Writing CFT correlation functions as AdS scattering amplitudes

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JP, arXiv:1011.1485
Okuda, JP, arXiv:1002.2641
Heemskerk, JP, Polchinski, Sully, arXiv:0907.0151
Gary, Giddings, JP, arXiv:0903.4437

Rutgers, March 22nd, 2011

Introduction

The AdS description of a CFT is useful if it is

- **Weakly coupled** $\longleftrightarrow$ "large N" factorization

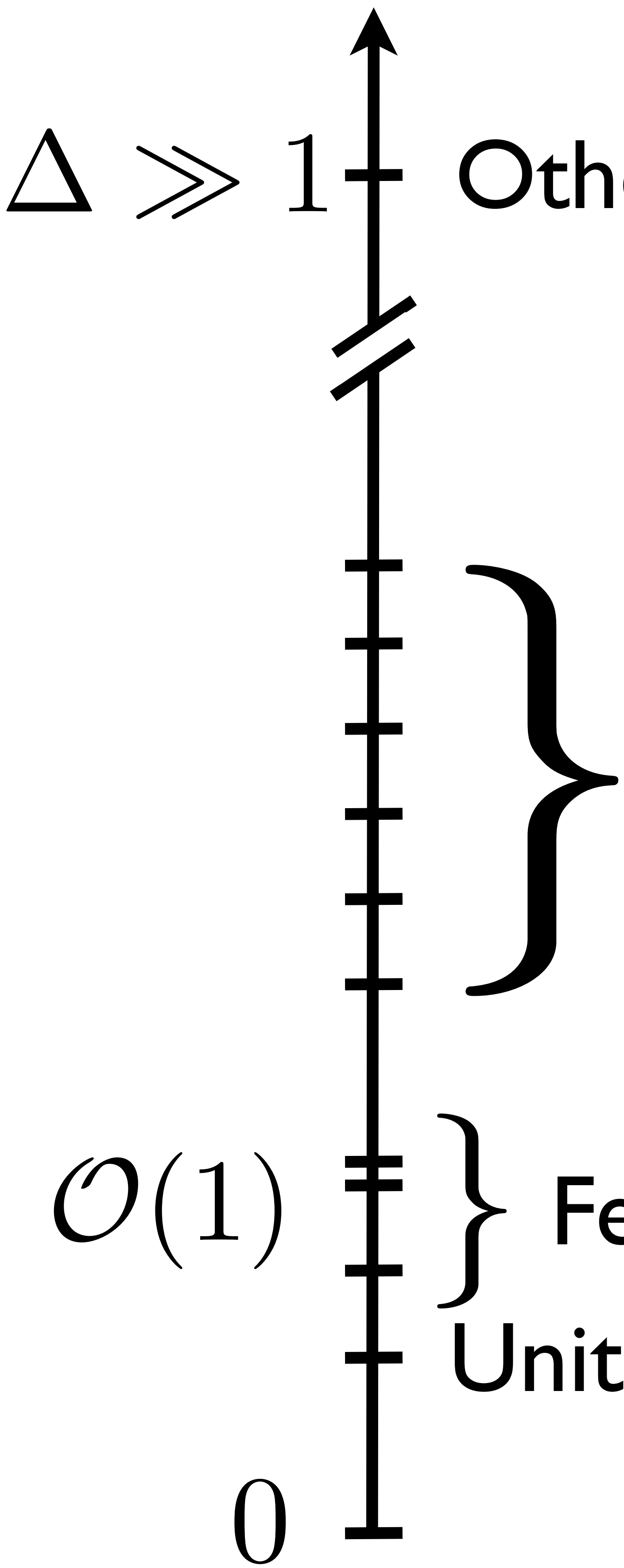
$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\text{connected}} \sim \kappa^{n-2} \, , \qquad \kappa^2 \sim G R^{1-d} \ll 1$$

- **Local** - effective field theory in AdS with small number of fields valid up to some UV cutoff ℓ much smaller than the AdS radius R

$$\ell \sim \frac{1}{\text{mass}} \sim \frac{R}{\Delta}$$

$\longrightarrow$ Large gap in spectrum of dimensions $\Delta \gg 1$

Spectrum of Effective CFT



Other single-trace

AdS coupling constant

Multi-trace primaries

$$\dim\left(\mathcal{O}_1 \partial^n \mathcal{O}_2\right) = \Delta_1 + \Delta_2 + n + \mathcal{O}\left(1/N^2\right)$$

Few single-trace primaries   spin ≤ 2

Unitary bound

[Heemskerk, JP, Polchinski, Sully ‘09]
[Fitzpatrick, Katz, Poland, Simmons-Duffin ‘10]
[El-Showk, Papadodimas ’11]

A Conjecture

[Heemskerk, JP, Polchinski, Sully '09]

Any CFT that has a large-N expansion, and in which all single-trace operators of spin greater than two have parametrically large dimensions, has a local bulk dual.

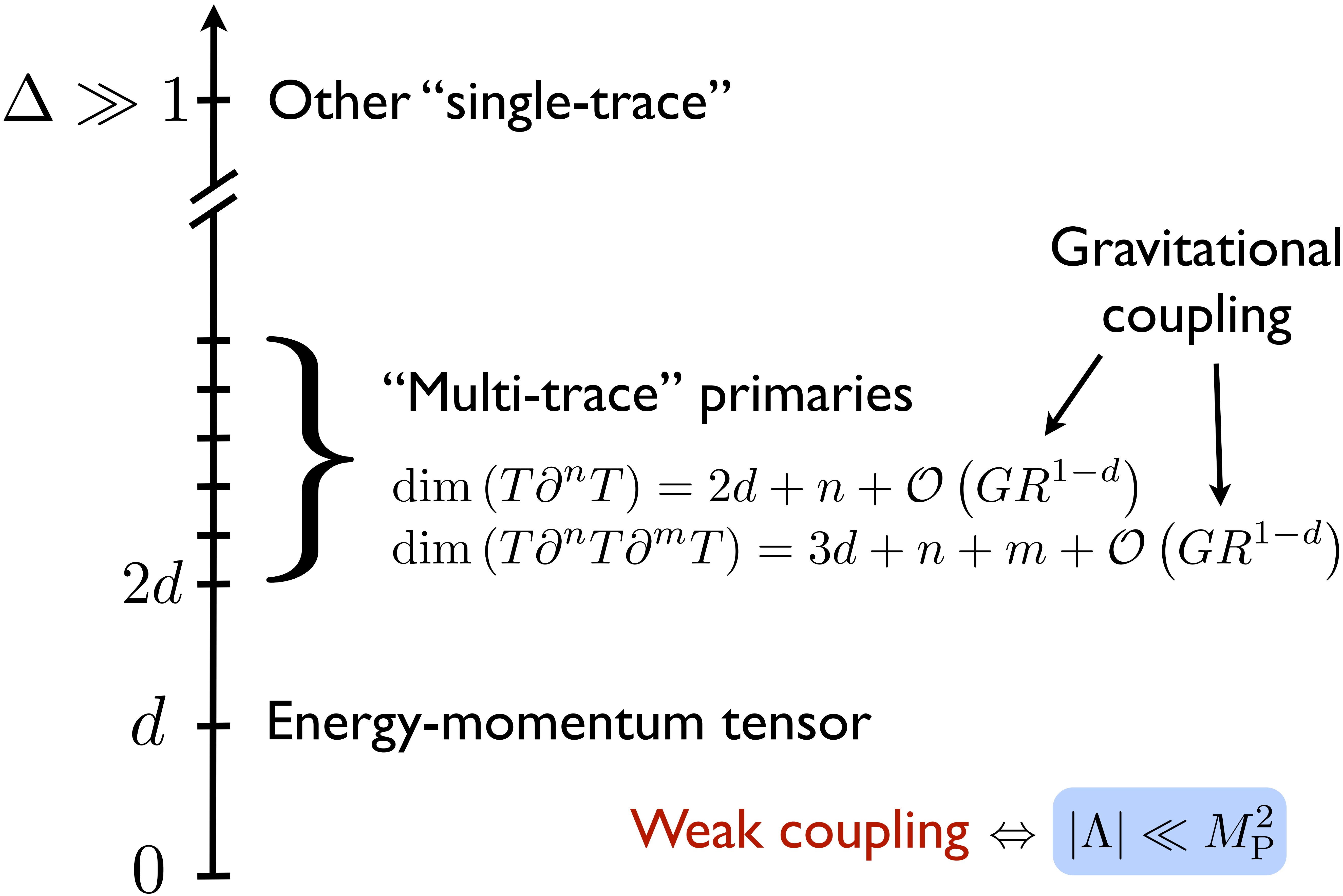
Large-N expansion $\longleftrightarrow$ Weakly coupled bulk dual $\frac{l_P}{R} \ll 1$

Single-trace operator $\longleftrightarrow$ Single-particle state

Large-N vector models have weakly coupled non-local bulk duals (AdS higher spin theories).

