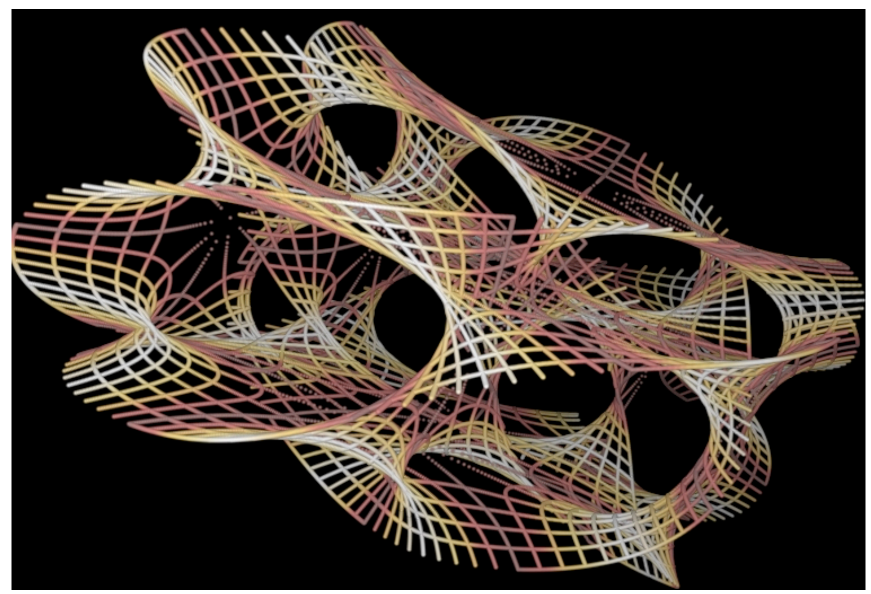
[Piatetski-Shapiro][Shalika][Vinogradov,Takhtajan][Miller, Sahi]

[Pioline, D.P][Bao, Kleinschmidt, Nilsson, D.P., Pioline][Alexandrov, D.P., Pioline]

3. BPS-States in N=2 Theories and Special Automorphic Representations

In the remainder of the talk I want to focus on a specific physical setting which is largely unexplored from the automorphic perspective

→ Type II string theory on a Calabi-Yau threefold

$\mathbb{R}^{1,3}\times$ 

Can we use automorphic techniques also in this setting?

BPS-states arise geometrically from D3-branes wrapping special Lagrangian 3-cycles inside the CY 3-fold X

→ Lattice of electric-magnetic charges $\Gamma = H_3(X, \mathbb{Z})$

"charge vector" $\gamma = (p^\Lambda, q_\Lambda) \in \Gamma$ $\Lambda = 0, 1, \ldots, h_{2,1}$

→ BPS-index $\Omega : \Gamma \to \mathbb{Q}$

appropriate "count" of BPS-states with charge γ

Mathematically, this index should coincide with the **generalized Donaldson-Thomas invariants** defined by Joyce & Kontsevich-Soibelman

[Denef, Moore][Gaiotto, Moore, Neitzke]
[Alexandrov, Saueressig, Pioline, Vandoren][Alexandrov, D.P., Pioline]

We wish to study the **"partition function"** of these BPS-states.
Schematically this would be a formal generating function:

[Ooguri, Strominger, Vafa][de Wit, Kappeli, Lopes Cardoso, Mohaupt][Denef, Moore]...

$$\sum_{\gamma \in \Gamma} \Omega(\gamma) e^{2\pi i (q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda)}$$

where we introduce "chemical potentials" $(\zeta^\Lambda, \tilde{\zeta}_\Lambda) \in \Gamma^\star \otimes \mathbb{R}/(2\pi\mathbb{Z})$

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Could one "resum" this series?

Or, rather, does the theory exhibit a **discrete U-duality symmetry** $G(\mathbb{Z})$ such that the BPS-index $\Omega(\gamma)$ arises as the Fourier coefficient of some automorphic form?

