Writing CFT correlation functions as AdS scattering amplitudes

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JP, arXiv:1011.1485 Okuda, JP, arXiv:1002.2641 Heemskerk, JP, Polchinski, Sully, arXiv:0907.0151 Gary, Giddings, JP, arXiv:0903.4437

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Introduction

The AdS description of a CFT is useful if it is

• **Weakly coupled 4 •••** "large N" factorization

 $\langle O_1 \dots O_n \rangle$ connected ∼ κ^{n-2} , $\kappa^2 \sim GR^{1-d} \ll 1$

• **Local** - effective field theory in AdS with small number of fields valid up to some UV cutoff ℓ much smaller than the AdS radius *R*

$$
\ell \sim \frac{1}{\text{mass}} \sim \frac{R}{\Delta}
$$

 \longrightarrow Large gap in spectrum of dimensions $\Delta \gg 1$

A Conjecture

Any CFT that has a large-N expansion, and in which all single-trace operators of spin greater than two have parametrically large dimensions, has a local bulk dual.

 $Large-N expansion \longleftrightarrow$ Weakly coupled bulk dual Single-trace operator \longleftrightarrow Single-particle state *lP* $\frac{P}{R} \ll 1$

Large-N vector models have weakly coupled non-local bulk duals (AdS higher spin theories). [Klebanov, Polyakov 02] [Fradkin,Vasiliev 87, ...] [Sezgin, Sundell 02]

Bulk Locality

Main difficulty: local bulk physics is encoded in CFT correlation functions in a non-trivial way. How to extract the bulk S-matrix?

Idea: prepare wave-packets that scatter in small region of AdS

> [Polchinski 99] [Susskind 99] [Gary, Giddings, JP '09] [Okuda, JP '10]

Best language: Mellin amplitudes

Outline

- **•** Introduction
- Mellin amplitudes
- Flat space limit of AdS
- Testing the conjecture
- Open questions

Mellin Amplitudes

Mellin Amplitudes **Mellin Amplitudes**

Euclidean correlator of primary scalar operators Correlation function of scalar primary operators

$$
A(x_i) = \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle
$$

$$
A(x_i) = \mathcal{N} \int\limits_{-i\infty}^{i\infty} [d\delta] \underbrace{M(\delta_{ij})}_{i
$$

Construction:
$$
\sum_{j \neq i}^{n} \delta_{ij} = \Delta_i = \dim[\mathcal{O}_i]
$$

the integration variables are constrained by $\frac{n(n-3)}{2}$ # integration variables = # ind. cross-ratios = $\frac{1}{2}$

δ*ij* = ∆*ⁱ ,* (3)

constructive to solve the solve the next section. It is instructive to solve the construction of the constructio
The next section of the constraints (3) and construction of the construction of the construction of the const Introduce k_i such that $-k_i^2 = \Delta_i$ and $\sum_{i=1}^n k_i = 0$ $\Lambda_i = 2\delta_i$ ∆*ⁱ* + ∆*^j* − *sij* ∆*ⁱ* + ∆*^j* − *sij* A palacy with Scattoring A mplitudes using *n* Lorentzian vectors *kⁱ* subject to !*ⁿ ⁱ*=1 *kⁱ* = 0 and −*k*² *ⁱ* = ∆*i*. Then $\delta_{ij} = k_i \cdot k_j$ at *i* − $\frac{1}{2}$ − $\frac{1}{2}$
i omatically so then $\delta_{ij} = k_i \cdot k_j$ automatically solves the constraints. Analogy with Scattering Amplitudes *ith Scattering Amplitudes* ∆*ⁱ* + ∆*^j* − *sij* \int **htroduce** k_i such that $-k_i^2 = \Delta_i$ and $\sum_{i=1}^n$ δ*ij* = *kⁱ · k^j* = ∆*ⁱ* + ∆*^j* − *sij* vich Stattening Aniipheud δ then $\delta_{ij} = k_i \cdot k_j$ automat

