RESURGENCE, WKB AND STRINGS

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The (exact) WKB method: a little bit of history

Shortly after the discovery of quantum mechanics, it was clear that the one-dimensional Schroedinger equation

$$-\frac{\hbar^2}{2m}\psi''(x) + (V(x) - E)\,\psi(x) = 0$$

can be solved in closed form only in very few cases. One needs approximation methods.

One such method was introduced as early as 1926 by Wentzel, Kramers and Brillouin.

The idea of the WKB method is to solve for the wavefunction as an asymptotic expansion in powers of $\hbar$

One considers the following ansatz

$$\psi(x,\hbar) \sim \frac{1}{\sqrt{p(x,\hbar)}} \exp\left(\frac{\mathrm{i}}{\hbar}\int^{x} p(x',\hbar)\mathrm{d}x'\right)$$

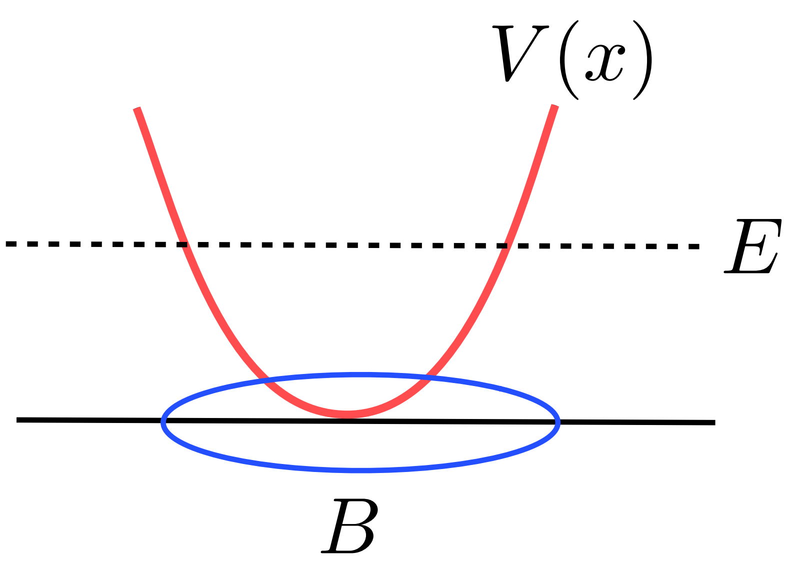
and solves it with $$p(x,\hbar) \sim p(x) + \sum_{n\geq 1} p_n(x)\hbar^{2n}$$

$$p(x) = \sqrt{2m(E-V(x))}$$

This defines a "quantum" Liouville one-form $p(x,\hbar)\mathrm{d}x$

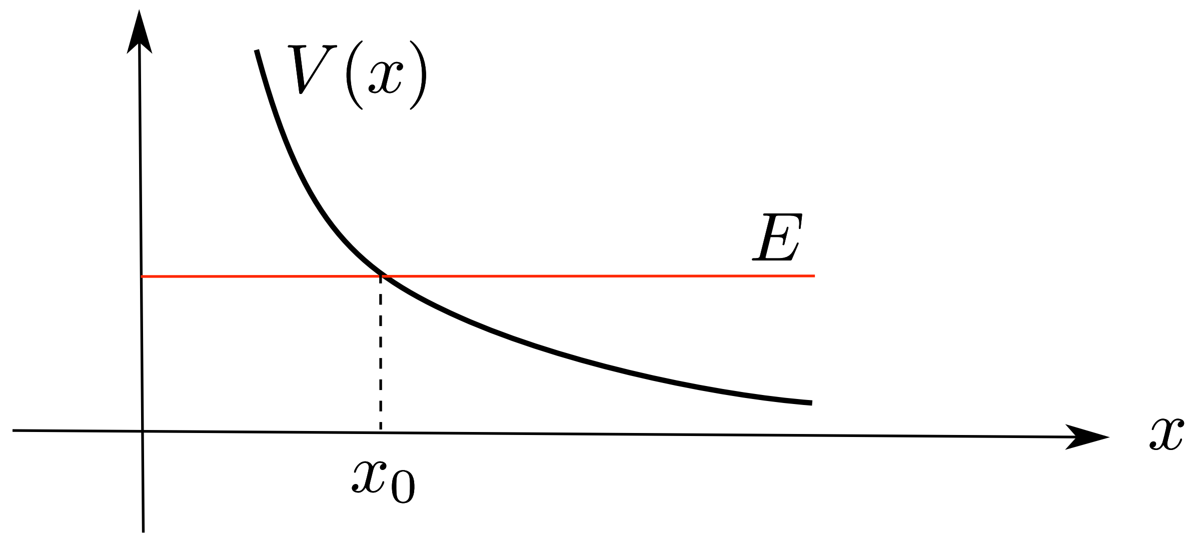
The WKB method quickly became a central tool in quantum mechanics.