We know that the theory is invariant under the **Jacobi group**

$$G^J(\mathbb{Z}) = G_4(\mathbb{Z}) \ltimes U(\mathbb{Z})$$

→ $G_4(\mathbb{Z})$ should contain the monodromy group of $X \subset Sp(2h_{2,1}+2;\mathbb{Z})$ and the S-duality group $SL(2,\mathbb{Z})$

→ $U(\mathbb{Z})$ discrete Heisenberg group

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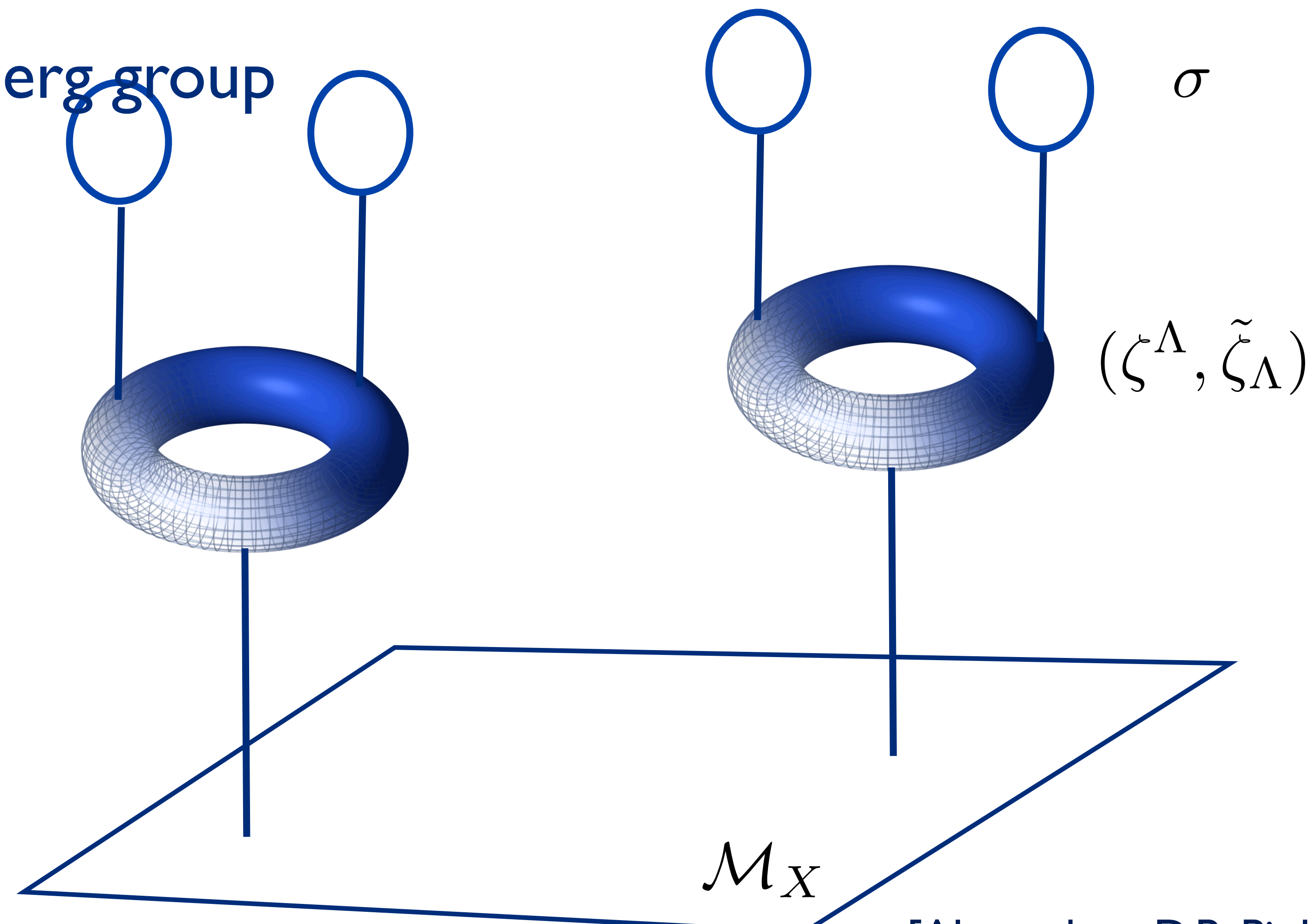
this comes from the structure of the moduli space as a torus fibration



[Alexandrov, D.P., Pioline]

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→ $U(\mathbb{Z})$ discrete Heisenberg group

Let us now assume that these symmetries combine into a bigger duality group

$$G_3(\mathbb{Z}) \supset G_4(\mathbb{Z}) \ltimes U(\mathbb{Z})$$

Supersymmetry suggests that this should be a discrete subgroup of a Lie group $G_3(\mathbb{R})$ in its **quaternionic real form**

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Can we construct a $G_3(\mathbb{Z})$-invariant automorphic form whose abelian Fourier coefficients give the degeneracies $\Omega(\gamma)$?