Define
$$
s_{ij} = -(k_i + k_j)^2 = \Delta_i + \Delta_j - 2\delta_{ij}
$$
 [Mack '09]

with *sij* = −(*kⁱ* + *k^j*)², automatically solves the constraints (3). M_n is a strong simple poles at $(n-1)$ with *sij* = −(*kⁱ* + *k^j*)², automatically solves the constraints (3). \blacksquare Meromorphic with simple poles at $(n=4)$ \mid The Mellin amplitude is crossing symmetric and *n* intromorphic with simple poles at $(n = 4)$ The Mellin amplitude is crossing symmetric and meromorphic with simple poles at $(n = 4)$

$$
M(s_{ij}) \approx \frac{C_{13k}C_{24k}P_{l_k}^m(\gamma_{13})}{s_{13}-(\Delta_k - l_k + 2m)} \qquad m = 0, 1, 2, ...
$$

Here, ∆*^k* and *l^k* are the scaling dimension and spin of an operator *O^k* present in the operator

 $\mathcal{O}_1\mathcal{O}_3 \sim C_{13k}\mathcal{O}_k$ *s*¹³ = ∆*^k* − *l^k* + 2*m, m* = 0*,* 1*,* 2*,... .* (5) meromorphic with simple poles at $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{2}$ $\mathcal{O}_1 \mathcal{O}_3 \sim C_{13k} \mathcal{O}_k$ $\gamma_{13} = \frac{1}{6} (s_{12} - s_{14})$ $\mathcal{O}_2\mathcal{O}_4 \sim C_{24k}\mathcal{O}_k$ 2 - 2 $\gamma_{13} =$ 1 2 $(s_{12} - s_{14})$

meromorphic with simple poles at

position of the four point function, Mack showed that *M*(*sij*) is crossing symmetric and

ⁱ=1 *kⁱ* = 0 and −*k*²

we find that the two particle state exchanged in the loop gives rise to poles of the Mellinnia to poles of the Example: Graviton exchange in AdS5 Example: Graviton exchange in AdS5

 λ is the graviton exchange between minimal mass scalars in AdS5 (λ ⁴ \vert This was computed in assiss states in terms of \vert Minimally coupled massless scalars

 $\sum_{i=1}^n$ Thoker, it eedinal, madium, madiusis, Nastelli (15) in terms of $\sum_{i=1}^n$ *^A*(*xi*) [∝] ⁹*D*4444(*xi*) [−] ⁴ <mark>1</mark>athus *^D*1414(*xi*) [−] ²⁰ [D'Hoker, Freedman, Mathur, Mathusis, Rastelli '99]

6γ²

¹³ + 2

55

dy d¯y e−2p·y−2¯p·¯y

decomposition of the tree–level diagram in figure 4.6. The corresponding Euclidean amplitude

$$
A(x_i) \propto 9D_{4444}(x_i) - \frac{4}{3x_{13}^6}D_{1414}(x_i) - \frac{20}{9x_{13}^4}D_{2424}(x_i) - \frac{23}{9x_{13}^2}D_{3434}(x_i) + \frac{16(x_{14}^2x_{23}^2 + x_{12}^2x_{34}^2)}{3x_{13}^6}D_{2525}(x_i) + \frac{64(x_{14}^2x_{23}^2 + x_{12}^2x_{34}^2)}{9x_{13}^4}D_{3535}(x_i) + \frac{8(x_{14}^2x_{23}^2 + x_{12}^2x_{34}^2 - x_{24}^2x_{13}^2)}{x_{13}^2}D_{4545}(x_i).
$$
 D-function

$$
M(s_{ij}) \propto \frac{6\gamma_{13}^2 + 2}{s_{13} - 2} + \frac{8\gamma_{13}^2}{s_{13} - 4} + \frac{\gamma_{13}^2 - 1}{s_{13} - 6} - \frac{15}{4}s_{13} + \frac{55}{2}
$$

looks quite cumbersome but the associated Mellin amplitude is a simple rational function,

*^s*¹³ [−] ⁴ ⁺ ^γ²

¹³ − 1

*^s*¹³ [−] ⁶ [−] ¹⁵

 $\overline{\mathcal{O}}(\mathcal{O})$ the other hand, from the term of order g 2 in (4.14), we have the term of order g 2 in (4.14), we have the term of $\mathcal{O}(\mathcal{O})$

8γ²

Double-trace operators

The double-trace operators $O_i \partial^n O_j$ (normal ordered product of external operators) do **not** give rise to poles in the Mellin amplitude.