As a first application, the WKB method explained the Bohr-Sommerfeld quantization condition as the leading approximation to a more complicated quantization condition, involving corrections in $\hbar$



$$\oint_B p(x,\hbar)\mathrm{d}x = \oint_B p(x)\mathrm{d}x + \mathcal{O}(\hbar^2) = 2\pi\left(k+\frac{1}{2}\right)$$

However, in the period 1930-1970 the understanding of the method was plagued with ambiguities and difficulties.
A particular vexing issue was the "connection problem" relating WKB wavefunctions on the two sides of a turning point.



two principles outlined in §3.1. The method has two main drawbacks: certain quantities (eg γ in (3.23) and δ in (3.57)) cannot be determined, and we are left in complete ignorance of the behaviour of wave functions in the neighbourhood of turning points. It was to remedy these defects that the method of 'uniform

Berry-Mount, 1972

The situation was only clarified in 1980-1990 thanks to the work of Voros and Silverstone (building up on previous work by Dingle). This led to the "exact" WKB method.

JWKB Connection-Formula Problem Revisited via Borel Summation

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The traditional version of the JWKB connection formula at a linear turning point is incorrect.

The heritage of the previous confusions is that (almost) all standard textbooks and courses on quantum mechanics are *incorrect* when it comes to the WKB method!

WKB becomes exact and complex

The reformulation of WKB in the 1980s-1990s was based on two (related) ideas:

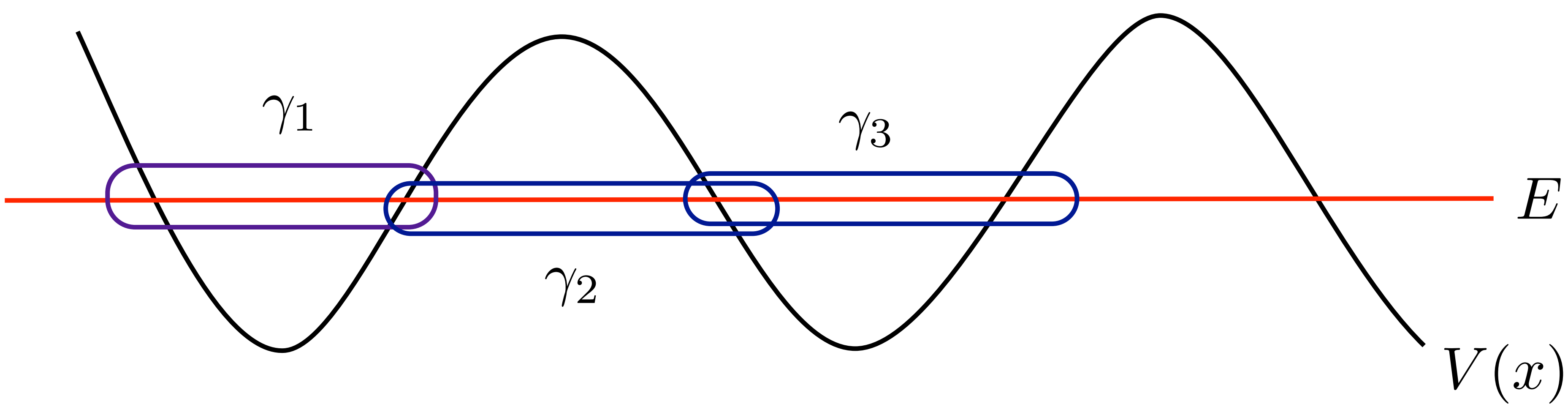
1) the right objects to consider are *Borel resummations* of asymptotic expansions
2) one should extend the Schroedinger equation to the complex realm