[Sezgin, Sundell 02]
[Klebanov, Polyakov 02]
[Fradkin, Vasiliev 87, ...]

Is there a CFT dual of GR?



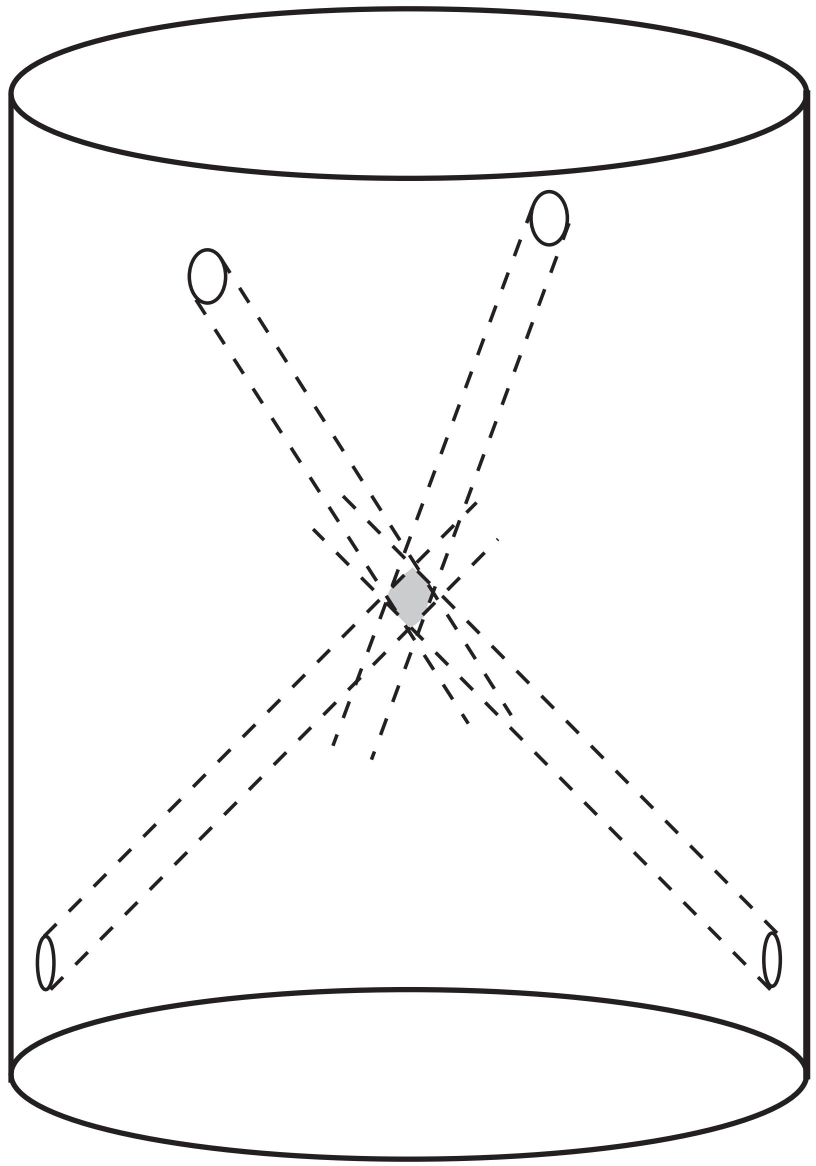
Bulk Locality

Main difficulty: local bulk physics is encoded in CFT correlation functions in a non-trivial way.

How to extract the bulk S-matrix?

Idea: prepare wave-packets that scatter in small region of AdS

[Polchinski 99]
[Susskind 99]
[Gary, Giddings, JP '09]
[Okuda, JP '10]



Best language: Mellin amplitudes

Outline

- Introduction
- Mellin amplitudes
- Flat space limit of AdS
- Testing the conjecture
- Open questions

Mellin Amplitudes

Mellin Amplitudes

[Mack '09]

Correlation function of scalar primary operators

$$A(x_i) = \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$$

$$A(x_i) = \mathcal{N} \int_{-i\infty}^{i\infty} [d\delta] \boxed{M(\delta_{ij})} \prod_{i<j}^{n} \Gamma(\delta_{ij}) \left(x_{ij}^2\right)^{-\delta_{ij}}$$

Constraint $\displaystyle \sum_{j \neq i}^{n} \delta_{ij} = \Delta_i = \dim[\mathcal{O}_i]$

integration variables = # ind. cross-ratios = $\dfrac{n(n-3)}{2}$

Analogy with Scattering Amplitudes

Introduce k_i such that $-k_i^2 = \Delta_i$ and $\sum_{i=1}^n k_i = 0$

then $\delta_{ij} = k_i \cdot k_j$ automatically solves the constraints.

Define $s_{ij} = -(k_i + k_j)^2 = \Delta_i + \Delta_j - 2\delta_{ij}$ [Mack '09]

The Mellin amplitude is crossing symmetric and meromorphic with simple poles at ($n = 4$)

$$M(s_{ij}) \approx \frac{C_{13k} C_{24k} P_{l_k}^m(\gamma_{13})}{s_{13} - (\Delta_k - l_k + 2m)} \qquad m = 0, 1, 2, \ldots$$

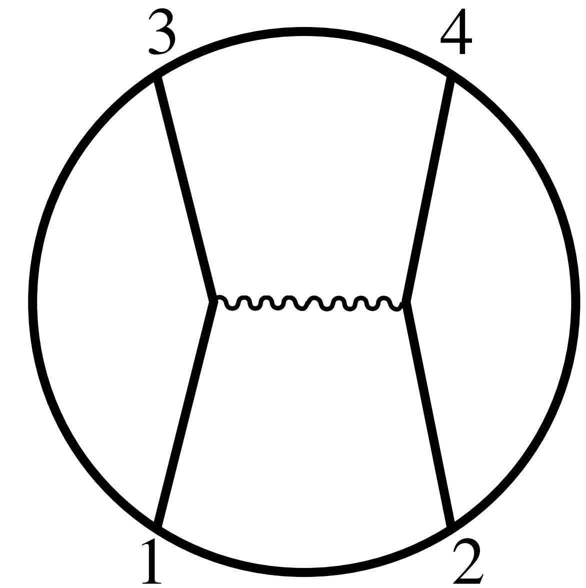
$$\mathcal{O}_1 \mathcal{O}_3 \sim C_{13k} \mathcal{O}_k$$
$$\mathcal{O}_2 \mathcal{O}_4 \sim C_{24k} \mathcal{O}_k$$

$$\gamma_{13} = \frac{1}{2}(s_{12} - s_{14})$$

Example: Graviton exchange in AdS5

Minimally coupled massless scalars

$$\Delta_i = d = 4$$



[D'Hoker, Freedman, Mathur, Mathusis, Rastelli '99]

$$A(x_i) \propto 9D_{4444}(x_i) - \frac{4}{3x_{13}^6} D_{1414}(x_i) - \frac{20}{9x_{13}^4} D_{2424}(x_i) - \frac{23}{9x_{13}^2} D_{3434}(x_i)$$
$$+ \frac{16(x_{14}^2 x_{23}^2 + x_{12}^2 x_{34}^2)}{3x_{13}^6} D_{2525}(x_i) + \frac{64(x_{14}^2 x_{23}^2 + x_{12}^2 x_{34}^2)}{9x_{13}^4} D_{3535}(x_i)$$
$$+ \frac{8(x_{14}^2 x_{23}^2 + x_{12}^2 x_{34}^2 - x_{24}^2 x_{13}^2)}{x_{13}^2} D_{4545}(x_i) \; .$$