Chapter 4. Eikonal Approximation in AdS/CFT All poles are associated with on-shell internal states.

 Γ Contact diagrams in AdS give polyn Contact diagrams in AdS give polynomial Mellin amplitudes

Figure 1: Witten diagram for a tree level *n*-point contact interaction in AdS.

X^Aⁱ

4.7 T–channel Decomposition It is important to test our main formula (9) beyond tree level diagrams. To this end, we Mellin amplitudes are specially nice in planar CFT's \mathbf{S} is the 1-loop diagram of \mathbf{S} and \mathbf{S} and \mathbf{S} in a subset in anglicular in anglicular in anglicular in anglicular in anglicular in a social in a socia (dual to tree level string theory in AdS). One can also describe this language. A tensor field in AdS can be the AdS can be the AdS can be the AdS can be

Flat Space Limit of AdS

Flat space limit of AdS \overline{f} \overline{f} of Ad 8γ² s¹³ + *4* **Flat Space IIMI**
Anti-de Sitter

¹³ − 1

momentum tensor. In this particular case, there is an extra simplification and the residues $\mathcal{L}(\mathcal{L})$

vanish for *m* ≥ 3. Furthermore, notice that the residues of the poles are quadratic polyno-

² *.* (8)

, sij % 1 *,* (9)

and *R* is the AdS radius. This formula assumes that all external particles become massless

2

55

$$
M(s_{ij}) \approx \frac{R^{n(1-d)/2+d+1}}{\Gamma(\frac{1}{2}\sum_{i}\Delta_{i}-\frac{d}{2})} \int_{0}^{\infty} d\beta \beta^{\frac{1}{2}\sum_{i}\Delta_{i}-\frac{d}{2}-1} e^{-\beta} T\left(S_{ij} = \frac{2\beta}{R^{2}} s_{ij}\right)
$$

Mellin amplitude
for $s_{ij} \gg 1$

55

1 $n - 1$ *n* ... *g* Figure 1: Witten diagram for a tree level *n*-point contact interaction in AdS. $\mathcal{O}_{\mathcal{A}}$ (a) $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{A}}$ in Ads $\mathcal{O}_{\mathcal{A}}$ in Ads can find in Ads can find in Addis for field in Addisor for fiel $M(s_{ij}) \approx gR^{n(1-d)/2+d+1-2N} \frac{1 \sqrt{2 \ln 4} \cdot 2^{i(1-d)} \cdot 2$ *T^A*1*...A^l* (*X*)=0 *.* (17) $T(S_{ij})=g$ $\overline{\mathsf{H}}$ *n* i $<$ j $\big($ S_{ij} 2 \bigwedge ^{α_{ij}} $g\nabla \ldots \nabla \phi_1 \nabla \ldots \nabla \phi_2 \ldots \nabla \ldots \nabla \phi_n$ dimension less Γ $\left(\frac{1}{2}\right)$ 2 $\sum_i \Delta_i - \frac{d}{2} + N$ Γ $\sqrt{\frac{1}{2}}$ 2 $\frac{i\,\Delta_i-\frac{a}{2} +N \,)}{\sum_i \Delta_i-\frac{d}{2})} \,\prod_{i < j}$ *n* i $\lt j$ $(s_{ij})^{\alpha_{ij}}$ dimensionless Evidence for *M* ≈ *...T* 1) Works for an infinite set of interactions α_{12} contractions # derivatives = $2 \sum \alpha_{ij} = 2N$ *n* $i < j$

Evidence for *M* ≈ *...T*

2) Agrees with previous results based on wave-packet constructions

[Gary, Giddings, JP '09] [Okuda, JP '10]

3) Works in several non-trivial examples

UV and IR divergences

UV divergences are the same in AdS and in flat space.