This reformulation (at least in its French version) was heavily influenced by Ecalle's theory of resurgence. Let me now present the basic ingredients of this exact or "resurgent" WKB method

WKB curve and quantum periods

The starting point of the method is to regard the classical Hamiltonian as defining a *complex* curve, which I will call the *WKB curve*

$$\Sigma(x,p) = H(x,p) - E = 0$$



γ_a one-cycles of the WKB curve

We can integrate the quantum one-form against the one-cycles of the curve to obtain *quantum periods* (aka *Voros symbols*), which are formal power series in $\hbar^2$

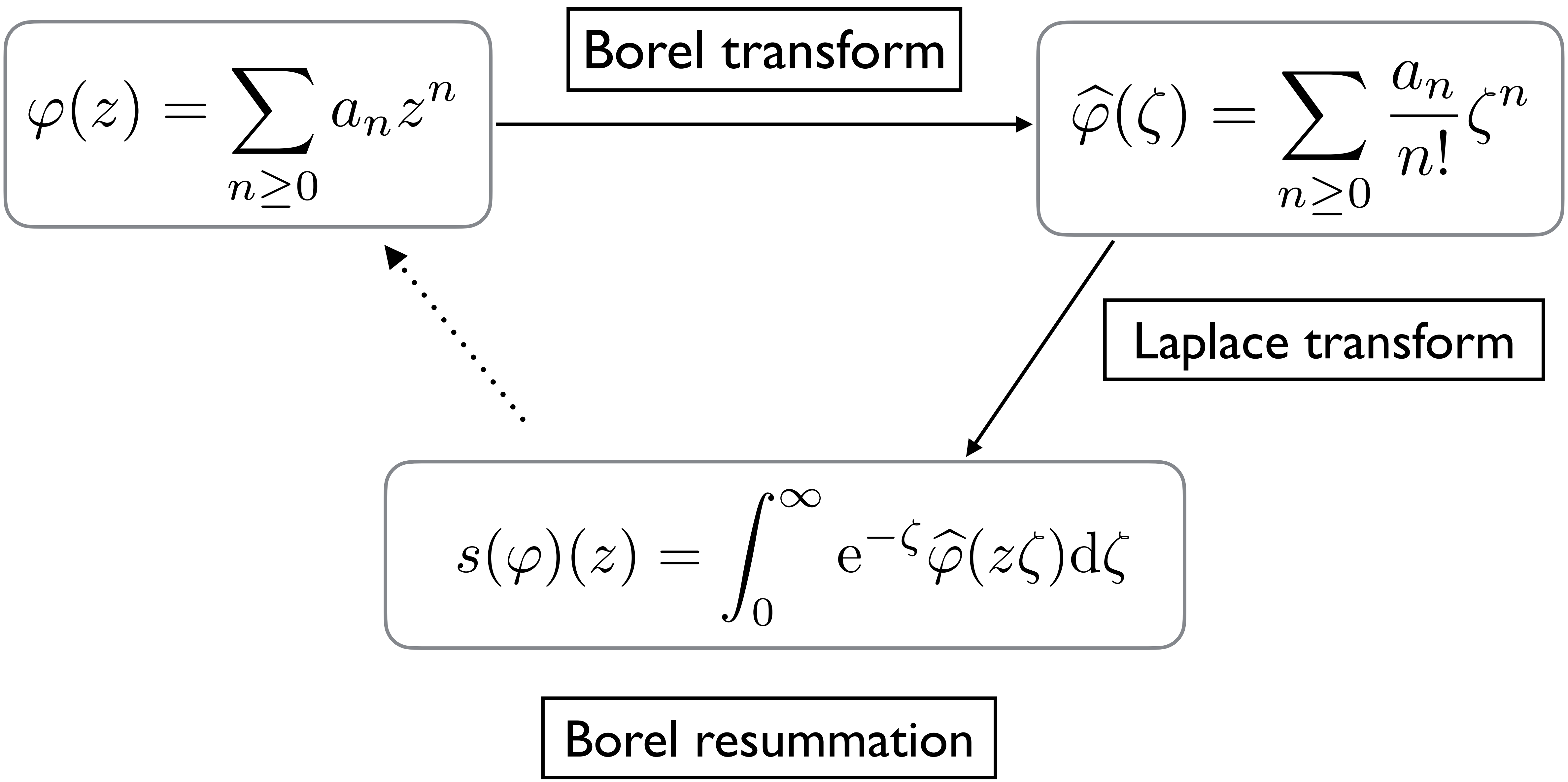
$$\Pi_a(\hbar) = \oint_{\gamma_a} p(x, \hbar)\mathrm{d}x \sim \sum_{n \geq 0} \Pi_a^{(n)} \hbar^{2n}$$