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		U-duality	classical symmetry
terminology and motivation comes from compactification	$D=4$	$G_4(\mathbb{Z})$	$G_4(\mathbb{R})$
	$\downarrow S^1$		
	$D=3$	$G_3(\mathbb{Z})$	$G_3(\mathbb{R})$

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→ $U(\mathbb{Z})$ discrete Heisenberg group

Examples of groups that we know occur in this context are

$$G_3(\mathbb{R}) = SU(2,1) \qquad \text{or} \qquad G_{2(2)}(\mathbb{R})$$

What singles out the candidate automorphic form?

→ Use **constraints from supersymmetry** combined with **representation theory**!

Any semi-simple Lie algebra $\mathfrak{g}$ exhibits a 5-grading:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

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$\mathfrak{g}_2 = \mathbb{R}E_\alpha$ — α = highest root

$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathbb{R}H_\alpha$ — Levi subalgebra

$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ — Heisenberg parabolic subalgebra

$\mathfrak{u} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ — Heisenberg subalgebra $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$

[Kazhdan, Savin]

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Heisenberg parabolic subgroup: $P = LU = MAU$

U is the unipotent radical of P

[Kazhdan, Savin]

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Physical interpretation: the 5-grading adapted to the decompactification limit $R \to \infty$

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Physical interpretation: the 5-grading adapted to the decompactification limit $R \longrightarrow \infty$

Let $\mathfrak{g}$ be the Lie algebra of the symmetry group $G_3(\mathbb{R})$

⟶ H_α is the Cartan generator associated with the radial direction R

⟶ $\mathfrak{m}$ is the Lie algebra of the 4d symmetry group $G_4(\mathbb{R})$

⟶ $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the Lie algebra of the Heisenberg group $U(\mathbb{R})$

⟶ $\mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the Lie algebra of the Jacobi group $G^J(\mathbb{R})$

Degenerate principal series

Introduce a quasi-character on the Heisenberg parabolic

$$\chi_s \; : \; P(\mathbb{Z})\backslash P(\mathbb{R}) \to \mathbb{C}^\times$$

defined by its restriction to A : $s \in \mathbb{C}$

$$\chi_s(p) = \chi_s(mau) = \chi_s(a) = \chi_s(e^{yH_\alpha}) = y^s$$

Extend to all of G_3 : $\chi_s(g) = \chi_s(pk) = \chi_s(p)$ $k \in K_3(\mathbb{R})$

Associated with this character we have the **degenerate principal series**

$$\mathrm{Ind}_P^{G_3}\chi_s$$

and the **Eisenstein series**

$$E(\chi_s, P, g) = \sum_{\gamma\in P(\mathbb{Z})\backslash G_3(\mathbb{Z})} \chi_s(\gamma g)$$

The **Gelfand-Kirillov dimension** is

$$\mathrm{GKdim}\,\mathrm{Ind}_P^{G_3}\chi_s = \dim_{\mathbb{R}} P\backslash G_3 = \dim_{\mathbb{R}}\mathfrak{g}_1 + \dim_{\mathbb{R}}\mathfrak{g}_2$$

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The functional dimension gives a measure of the number of independent summation variables occurring in the Fourier expansion of the automorphic form

Physically, these are the **charges** of the BPS-states!

$$\dim_{\mathbb{R}} \mathfrak{g}_1 = 2(n+1) \quad \longleftrightarrow \quad (p^\Lambda, q_\Lambda) \quad \Lambda = 0, 1, \ldots, n$$

D-brane charges
e.g. $(p^\Lambda, q_\Lambda) \in H_3(X, \mathbb{Z})$

number of vector fields in the original 4d theory
e.g. $n = h_{2,1}(X)$

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$$\dim_{\mathbb{R}} \mathfrak{g}_2 = 1 \quad \longleftrightarrow \quad k$$ **NS5-brane charge**

The total number of physical charges $(p^\Lambda, q_\Lambda, k)$ corresponds to the functional dimension of $\mathrm{Ind}_P^{G_3}\chi_s$

Does the Eisenstein series $E(\chi_s, P, g)$ have the right properties for an instanton partition function that captures all these effects?