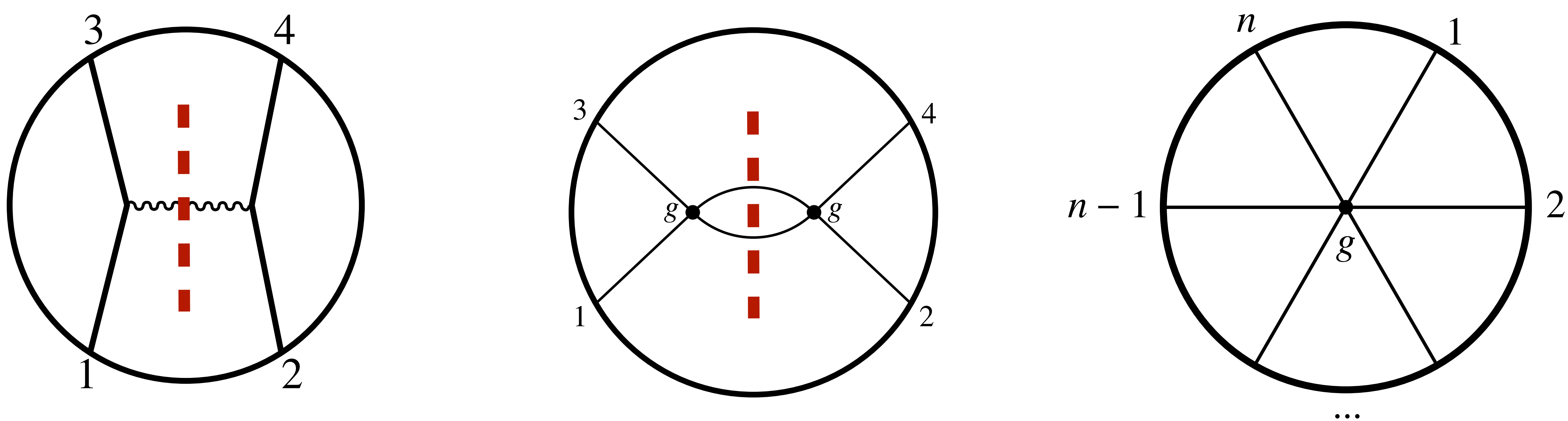
D-function

$$M(s_{ij}) \propto \frac{6\gamma_{13}^2 + 2}{s_{13} - 2} + \frac{8\gamma_{13}^2}{s_{13} - 4} + \frac{\gamma_{13}^2 - 1}{s_{13} - 6} - \frac{15}{4} s_{13} + \frac{55}{2}$$

Double-trace operators

The double-trace operators $\mathcal{O}_i \partial^n \mathcal{O}_j$ (normal ordered product of external operators) do **not** give rise to poles in the Mellin amplitude.

All poles are associated with on-shell internal states.



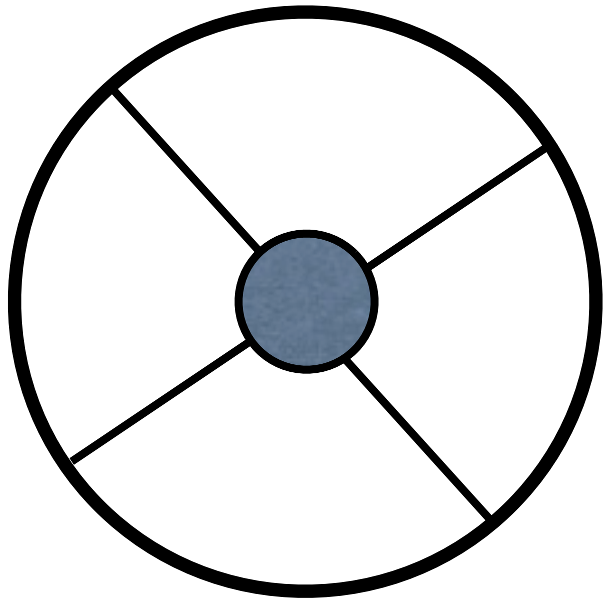
Contact diagrams in AdS give polynomial Mellin amplitudes

Mellin amplitudes are specially nice in planar CFT's (dual to tree level string theory in AdS).

Flat Space Limit of AdS

Flat space limit of AdS

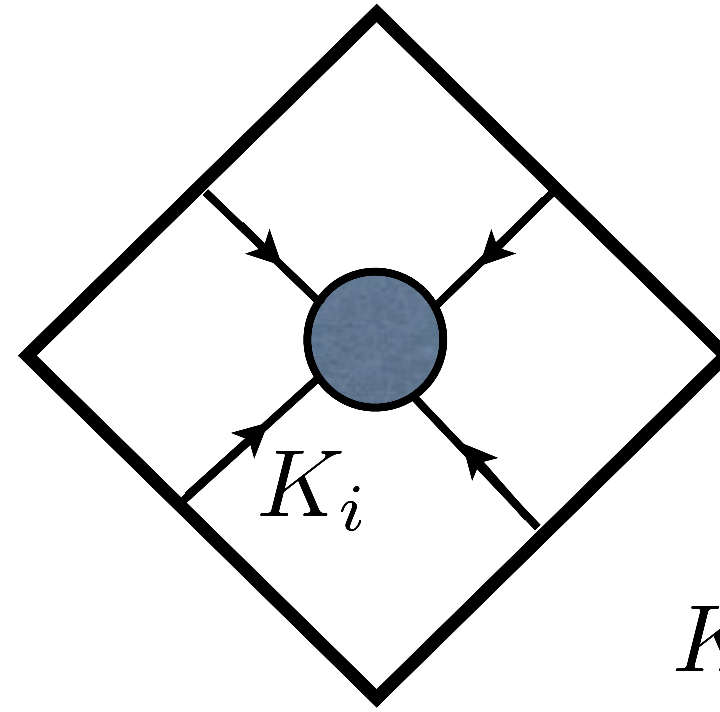
Anti-de Sitter



$$M_i^2 = \frac{\Delta_i(\Delta_i - d)}{R^2}$$

$$R \to \infty$$

Minkowski



$$K_i^2 = 0$$

$$S_{ij} = -(K_i + K_j)^2$$

$$M(s_{ij}) \approx \frac{R^{n(1-d)/2+d+1}}{\Gamma\left(\frac{1}{2}\sum_i \Delta_i - \frac{d}{2}\right)} \int_0^\infty d\beta\, \beta^{\frac{1}{2}\sum_i \Delta_i - \frac{d}{2} - 1} e^{-\beta}\, T\left(S_{ij} = \frac{2\beta}{R^2} s_{ij}\right)$$