We can think about the different quantum periods as different "sectors" of the theory

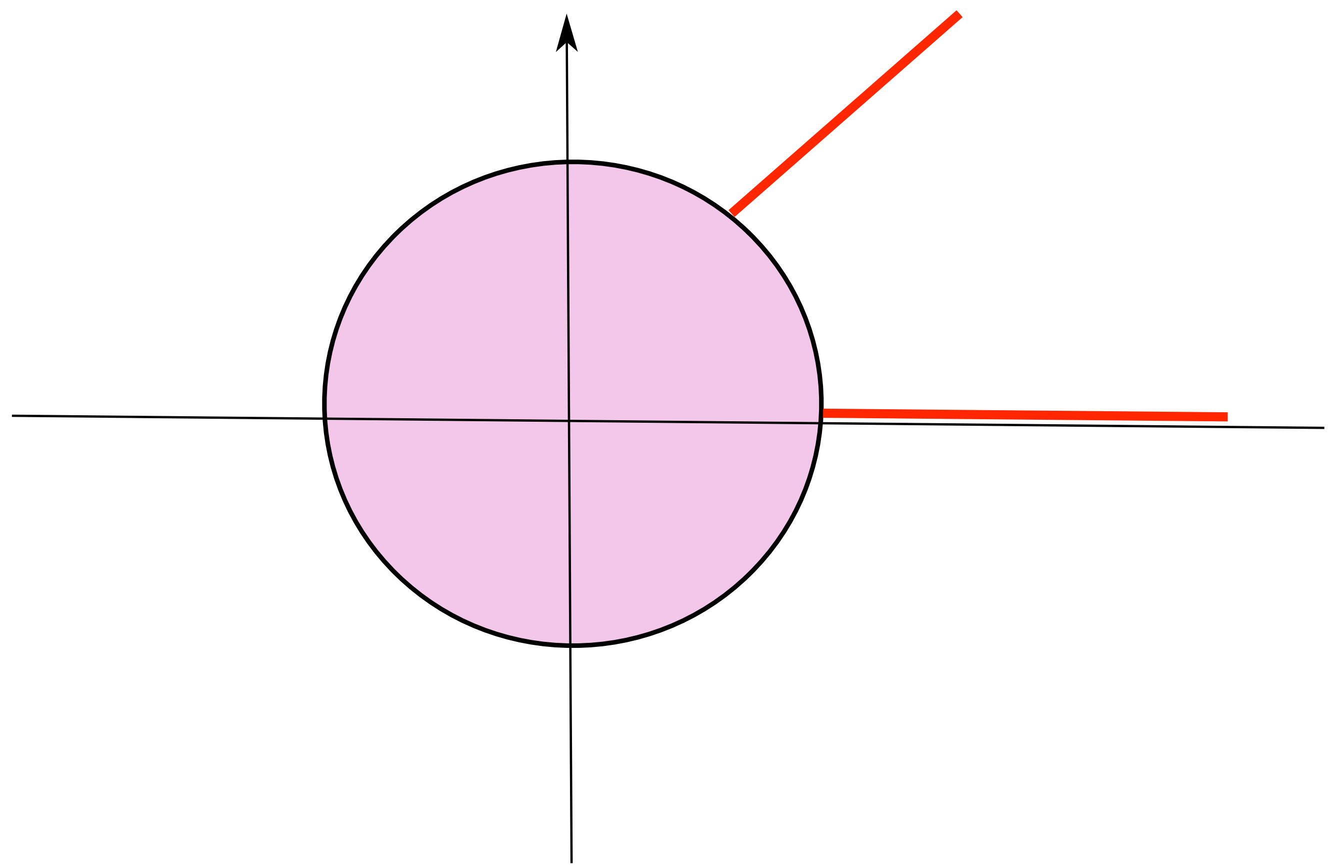
It is well-known that these series are asymptotic and do *not* define functions: their coefficients grow as $(2n)!$ Can we make sense of them?