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→ Compute the **non-abelian Fourier expansion**!

1. First extract the Fourier coefficients along the center $Z = [U, U]$

$$W_{\psi_Z}(g) = \int_{Z(\mathbb{Z})\backslash Z(\mathbb{R})} E(\chi_s, P, zg)\overline{\psi_Z(z)}dz$$

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$$W_{\psi_Z}(g) = \int_{Z(\mathbb{Z})\backslash Z(\mathbb{R})} E(\chi_s, P, zg)\overline{\psi_Z(z)}dz$$

2. Then expand the constant term along $U_{\text{ab}} = Z\backslash U$

$$\int_{Z(\mathbb{Z})\backslash Z(\mathbb{R})} E(\chi_s, P, zg)dz = \sum_{\psi : U_{\text{ab}}(\mathbb{Z})\backslash U_{\text{ab}}(\mathbb{R}) \to U(1)} W_\psi(g)$$

The result is of the general form:

$$E(\chi_s, P, g) = E_{\mathrm{const}}(s) + \sum_{\gamma\in\Gamma} C_s(\gamma) W_\gamma(s; z, R) e^{2\pi i(q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda)}$$

$$+ \sum_{k\neq 0} \Psi_k(s; z, R, \zeta, \tilde{\zeta}) e^{i\pi k\sigma}$$

[Pioline, D.P.][Bao, Kleinschmidt, Nilsson, D.P., Pioline][Fleig, Gustafsson, Kleinschmidt, D.P.]

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$$W_\gamma(s; z, R) \underset{R \to \infty}{\sim} e^{\pi R |Z(\gamma, z)|}$$

BPS-instantons
(D-brane instantons)

$$|Z(\gamma, z)| = \text{Mass}$$

[Pioline, D.P.][Bao, Kleinschmidt, Nilsson, D.P., Pioline][Fleig, Gustafsson, Kleinschmidt, D.P.]

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gravitational instantons
(NS5-instantons)

$$\Psi_k(s; z, R, \zeta, \tilde{\zeta}) \underset{R\to\infty}{\sim} c_k e^{\pi R^2 |k|}$$

[Pioline, D.P.][Bao, Kleinschmidt, Nilsson, D.P., Pioline][Fleig, Gustafsson, Kleinschmidt, D.P.]

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The numbers $C_s(\gamma)$ should follow from the p-adic Whittaker vector on the unipotent radical U:

$$C_s(\gamma) = \prod_{p<\infty} W_{\psi_{U,p}}(s; 1)$$

Unfortunately there is no general formula for $W_{\psi_{U,p}}(s; 1)$

Example: rigid Calabi-Yau 3-folds

X rigid $\longrightarrow$ $h_{2,1} = 0$

When the intermediate Jacobian is of the form

$$H^3(X,\mathbb{R})/H^3(X,\mathbb{Z}) = \mathbb{C}/\mathcal{O}_d$$

ring of integers: $\mathcal{O}_d \subset \mathbb{Q}(\sqrt{-d})$ ($d > 0$ and square-free)

the group $G_3(\mathbb{Z})$ is conjectured to be the **Picard modular group**:

$$PU(2,1;\mathcal{O}_d) := U(2,1) \cap PGL(3,\mathcal{O}_d)$$

[Bao, Kleinschmidt, Nilsson, D.P., Pioline]

The abelian Fourier coefficients of the Eisenstein series

$$E(\chi_s, P, g) = \sum_{\gamma \in P(\mathcal{O}_d)\backslash PU(2,1;\mathcal{O}_d)} \chi_s(\gamma g)$$

are given by the **double divisor sum**

$$C_s(\gamma) = \sum_{\substack{\omega \in \mathcal{O}_d \\ \gamma/\omega \in \mathcal{O}_d^\star}} \left|\frac{\gamma}{\omega}\right|^{2s-2} \sum_{\substack{z \in \mathcal{O}_d \\ \gamma/(z\omega) \in \mathcal{O}_d^\star}} |z|^{4-4s}$$

[Bao, Kleinschmidt, Nilsson, D.P., Pioline]

What is the physical interpretation of these numbers?