Mellin amplitude for $s_{ij} \gg 1$

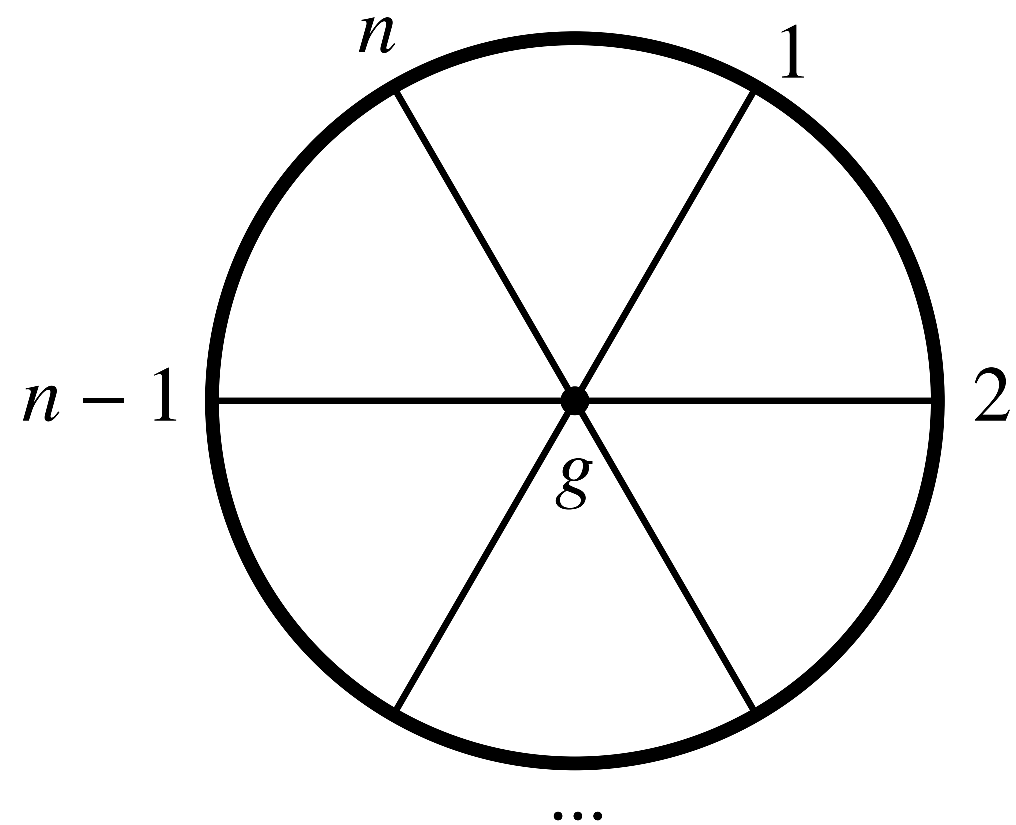
Scattering amplitude

Evidence for $M \approx \int \ldots T$

1) Works for an infinite set of interactions

$$g\nabla \ldots \nabla\phi_1\nabla \ldots \nabla\phi_2 \ldots \nabla \ldots \nabla\phi_n$$

α_{12} contractions



$$\#\text{ derivatives} = 2\sum_{i<j}^{n}\alpha_{ij} = 2N$$

$$T(S_{ij}) = g\prod_{i<j}^{n}\left(\frac{S_{ij}}{2}\right)^{\alpha_{ij}}$$

$$M(s_{ij}) \approx \underbrace{gR^{n(1-d)/2+d+1-2N}}_{\text{dimensionless}}\frac{\Gamma\left(\frac{1}{2}\sum_i\Delta_i - \frac{d}{2} + N\right)}{\Gamma\left(\frac{1}{2}\sum_i\Delta_i - \frac{d}{2}\right)}\prod_{i<j}^{n}(s_{ij})^{\alpha_{ij}}$$

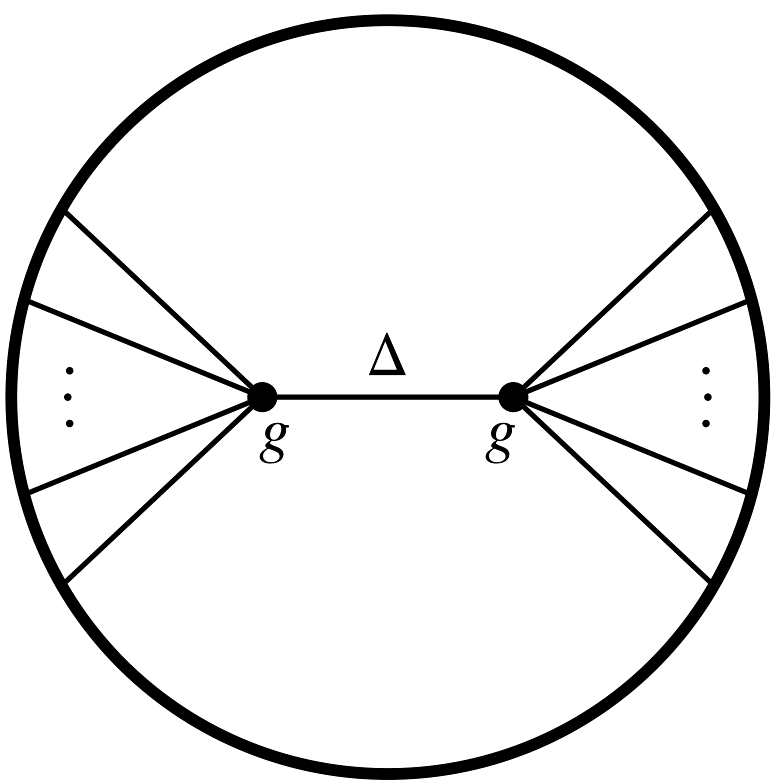
Evidence for $M \approx \int \ldots T$

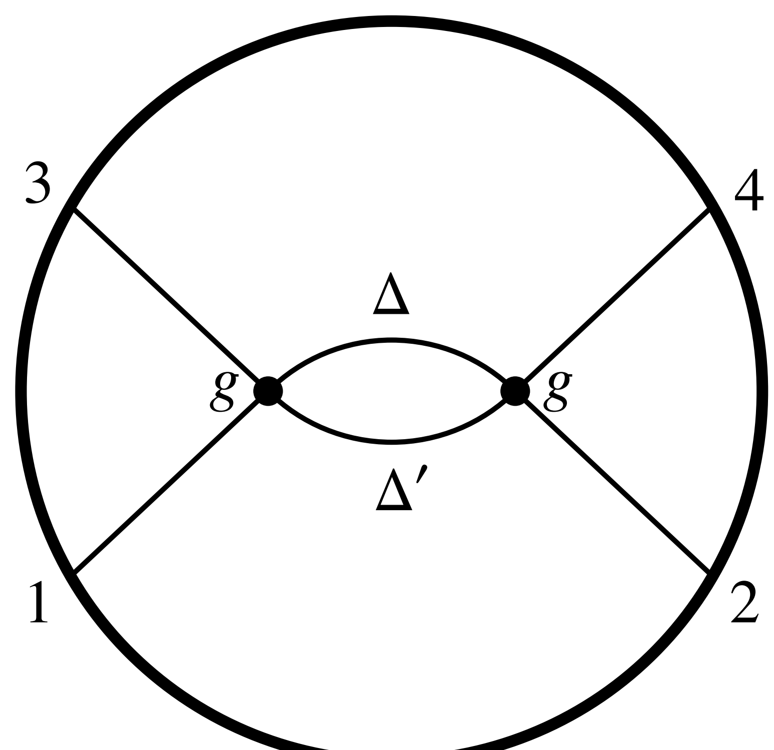
2) Agrees with previous results based on wave-packet constructions

[Gary, Giddings, JP '09]

[Okuda, JP '10]

3) Works in several non-trivial examples



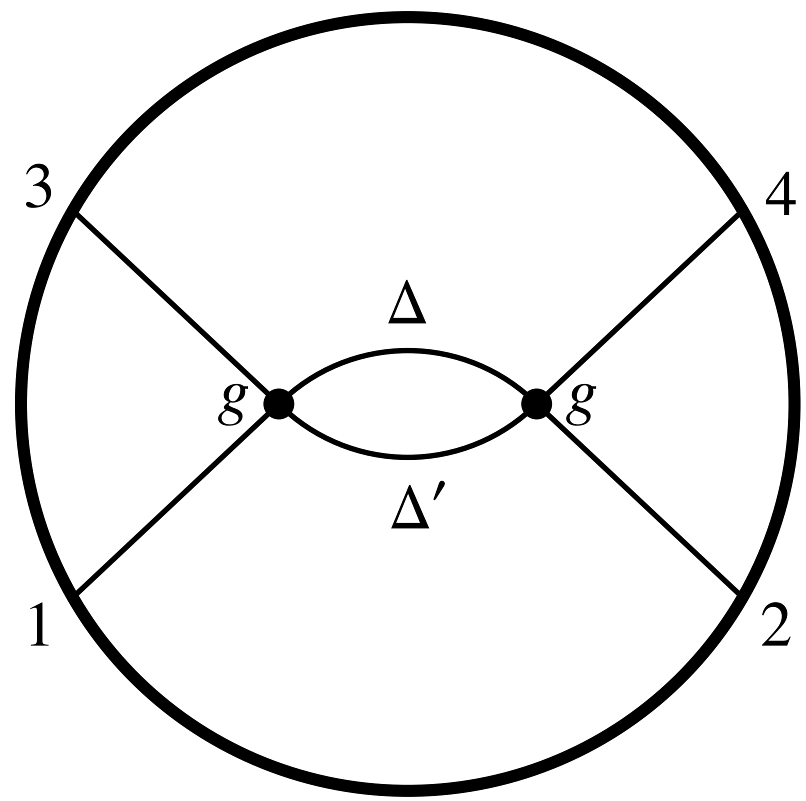


UV and IR divergences

UV divergences are the same in AdS and in flat space.