The Borel triangle

The Borel method is a systematic (and traditional) way of making sense of factorially divergent formal power series



The Borel transform $\widehat{\varphi}(\zeta)$ is analytic at the origin. Very often it can be analytically continued to the complex plane, displaying *singularities* (poles, branch cuts).

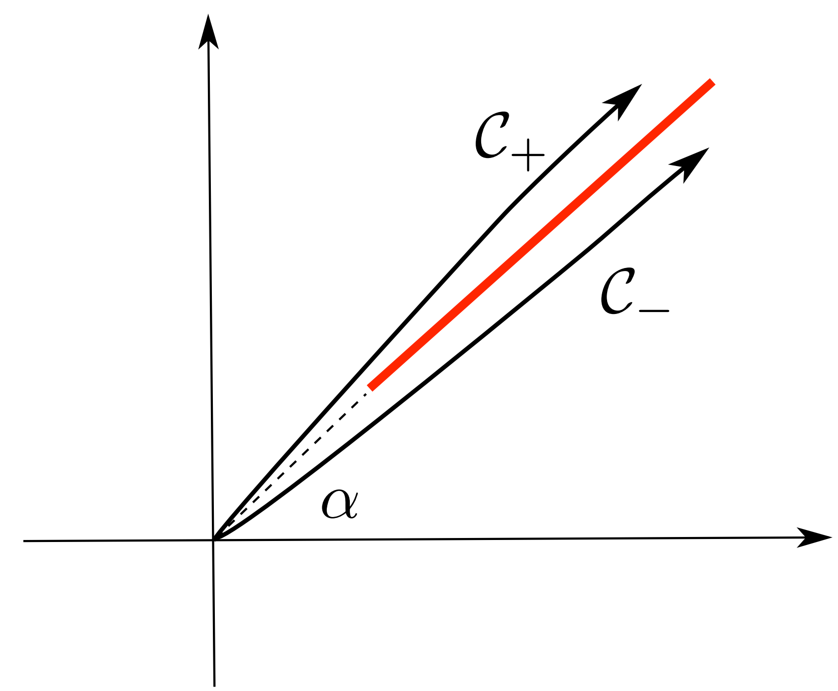


Singularities along the positive real line are obstructions to Borel resummation

Resurgence

Don't be afraid of Borel singularities: do lateral resummations!

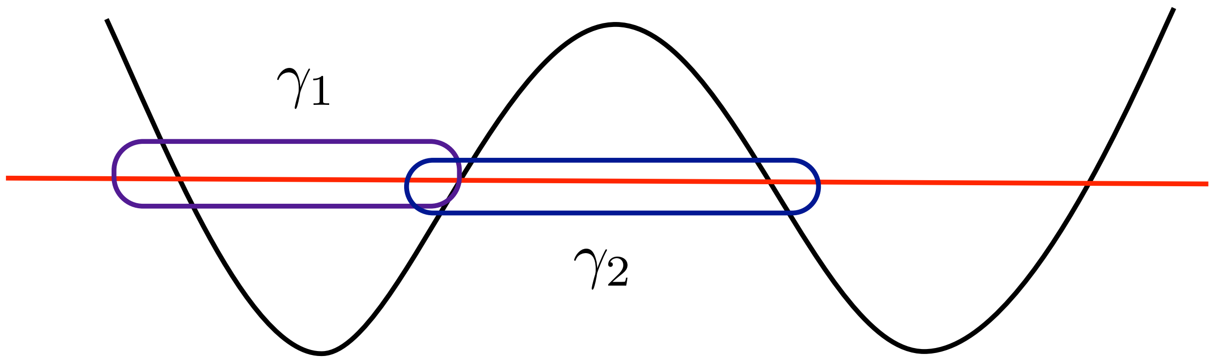
$$s_{\pm\alpha}(\varphi)(z) = \int_{\mathcal{C}_\pm} \mathrm{e}^{-\zeta}\widehat{\varphi}(z\zeta)\mathrm{d}\zeta$$



Stokes discontinuity (or Stokes automorphism)

$$\mathrm{disc}_\alpha(\varphi) = s_{+\alpha}(\varphi) - s_{-\alpha}(\varphi)$$

A quantum theory is **resurgent** if the Stokes discontinuity of the perturbative series in a given sector is a function of the series in other sectors (*and nothing else*).