IR divergences are absent in AdS.

The Mellin amplitudes can be thought as IR regulated scattering amplitudes.

Application: from SYM to IIB strings

 $\mathcal{N}=4$ SYM $g_{\text{YM}}^2 = 4\pi g_s$ $g_{\rm YM}^2 N = \lambda = (R/\ell_s)^4$ type IIB strings

$$
O(x)
$$
 = Lagrangian density

 $\phi =$ Dilaton

 $\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle$ $R \to \infty$ 4pt function $2 \rightarrow 2$ scattering amplitude Mellin amplitude lim $\lambda \rightarrow \infty$ $\lambda^{\frac{3}{2}} M(g_{\text{YM}}^2, \lambda, s_{ij} = \sqrt{\lambda} \alpha_{ij}) = \frac{1}{120\pi}$ $120\pi^{3}\ell_{s}^{6}$!
!
! ∞ 0 $d\beta\beta^5e^{-\beta}T_{10} \left(g_s, \ell_s, S_{ij}\right) =$ 2β ℓ_s^2 $\alpha_{ij}^{\text{}}$

Testing the Conjecture

Any CFT that has a large-N expansion, and in which all single-trace operators of spin greater than two have parametrically large dimensions, has a local bulk dual.

Scalar Toy Model We will further assume a ^Z² symmetry ^O → −O, so that the operator ^O does not itself appear in the OOPE. The OPE. The Contract dimension of the United States (S, J), Toleriniski, Suny 07

Consider a "CFT" in which the only low dimension Single-trace operator is a scalar O of dimension Δ

OPE
$$
OO \sim \mathbb{I} + \mathbb{X} + \sum_{\mathbb{Z}_2 \text{ symmetry}} O_{n,l} + \dots
$$

\mathbb{Z}_2 symmetry double-trace set of primary double $O_{n,l}\equiv O$ $\overleftrightarrow{\partial}_{\mu_1}\ldots \overleftrightarrow{\partial}_{\mu_l} ($ \leftrightarrow ∂_ν \leftrightarrow $\overline{\partial}^{\nu})^n \mathcal{O} - \text{traces}$ Double-trace primary operators

such as to be traceless on the µ's. This has spin l and α +2n+l+O(1/N2). This has spin l and α is a α +2n+l+O(1/N2). This has spin l and α The contribution of higher-trace operators in the OPE is absent at the OPE is absent at the order in $1/2$ in 1/N2 in CFT \Rightarrow Four point Mellin amplitude is analytic

¹³Note that our focus is orthogonal to that in Ref. [21]. That paper is largely concerned with high-

\sim \sim Bulk quartic vertices generate all possible $\begin{array}{c}\n \sim \text{S}\n\end{array}$ two-point function and disconnected function are of \sim Bulk quartic vertices generate all possible $AdS \Rightarrow$ Duik qual tic vertices generated
polynomial Mellin amplitudes

∂ =

Inclusion of *Tµ*^ν

 $\textsf{First drop the } \mathbb{Z}_2 \textbf{ symmetry } \quad \mathcal{OO} \sim \mathbb{I} + \mathcal{O} + \sum \mathcal{O}_{n,l} + \ldots$ *n,l*

Gives rise to poles in the Mellin amplitude, whose positions are fixed by Δ and residues are fixed by $c_{\mathcal{O}\mathcal{O}\mathcal{O}}$

Any two solutions differ by an analytic Mellin amplitude already studied

Counting of solutions agrees with bulk expectations.

We can also consider other operators (like $T_{\mu\nu}$) in the OO OPE and the same reasoning applies.