IR divergences are absent in AdS.

The Mellin amplitudes can be thought as IR regulated scattering amplitudes.



Application: from SYM to IIB strings

$\mathcal{N} = 4$ SYM $\longleftrightarrow$ type IIB strings

$$g_{\rm YM}^2 = 4\pi g_s$$

$$g_{\rm YM}^2 N = \lambda = (R/\ell_s)^4$$

$\mathcal{O}(x) =$ Lagrangian density

$\phi =$ Dilaton

4pt function
$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$

$\xrightarrow{\;R \to \infty\;}$

2 → 2 scattering amplitude

$$\lim_{\lambda\to\infty} \lambda^{\frac{3}{2}} M(g_{\rm YM}^2, \lambda, s_{ij} = \sqrt{\lambda}\alpha_{ij}) = \frac{1}{120\pi^3 \ell_s^6} \int_0^\infty d\beta \beta^5 e^{-\beta} T_{10}\left(g_s, \ell_s, S_{ij} = \frac{2\beta}{\ell_s^2}\alpha_{ij}\right)$$

Mellin amplitude

Testing the Conjecture

Any CFT that has a large-N expansion, and in which all single-trace operators of spin greater than two have parametrically large dimensions, has a local bulk dual.

Scalar Toy Model

[Heemskerk, JP, Polchinski, Sully '09]

Consider a "CFT" in which the only low dimension single-trace operator is a scalar $\mathcal{O}$ of dimension Δ

OPE $$\mathcal{O}\mathcal{O} \sim \mathbb{I} + \not{\mathcal{O}} + \sum_{n,l} \mathcal{O}_{n,l} + \ldots$$

$\mathbb{Z}_2$ symmetry

Double-trace primary operators

$$\mathcal{O}_{n,l} \equiv \mathcal{O} \overleftrightarrow{\partial}_{\mu_1} \ldots \overleftrightarrow{\partial}_{\mu_l} (\overleftrightarrow{\partial}_\nu \overleftrightarrow{\partial}^\nu)^n \mathcal{O} - \text{traces}$$

CFT $\Rightarrow$ Four point Mellin amplitude is analytic

AdS $\Rightarrow$ Bulk quartic vertices generate all possible polynomial Mellin amplitudes

Inclusion of $T_{\mu\nu}$

First drop the $\mathbb{Z}_2$ symmetry $\quad \mathcal{O}\mathcal{O} \sim \mathbb{I} + \mathcal{O} + \sum_{n,l} \mathcal{O}_{n,l} + \ldots$

Gives rise to poles in the Mellin amplitude, whose positions are fixed by Δ and residues are fixed by $c_{\mathcal{O}\mathcal{O}\mathcal{O}}$

Any two solutions differ by an analytic Mellin amplitude already studied → **Only one new parameter: $c^2_{\mathcal{O}\mathcal{O}\mathcal{O}}$**

Counting of solutions agrees with bulk expectations.

We can also consider other operators (like $T_{\mu\nu}$) in the $\mathcal{O}\mathcal{O}$ OPE and the same reasoning applies.

Open Questions

- Generalize to external massive particles (work in progress)
  ⟶ 3pt-functions of SYM at strong coupling

- Mellin amplitudes for external operators with spin (helicity)

- Build n-pt functions by “gluing” 3pt functions of single-trace operators (analogous to BCFW) [Raju ‘10]

- Feynman rules for Mellin amplitudes?

- Unitarity for Mellin amplitudes? [Fitzpatrick, Katz, Poland, Simmons-Duffin ‘10]
  Renormalizable vs non-renormalizable bulk interactions

- Bootstrap for CFT in higher dimensions (d>2)

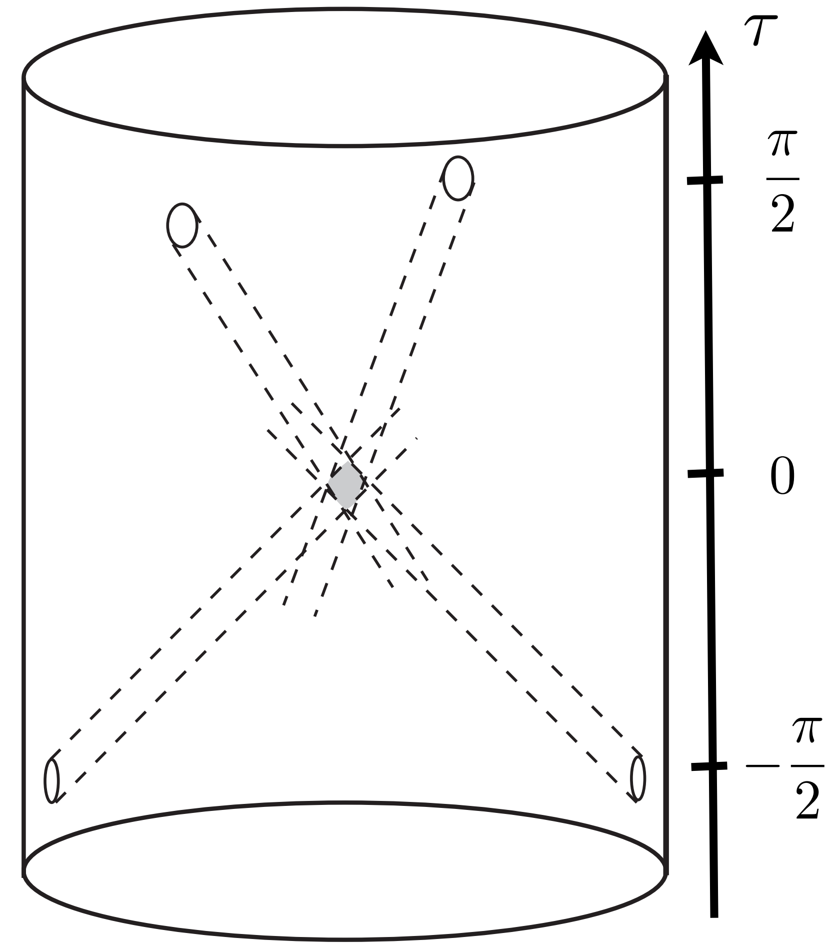
- Mellin amplitudes without conformal invariance?

Thank you!

Sharp Locality in the 4pt-function

Conformal invariance gives

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{\mathcal{A}(z,\bar{z})}{x_{12}^{2\Delta}x_{34}^{2\Delta}}$$

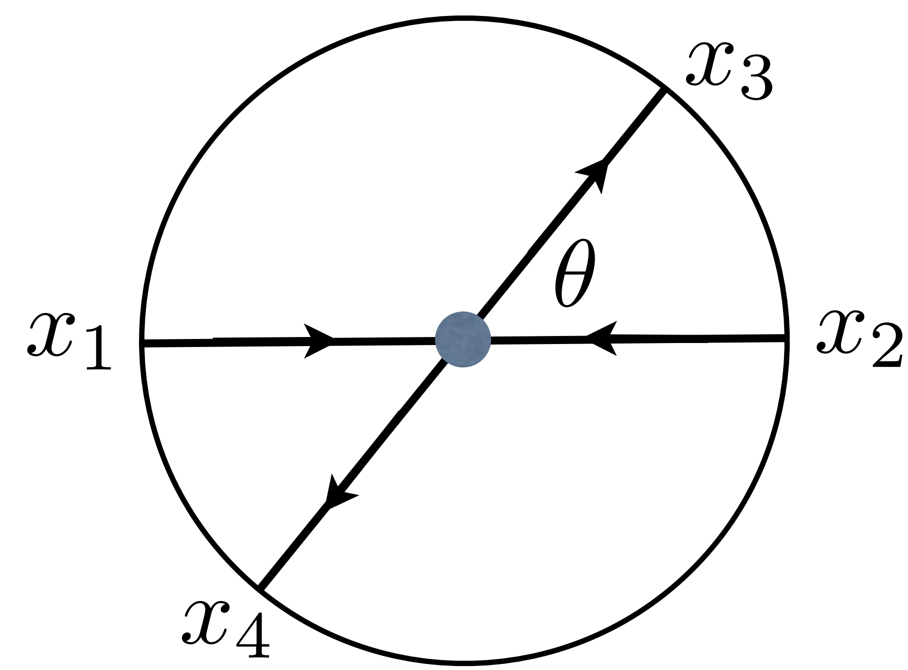


Cross ratios

$$z\bar{z} = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2}$$

Sharp locality in the bulk implies a singularity in $\mathcal{A}$ for $z = \bar{z}$.