Π_1 perturbative

Π_2 non-perturbative

In the case of the symmetric double-well in quantum mechanics, we have

$$s_+(\Pi_1) - s_-(\Pi_1) = -\mathrm{i}\hbar \log\left(1 + \mathrm{e}^{-s(\Pi_2)/\hbar}\right)$$

The Borel singularities of the Borel transform of Π_1 are located at multiples of the instanton action $\Pi_2^{(0)}$

The perturbative sector knows about the non-perturbative sector!

Exact quantization conditions

What is the use of quantum periods? One beautiful consequence of the exact WKB method is that **exact** quantization conditions (EQC) for the spectrum can be obtained as **vanishing conditions for Borel-resummed quantum periods**

[Voros, Zinn-Justin]

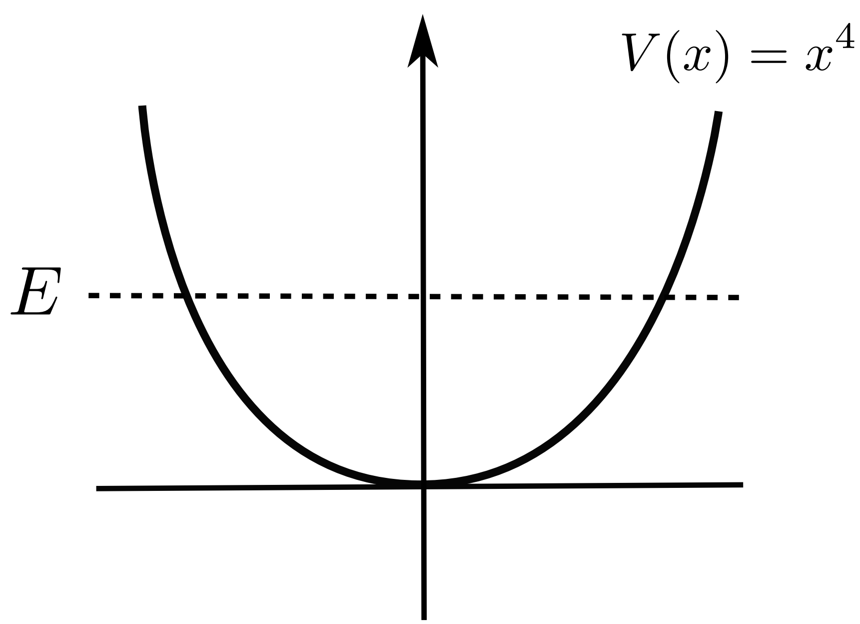
In the case of the double-well potential, one finds

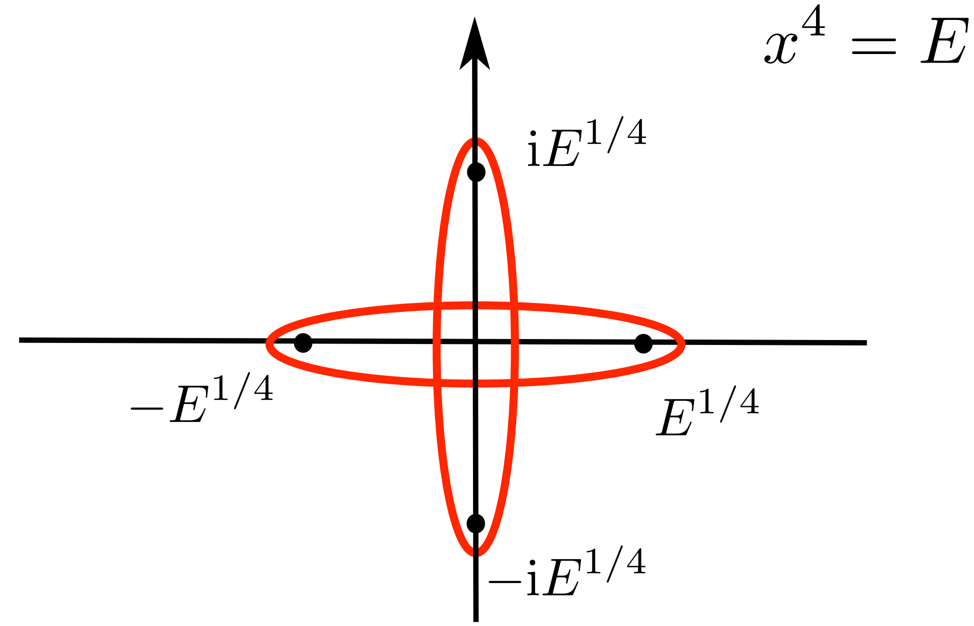
$$\frac{1}{\hbar}\underbrace{\left(s_+(\Pi_1)(\hbar)+s_-(\Pi_1)(\hbar)\right)}_{\text{perturbative}} \pm \underbrace{\tan^{-1}\left(\mathrm{e}^{-\frac{1}{2\hbar}s(\Pi_2)(\hbar)}\right)}_{\text{instantons}} = 2\pi\left(k+\frac{1}{2}\right)$$