Open Questions

- Generalize to external massive particles (work in progress) \rightarrow 3 pt-functions of SYM at strong coupling
- Mellin amplitudes for external operators with spin (helicity)
- Build n-pt functions by "gluing" 3pt functions of single-trace operators (analogous to BCFW) [Raju '10]
- Feynman rules for Mellin amplitudes?
- Unitarity for Mellin amplitudes? Renormalizable vs non-renormalizable bulk interactions [Fitzpatrick, Katz, Poland, Simmons-Duffin '10]
- Bootstrap for CFT in higher dimensions (d>2)
- Mellin amplitudes without conformal invariance?

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Thank you!

Sharp Locality in the 4pt-function

Conformal invariance gives

$$
\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{\mathcal{A}(z,\bar{z})}{x_{12}^{2\Delta}x_{34}^{2\Delta}}
$$

Cross ratios

$$
z\overline{z} = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \qquad (1-z)(1-\overline{z}) = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2}
$$

Sharp locality in the bulk implies a singularity in
$$
A
$$
 for $z = \overline{z}$.

\nOutput

\nDescription:

$$
\mathcal{A}(z,\bar{z}) \sim \frac{\mathcal{F}(\sigma)}{\rho^{4\Delta+2k-3}}
$$

$$
z = \sigma e^{\rho}
$$

$$
\bar{z} = \sigma e^{-\rho} \qquad \sigma = \sin^2 \frac{\theta}{2} = -\frac{t}{s}
$$

Flat Space S-matrix from AdS/CFT

The strength of the singularity is fixed by dimensional analysis

$$
\mathcal{A}(z,\bar{z}) \sim g^2 R^{3-d-2k} \frac{\mathcal{F}(\sigma)}{\rho^{4\Delta+2k-3}}
$$

$$
z = \sigma e^{\rho} \qquad \bar{z} = \sigma e^{-\rho}
$$

$$
x_1 \left(\frac{x_3}{x_4} \right)^{x_3}
$$

Example:
$$
\mathcal{L}_I = g^2 \phi^2 (\nabla^2)^k \phi^2
$$

Witten diagram in AdS_{d+1}

The bulk flat space S-matrix determines the residue of the singularity of the CFT 4pt-function.

$$
T(s,t) = g^2 s^k \frac{\mathcal{F}(\sigma)}{\sigma^{1-2\Delta-k}(1-\sigma)^{2\Delta-2+k}} \qquad \sigma = \sin^2 \frac{\theta}{2} = -\frac{t}{s}
$$

CFT Constraints

- Conformal invariance
- Operator product expansion (OPE)
- **Crossing**
- Unitarity
- Generalized modular invariance [El-Showk, Papadodimas '11]

Scalar Toy Model which there is a light scalar with self-interaction much self-interaction much stronger than gravity, working i **bealar loy IV**

Consider a "CFT" in which the only low dimension Single-trace operator is a scalar *O* of dimension \triangle

$$
\langle \mathcal{O}(0)\mathcal{O}(z,\bar{z})\mathcal{O}(1)\mathcal{O}(\infty)\rangle \equiv \mathcal{A}(z,\bar{z}) = \mathcal{A}(1-z,1-\bar{z})
$$

OPE
$$
OO \sim \mathbb{I} + \mathbb{X} + \sum_{\mathbb{Z}_2 \text{ symmetry}} O_{n,l} + \dots
$$

Double-trace primary operators

$$
\mathcal{O}_{n,l} \equiv \mathcal{O}\overleftrightarrow{\partial}_{\mu_1} \dots \overleftrightarrow{\partial}_{\mu_l} (\overleftrightarrow{\partial}_{\nu} \overleftrightarrow{\partial}^{\nu})^n \mathcal{O} - \text{traces}
$$

Conformal partial wave expansion 2→ 2→2

$$
(z\overline{z})^{\Delta} \mathcal{A}(z,\overline{z}) = 1 + \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} p(n,l) g_{\Delta(n,l),l}(z,\overline{z})
$$

↔

∂ =

Conformal Partial Waves

$$
\left\langle \frac{1}{(E,l)}\right\langle \left\langle g_{E,l}(z,\bar{z})\right\rangle
$$