$$\mathcal{A}(z,\bar{z}) \sim \frac{\mathcal{F}(\sigma)}{\rho^{4\Delta+2k-3}}$$



$$z = \sigma e^{\rho}$$
$$\bar{z} = \sigma e^{-\rho}$$

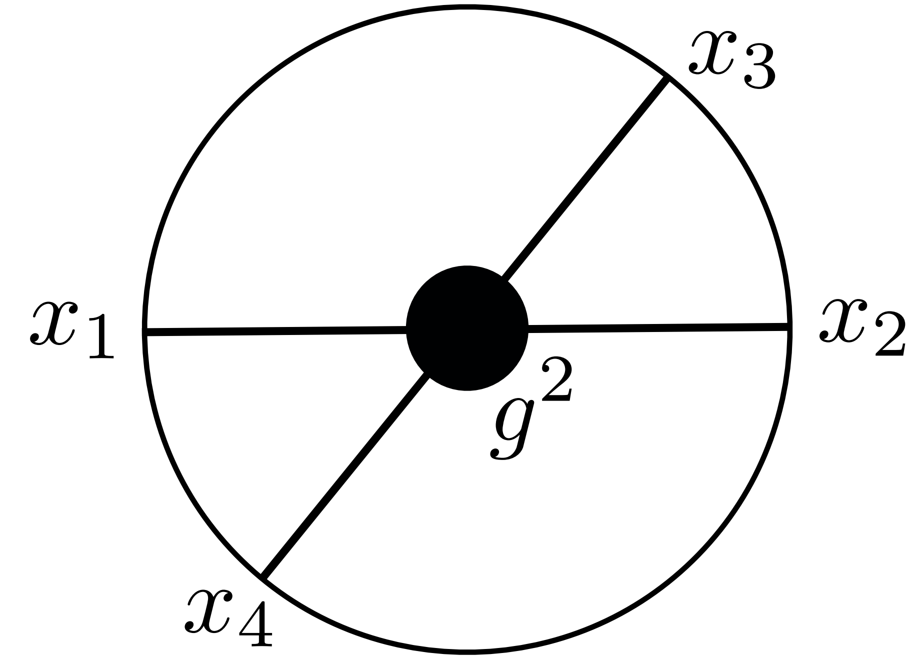
$$\sigma = \sin^2\frac{\theta}{2} = -\frac{t}{s}$$

Flat Space S-matrix from AdS/CFT

The strength of the singularity is fixed by dimensional analysis

$$\mathcal{A}(z,\bar{z}) \sim g^2 R^{3-d-2k} \frac{\mathcal{F}(\sigma)}{\rho^{4\Delta+2k-3}}$$

$z = \sigma e^{\rho}$ $\qquad \bar{z} = \sigma e^{-\rho}$



Witten diagram in AdS_{d+1}

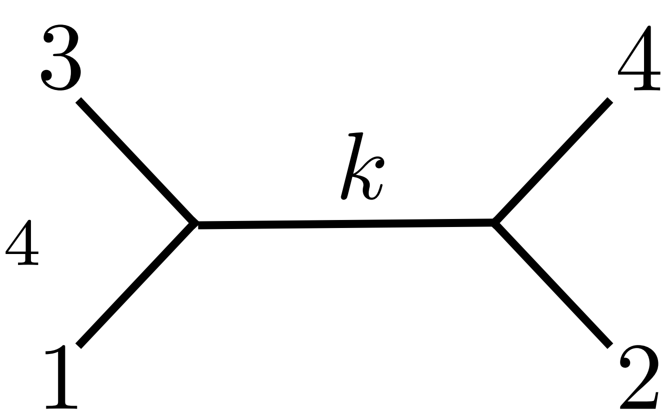
Example: $\mathcal{L}_I = g^2 \, \phi^2 (\nabla^2)^k \phi^2$

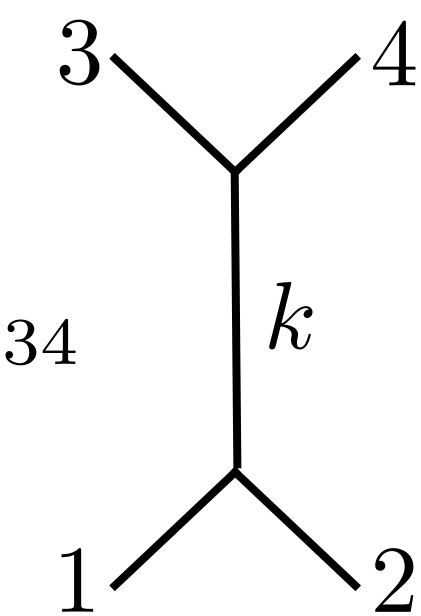
The bulk flat space S-matrix determines the residue of the singularity of the CFT 4pt-function.

$$T(s,t) = g^2 s^k \frac{\mathcal{F}(\sigma)}{\sigma^{1-2\Delta-k}(1-\sigma)^{2\Delta-2+k}} \qquad \sigma = \sin^2\frac{\theta}{2} = -\frac{t}{s}$$

CFT Constraints

- Conformal invariance
- Operator product expansion (OPE)
- Crossing
- Unitarity
- Generalized modular invariance [El-Showk, Papadodimas '11]

$$\sum_k c_{13}^{k} c_{k24}$$ 

$$= \sum_k c_{12}^{k} c_{k34}$$ 

Scalar Toy Model

Consider a "CFT" in which the only low dimension single-trace operator is a scalar $\mathcal{O}$ of dimension Δ

$$\langle \mathcal{O}(0)\mathcal{O}(z,\bar{z})\mathcal{O}(1)\mathcal{O}(\infty)\rangle \equiv \mathcal{A}(z,\bar{z}) = \mathcal{A}(1-z,1-\bar{z})$$

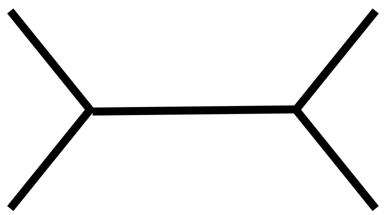
OPE
$$\mathcal{O}\mathcal{O} \sim \mathbb{I} + \cancel{\mathcal{O}} + \sum_{n,l} \mathcal{O}_{n,l} + \dots$$

$\mathbb{Z}_2$ symmetry

Double-trace primary operators

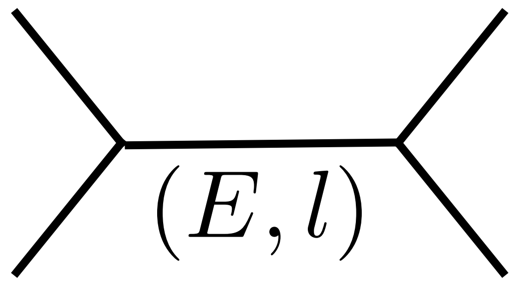
$$\mathcal{O}_{n,l} \equiv \mathcal{O}\overleftrightarrow{\partial}_{\mu_1}\dots\overleftrightarrow{\partial}_{\mu_l}(\overleftrightarrow{\partial}_\nu\overleftrightarrow{\partial}^\nu)^n\mathcal{O} - \text{traces}$$

Conformal partial wave expansion



$$(z\bar{z})^\Delta \mathcal{A}(z,\bar{z}) = 1 + \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} p(n,l)\, g_{\Delta(n,l),l}(z,\bar{z})$$