This requires the exact version of the connection formula due to Voros and Silverstone

Complex quantum periods turn out to be crucial in the exact WKB method, as shown by [Balian-Parisi-Voros, Voros] in the case of the pure quartic oscillator

$$\mathsf{H} = \mathsf{p}^2 + \mathsf{x}^4$$





real cycle: only *approximate* spectrum

The exact spectrum requires the real and the *complex* cycle: "complex tunneling"

Insights from strings and gauge theories

The basic ingredients of the exact WKB method are quantum versions of periods of complex curves. Periods of curves play an important role in other contexts.

N=2 susy gauge theory $\longrightarrow$ Seiberg-Witten (SW) curve $\Sigma(x, \mathrm{e}^p) = 0$

$$a_i = \oint_{A_i} p \mathrm{d}x \qquad a_{D,i} = \oint_{B_i} p \mathrm{d}x$$

These periods determine the masses of BPS solitons in the gauge theory toric Calabi-Yau manifold *X* $\longrightarrow$ mirror curve $\Sigma(\mathrm{e}^x, \mathrm{e}^p) = 0$

In this case, the periods determine the prepotential of topological string theory on *X*, which contains information about the counting of curves of genus zero on *X*

$$a_{D,i} = \frac{\partial F_0}{\partial a_i}$$

Quantum curves

How do we quantize this classical picture?
We can obtain a *quantum curve* by promoting *x*, *p* to Heisenberg operators

Example:
SW curve for SU(2), N=2 SYM

$$\Sigma(x, \mathrm{e}^p) = 2\Lambda^2 \cosh(p) + x^2 - u$$

$$\downarrow$$

$$\left(2\Lambda^2 \cosh(\mathsf{p}) + \mathsf{x}^2 - u\right) |\psi\rangle = 0$$

$$[\mathsf{x}, \mathsf{p}] = \mathrm{i}\hbar$$

By using a WKB ansatz for the wavefunction, one obtains again a quantum Liouville one-form

$$p(x,\hbar)\mathrm{d}x$$

This gives quantum versions of the periods appearing in gauge theory/topological strings, as in the conventional WKB method

$$a_i(\hbar) = \sum_{k\geq 0} a_i^{(k)} \hbar^{2k} \qquad a_{D,i}(\hbar) = \sum_{k\geq 0} a_{D,i}^{(k)} \hbar^{2k}$$

What is the meaning of this quantization?

It turns out that it is related to the "Omega background" for the gauge/string theory, which involves two parameters

$$\epsilon_1,\ \epsilon_2$$

Quantization corresponds to the so-called Nekrasov-Shatashvili (NS) limit

$$\epsilon_1 = \hbar,\ \epsilon_2 = 0$$

Note that we have formulated the correspondence by using WKB quantization of a one-dimensional curve [cf. Mironov-Morozov]. This might be more fundamental than approaches based on the quantization of a higher-dimensional integrable system.

The correspondence between the Omega background and quantization is not fully understood. To complicate matters, we note that the self-dual Omega background,

$$\epsilon_1 = -\epsilon_2 = g_s$$

which gives the conventional genus expansion of topological string, corresponds to a *dual quantization* [Kallen-M.M.,Grassi-Hatsuda-M.M., ...]

$$g_s = \frac{1}{\hbar}$$

We will not develop this, however, and will restrict ourselves to the original story

The NS correspondence leads to some surprising consequences for the conventional WKB method. It suggests to define a "quantum prepotential"

$$a_D(\hbar) = \frac{\partial F(a(\hbar), \hbar)}{\partial a(\hbar)}$$

This "quantum prepotential" satisfies a version of the holomorphic anomaly equations (HAE) of topological string theory [Huang-Klemm,Krefl-Walcher]

It follows that the the quantum periods of the WKB method (even in ordinary quantum mechanics) are formal series of **quasi-modular forms on the WKB curve**, governed by the HAE [Codesido-M.M.]