We expect that for some **fixed value** $s = s_0$ we should have:

$$C_{s_0}(\gamma) = \Omega(\gamma)$$

BPS-index

But there are additional physical constraints on the numbers $\Omega(\gamma)$

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γ is the charge of a black hole with Bekenstein-Hawking entropy:

$$S(\gamma) = \log \Omega(\gamma) \sim \pi \sqrt{\mathcal{Q}(\gamma)} + \cdots$$

$\mathcal{Q}(\gamma)$ = quartic $G_4(\mathbb{Z})$ -invariant

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$\mathcal{Q}(\gamma)$ = quartic $G_4(\mathbb{Z})$ -invariant

BPS constraint: $\Omega(\gamma) = 0$ **unless** $\mathcal{Q}_4(\gamma) \geq 0$

[Ferrara, Maldacena]

To summarize, we need to satisfy the two constraints:

$$\mathrm{GKdim} = 2n + 3$$

1/2 BPS constraint: $\mathcal{Q}_4(\gamma) \geq 0$

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For groups $G_3(\mathbb{R})$ in their **quaternionic real form**, Gross & Wallach have constructed a unitary representation π_ν called the

Quaternionic Discrete Series.

- it depends on a single integral parameter ν
- $\mathrm{GKdim}\,\pi_\nu = 2n+3$
- it is a **submodule** of the **degenerate principal series**

$$\pi_\nu \subset \mathrm{Ind}_P^{G_3}\chi_s\Big|_{s=\nu-3/2}$$

Moreover, Wallach has shown that the Fourier coefficients

$$W_{\psi_U}(g) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(ug)\overline{\psi_U(u)}du \qquad f \in \pi_\nu$$

in the automorphic realization of π_ν have support only on charges:

$$\mathcal{Q}_4(\gamma) \geq 0$$

So this takes care of the constraint:

BPS constraint: $\Omega(\gamma) = 0$ **unless** $\mathcal{Q}_4(\gamma) \geq 0$

This leads to the following:

Conjecture: When a "U-duality" symmetry $G_3(\mathbb{Z})$ is present in an $\mathcal{N}=2$ theory, the associated BPS-instanton effects are captured by the Fourier coefficients of an automorphic form attached to the **quaternionic discrete series** of G_3

[Günaydin, Neitzke, Pioline, Waldron][Pioline, D.P.][Bao, Kleinschmidt, Nilsson, D.P., Pioline]

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[Günaydin, Neitzke, Pioline, Waldron][Pioline, D.P.][Bao, Kleinschmidt, Nilsson, D.P., Pioline]

If correct, this would also have interesting **mathematical implications**:

→ New connection between **Donaldson-Thomas invariants** of Calabi-Yau 3-folds and **automorphic representations**

→ Prediction on the **growth of the Fourier coefficients**:

$$\Omega(\gamma) \underset{\gamma\to\infty}{\sim} e^{\pi\sqrt{Q_4(\gamma)}}$$

4. Conclusions and Future Prospects

- Automorphic techniques connected to U-duality symmmetries extremely useful for counting BPS-states in theories with a large amount of susy

- General arguments suggest that there should exist a U-duality group also in N=2 theories

$$G_3(\mathbb{Z}) \supset G^J(\mathbb{Z}) = G_4(\mathbb{Z}) \ltimes U(\mathbb{Z})$$

- Constraints from N=2 susy points to a connection between instanton partition functions in D=3 and automorphic representations of G_3

- Preliminary results obtained for $SU(2,1)$ and $SL(3)$

[Pioline, D.P.][Bao, Kleinschmidt, Nilsson, D.P., Pioline]

Future Prospects

Interesting example is type II string theory CY3's with $h_{1,1}(X) = 1$

classical symmetry: $G_{2(2)}(\mathbb{R})$ [Bodner, Cadavid]

U-duality? $G_{2(2)}(\mathbb{Z})$

Future Prospects

- Interesting example is type II string theory CY3's with $h_{1,1}(X) = 1$

classical symmetry: $G_{2(2)}(\mathbb{R})$ [Bodner, Cadavid]

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do they contain information about **BPS-degeneracies**/**DT-invariants**?

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- Compactification to D=2 → automorphic forms on affine KM-groups!