Conformal Partial Waves



$$= g_{E,l}(z,\bar{z})$$

Only SO(1,3)
Not Virasoro

$d = 2$

$$g_{E,l}(z,\bar{z}) = \frac{(z\bar{z})^{E/2}}{1+\delta_{l,0}} \left[\left(\frac{z}{\bar{z}}\right)^{l/2} F_{E+l}(z)\, F_{E-l}(\bar{z}) + (z \leftrightarrow \bar{z}) \right]$$

$d = 4$

$$g_{E,l}(z,\bar{z}) = \frac{(z\bar{z})^{1+E/2}}{z-\bar{z}} \left[\left(\frac{z}{\bar{z}}\right)^{l/2} F_{E+l}(z)\, F_{E-l-2}(\bar{z}) - (z \leftrightarrow \bar{z}) \right]$$

where $F_a(z) = {}_2F_1\left(\frac{a}{2}, \frac{a}{2}, a, z\right)$

Explicit expressions in even dimension [Dolan, Osborn 01]

1/N Expansion

$$(z\bar{z})^{\Delta}\mathcal{A}(z,\bar{z}) = 1 + \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} p(n,l)\, g_{\Delta(n,l),l}(z,\bar{z}) = (z\bar{z})^{\Delta}\mathcal{A}(1-z,1-\bar{z})$$

Solve in the 1/N expansion

$$\mathcal{A}(z,\bar{z}) = \mathcal{A}_0(z,\bar{z}) + N^{-2}\mathcal{A}_1(z,\bar{z}) + \dots$$

$$p(n,l) = p_0(n,l) + N^{-2}p_1(n,l) + \dots$$

$$\Delta(n,l) = 2\Delta + 2n + l + N^{-2}\gamma(n,l) + \dots$$

anomalous dimensions

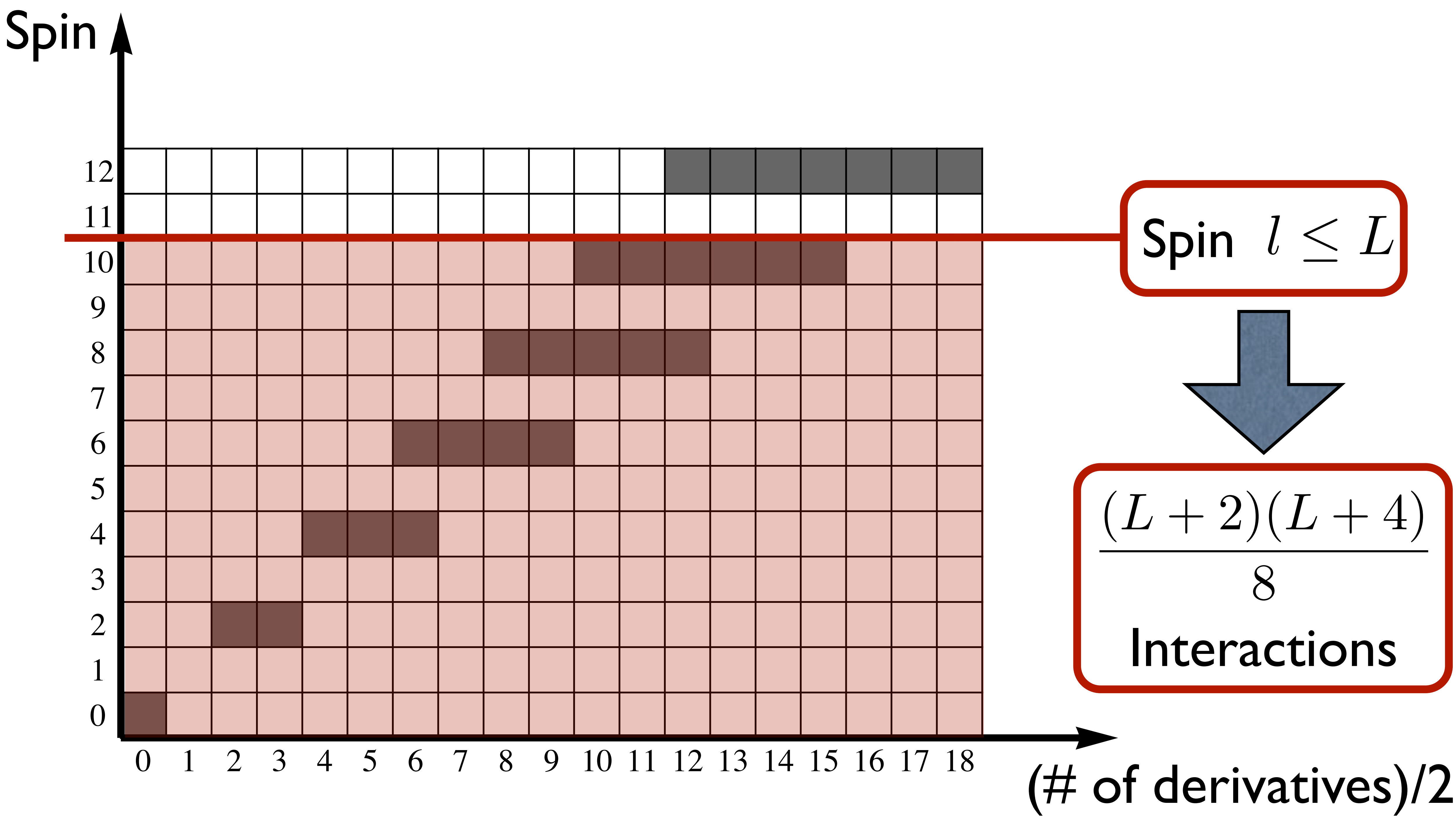
$$\mathcal{A}_1(z,\bar{z}) = \sum_{n,l}\left[p_1(n,l) + p_0(n,l)\gamma(n,l)\,\frac{1}{2}\frac{\partial}{\partial n}\right]\frac{g_{2\Delta+2n+l,l}(z,\bar{z})}{(z\bar{z})^{\Delta}} = \mathcal{A}_1(1-z,1-\bar{z})$$

unknowns

Counting bulk interactions

Any bulk quartic interaction gives a solution to crossing

$$\phi^4\ ,\ \phi^2(\nabla_\mu\nabla_\nu\phi)^2\ ,\ \phi^2(\nabla_\mu\nabla_\nu\nabla_\sigma\phi)^2\ ,\ \ldots$$



Counting solutions to crossing

$$\mathcal{A}_1(z,\bar{z}) = \sum_{l\leq L,\, n} \left[p_1(n,l) + p_0(n,l)\gamma(n,l)\,\frac{1}{2}\frac{\partial}{\partial n}\right] \frac{g_{2\Delta+2n+l,l}(z,\bar{z})}{(z\bar{z})^{\Delta}} = \mathcal{A}_1(1-z,1-\bar{z})$$

Eliminate $p_1(n,l)$ by expanding around $z=0$ and $\bar{z}=1$ and considering the terms with $\log z \log(1-\bar{z})$.