Only SO(1,3) Not Virasoro

$$
d = 2
$$

$$
g_{E,l}(z,\overline{z}) = \frac{(z\overline{z})^{E/2}}{1+\delta_{l,0}} \left[\left(\frac{z}{\overline{z}}\right)^{l/2} F_{E+l}(z) F_{E-l}(\overline{z}) + (z \leftrightarrow \overline{z}) \right]
$$

$$
d=4
$$

$$
g_{E,l}(z,\overline{z}) = \frac{(z\overline{z})^{1+E/2}}{z-\overline{z}} \left[\left(\frac{z}{\overline{z}}\right)^{l/2} F_{E+l}(z) F_{E-l-2}(\overline{z}) - (z \leftrightarrow \overline{z}) \right]
$$

where
$$
F_a(z) = {}_2F_1\left(\frac{a}{2}, \frac{a}{2}, a, z\right)
$$

Explicit expressions in even dimension [Dolan, Osborn 01]

1/N Expansion

$$
(z\overline{z})^{\Delta} \mathcal{A}(z,\overline{z}) = 1 + \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} p(n,l) g_{\Delta(n,l),l}(z,\overline{z}) = (z\overline{z})^{\Delta} \mathcal{A}(1-z,1-\overline{z})
$$

Solve in the 1/N expansion

$$
\mathcal{A}(z, \bar{z}) = \mathcal{A}_0(z, \bar{z}) + N^{-2}\mathcal{A}_1(z, \bar{z}) + \dots
$$

\n
$$
p(n, l) = p_0(n, l) + N^{-2}p_1(n, l) + \dots
$$

\n
$$
\Delta(n, l) = 2\Delta + 2n + l + N^{-2}\gamma(n, l) + \dots
$$

\n<sub>anomalous
\ndimensions</sub>

$$
\mathcal{A}_1(z,\bar{z}) = \sum_{n,l} \left[p_1(n,l) + p_0(n,l)\gamma(n,l) \frac{1}{2} \frac{\partial}{\partial n} \right] \frac{g_{2\Delta + 2n + l, l}(z,\bar{z})}{(z\bar{z})^{\Delta}} = \mathcal{A}_1(1-z, 1-\bar{z})
$$
unknowns

Counting bulk interactions Any bulk quartic interaction gives a solution to crossing ϕ^4 , $\phi^2 (\nabla_\mu \nabla_\nu \phi)^2$, $\phi^2 (\nabla_\mu \nabla_\nu \nabla_\sigma \phi)^2$, ... 0 1 2 3 4 5 6 7 8 9 10 11 12 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 *k l* Spin (# of derivatives)/2 $(L+2)(L+4)$ 8 Interactions $|\mathsf{Spin}\ \, l\leq L|$

Counting solutions to crossing

$$
\mathcal{A}_1(z,\bar{z}) = \sum_{l \leq L,n} \left[p_1(n,l) + p_0(n,l)\gamma(n,l) \frac{1}{2} \frac{\partial}{\partial n} \right] \frac{g_{2\Delta + 2n + l, l}(z,\bar{z})}{(z\bar{z})^{\Delta}} = \mathcal{A}_1(1-z, 1-\bar{z})
$$

Eliminate $p_1(n, l)$ by expanding around $z = 0$ and $\bar{z} = 1$ and considering the terms with $\log z \log(1 - \bar{z})$. Project onto a complete set to get recursion relation

There are more equations than unknowns...

Counting solutions to crossing

Crossing also determines

$$
p_1(n,l) = \frac{1}{2} \frac{\partial}{\partial n} \Big[p_0(n,l) \gamma(n,l) \Big]
$$

S-matrix Theory in AdS

[Giddings 99]

The boundary correlators for any weakly coupled unitary quantum field theory on AdS with a "small" number of fields should be produced by a local Lagrangian.

$$
\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \int d^2\delta\, M(\delta_{ij}) \prod_{i
$$

 $M(\delta_{ij})$ has very similar analytic properties to a flat space S-matrix.

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