[Garland][Kapranov][Braverman, Kazhdan][Fleig, Kleinschmidt][Garland, Miller, Patnaik]
[Bao, Carbone][Fleig, Kleinschmidt, D.P] (*to appear*)

Secret Slides

Casselman-Shalika formula

$$W_p(1) = \int_{N(\mathbb{Q}_p)} \chi(w_0 n)\overline{\psi(n)}dn$$

For the p-adic Whittaker function there exists a remarkable formula due to **Casselman-Shalika** (and Shintani,Kato):

$$W_p(1) = e^{-\langle w_0\lambda+\rho|H(a)\rangle} \prod_{\alpha>0} \frac{1-p^{-(\langle\lambda|\alpha\rangle+1)}}{1-p^{\langle\lambda|\alpha\rangle}}$$

$$\times \sum_{w\in W(\mathfrak{g})} (\det w) \prod_{\substack{\alpha>0\\ w\alpha<0}} p^{\langle\lambda|\alpha\rangle} e^{\langle w\lambda+\rho|H(a)\rangle}$$

where $\quad a = e^{\sum_{\alpha\in\Pi}(\log v_\alpha)H_\alpha}$

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$$a = e^{\sum_{\alpha\in\Pi}(\log v_\alpha)H_\alpha}$$

From a physics perspective the "instanton charges" are captured by

$$m_\alpha = \prod_{\beta\in\Pi} v_\alpha^{A_{\alpha\beta}}$$

[Fleig, Gustafsson, Kleinschmidt, D.P.] *(to appear)*

Example: $SL(3,\mathbb{R})$

Simple roots $\Pi = \{\alpha_1, \alpha_2\}$

Fundamental weights $\{\Lambda_1, \Lambda_2\}$

$$E(s_1, s_2, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(3,\mathbb{Z})} e^{\langle 2s_1\Lambda_1 + 2s_2\Lambda_2 | H(\gamma g)\rangle}$$

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This Eisenstein series occurs in string theory in a variety of places:

→ The automorphic membrane (M-theory on T^3) [Pioline, Waldron]

→ Type IIB string theory on T^2 [Kiritsis, Pioline]

→ Type IIB string theory on a Calabi-Yau 3-fold [Pioline, D.P.]

Of key physical interest are the numerical Fourier coefficients, a.k.a. p -adic Whittaker function!

Generic character on $N(\mathbb{Z})\backslash N(\mathbb{R})$

$$\psi(e^{x_1 E_{\alpha_1} + x_2 E_{\alpha_2}}) = \exp\left(2\pi i [m_1 x_1 + m_2 x_2]\right)$$

$$m_1, m_2 \neq 0$$

The p -adic Whittaker function is defined by the integral

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The Casselman-Shalika formula gives

$$W_p(1) = \Upsilon(p) \Big(|m_1|^{2s_1+2s_2-2} |m_2|^{2s_1+2s_2-2} - p^{2s_1-1} |m_1|^{2s_1-1} |m_2|^{2s_1+2s_2-2}$$

$$-p^{2s_2-1} |m_1|^{2s_1+2s_2-2} |m_2|^{2s_1-1} + p^{4s_1+2s_2-3} |m_1|^{2s_2-1}$$

$$+p^{2s_1+4s_2-3} |m_2|^{2s_1-1} - p^{4s_1+4s_2-4} \Big)$$

The "instanton measure" is then given by the Euler product:

$$\Omega(m_1, m_2) := \prod_{p<\infty} W_p(1) = |m_1|^{s_1+2s_2-1}|m_2|^{2s_1+s_2-1}\sigma_{1-3s_2,1-3s_1}(|m_1|, |m_2|)$$

where we defined the "double divisor sum" [Bump][Vinogradov, Takhtajan]

$$\sigma_{\alpha,\beta}(n, m) = \sum_{\substack{m=d_1 d_2 d_3 \\ d_1, d_2, d_3>0, \mathrm{gcd}(d_3, n)=1}} d_2^{\alpha} d_3^{\beta}$$

For $(s_1, s_2) = (3/2, -3/2)$ it was proposed in [Pioline, D.P.] that $\Omega(m_1, m_2)$ captures the BPS-degeneracies of D-branes on Calabi-Yau 3-folds with electric-magnetic charges $(p, q) = (m_1, m_2)$