Project onto a complete set to get recursion relation

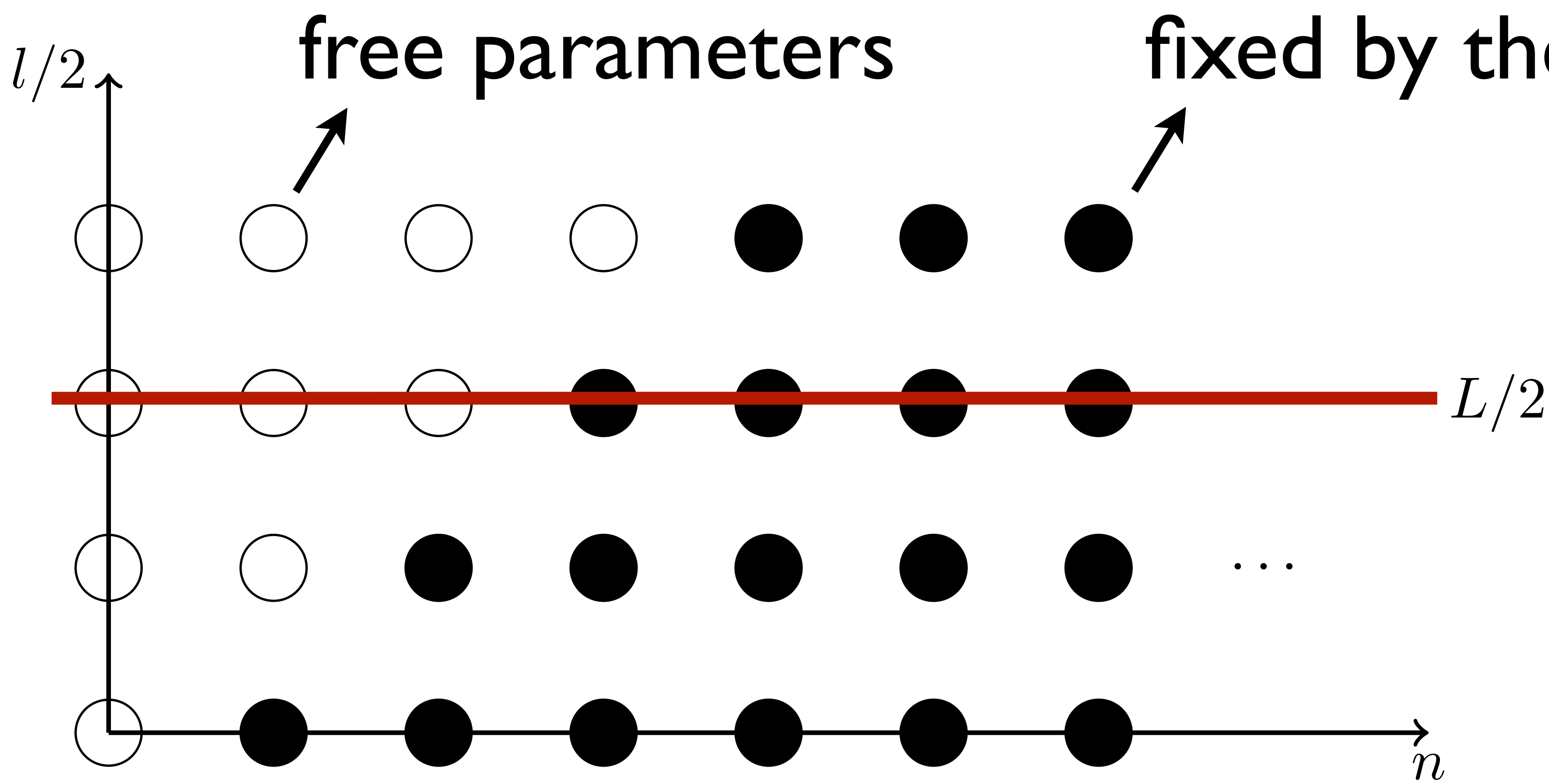
$$\sum_{\substack{l=0 \\ \text{even}}}^{L} \gamma(p,l) J(p+l,q) + \sum_{\substack{l=2 \\ \text{even}}}^{L} \gamma(p-l,l) J(p-l,q) = (p \leftrightarrow q)$$

known function

There are more equations than unknowns...

Counting solutions to crossing

$$\sum_{\substack{l=0 \\ \text{even}}}^{L} \gamma(p,l) J(p+l,q) + \sum_{\substack{l=2 \\ \text{even}}}^{L} \gamma(p-l,l) J(p-l,q) = (p \leftrightarrow q)$$



$$\frac{(L+2)(L+4)}{8}$$

Solutions

Crossing also determines

$$p_1(n,l) = \frac{1}{2}\frac{\partial}{\partial n}\Big[p_0(n,l)\gamma(n,l)\Big]$$

S-matrix Theory in AdS

[Giddings 99]

S-matrix elements	$\longleftrightarrow$	CFT correlators
Lorentz invariance	$\longleftrightarrow$	Conformal invariance
Unitarity	$\longleftrightarrow$	Reflection Positivity
Analiticity	$\longleftrightarrow$	OPE, Crossing

The boundary correlators for any weakly coupled unitary quantum field theory on AdS with a "small" number of fields should be produced by a local Lagrangian.

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \int d^2\delta\, M(\delta_{ij}) \prod_{i<j}^{4} \frac{\Gamma(\delta_{ij})}{\left(x_{ij}^2\right)^{\delta_{ij}}}$$

[Mack 09]

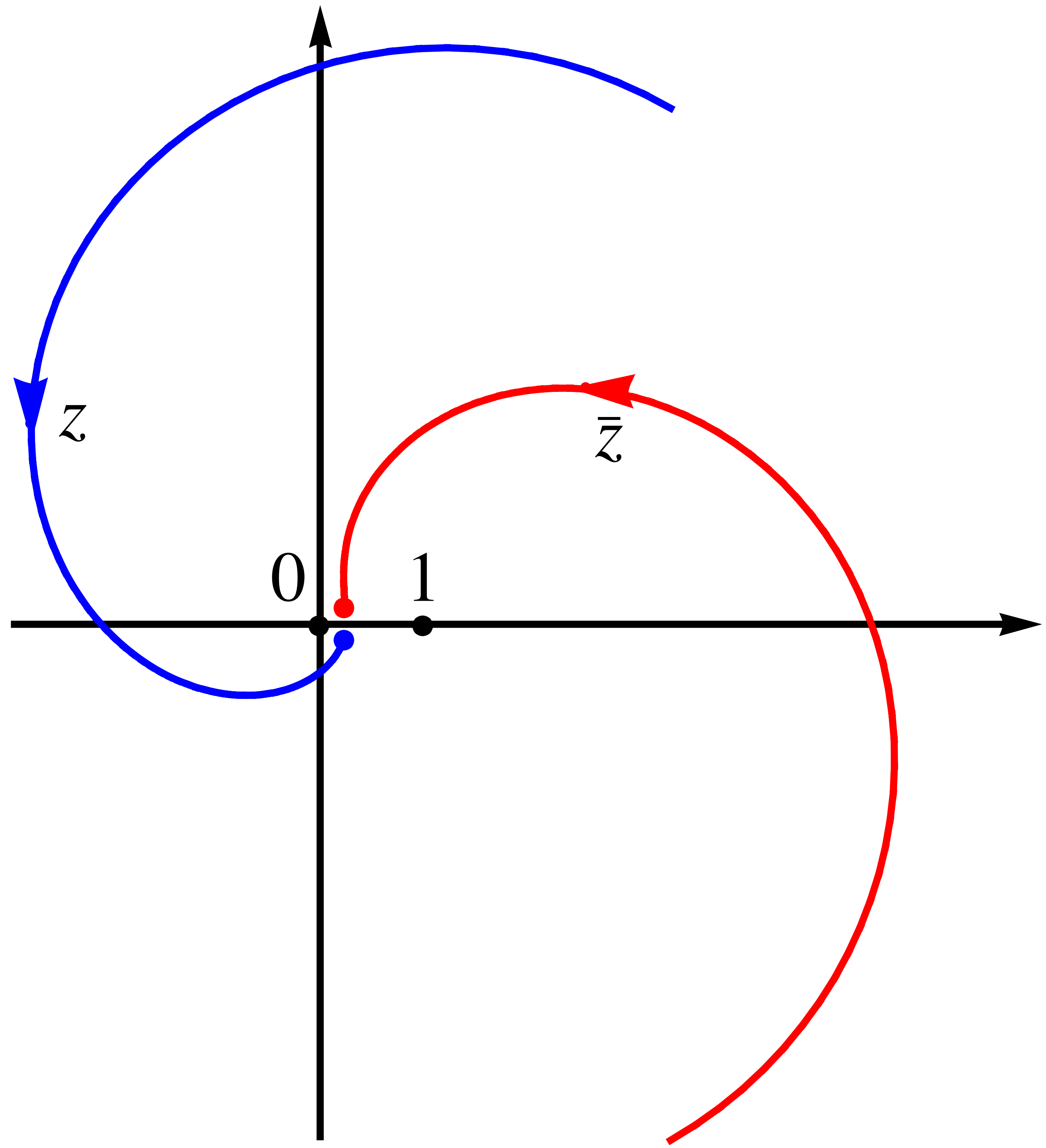
$M(\delta_{ij})$ has very similar analytic properties to a flat space S-matrix.

Analytic Continuation

Euclidean regime

$$\bar{z} = z^\star$$

$z = \bar{z} \quad \Leftrightarrow$ collinear points



Singularity only appears after analytic continuation to the Lorentzian regime.