In some cases the quantum periods can be computed by instanton calculus in the *N*=2 gauge theory. This expresses them as **convergent series** in an "instanton counting" parameter

SU(2), N=2 SYM
$$a(u,\hbar) = \sqrt{u}\left(1 + \frac{\Lambda^4}{u(4u+\hbar^2)} + \cdots\right)$$

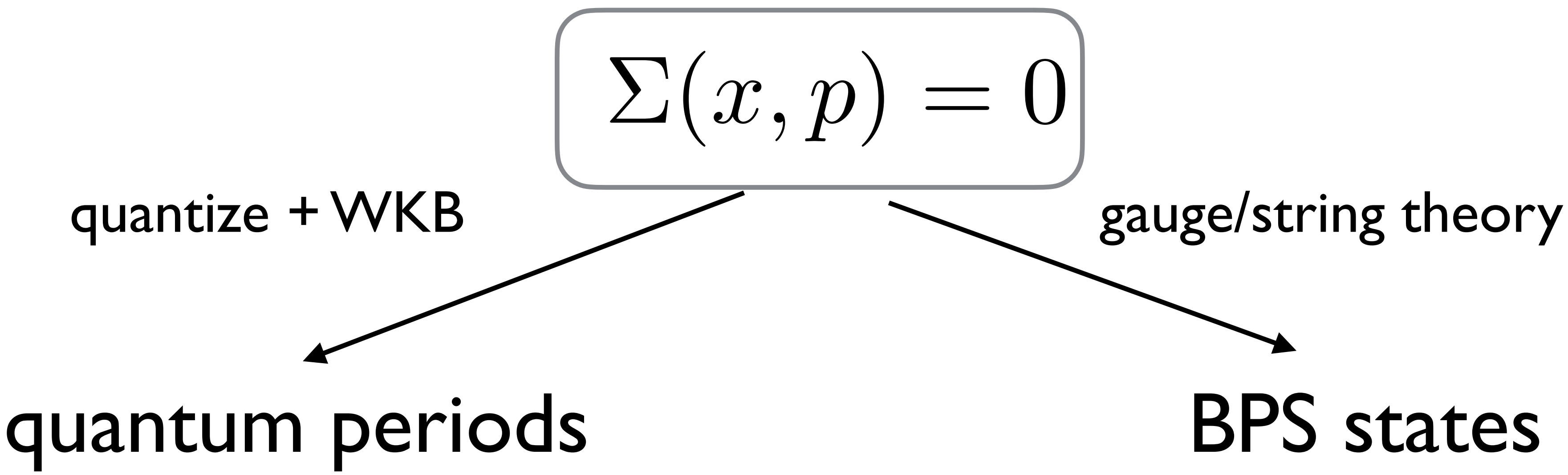
This can be regarded as a **different resummation** of the quantum periods. The relation to the standard Borel resummation is non-trivial [Kashani-Poor-Troost, Grassi-Gu-M.M.]

GMN

An important recent development in the interface of WKB/ string-gauge theory is the monumental work of Gaiotto-Moore-Neitzke (GMN) on BPS states in N=2 gauge theories.

It turns out that many ingredients in their theory are related in a precise way to the resurgent WKB method

From WKB to GMN



WKB	GMN
$\langle \gamma_a \rangle$	Γ
$\Pi_a^{(0)}$	$Z(\gamma_a)$
Π_a	X_{γ_a}
Borel singularities	BPS spectrum
Stokes discontinuities	KS morphisms

Voros' analytic bootstrap

Some of the tools introduced by GMN make it possible to solve old problems in the theory of resurgence. I will focus here on the "analytic bootstrap", an approach to quantization proposed by André Voros in 1983.

Suppose we have a resurgent quantum theory and we know (1) the Stokes discontinuities of the perturbative series in all sectors, and (2) their classical limit.

Can we then reconstruct the *exact (resummed)* series?

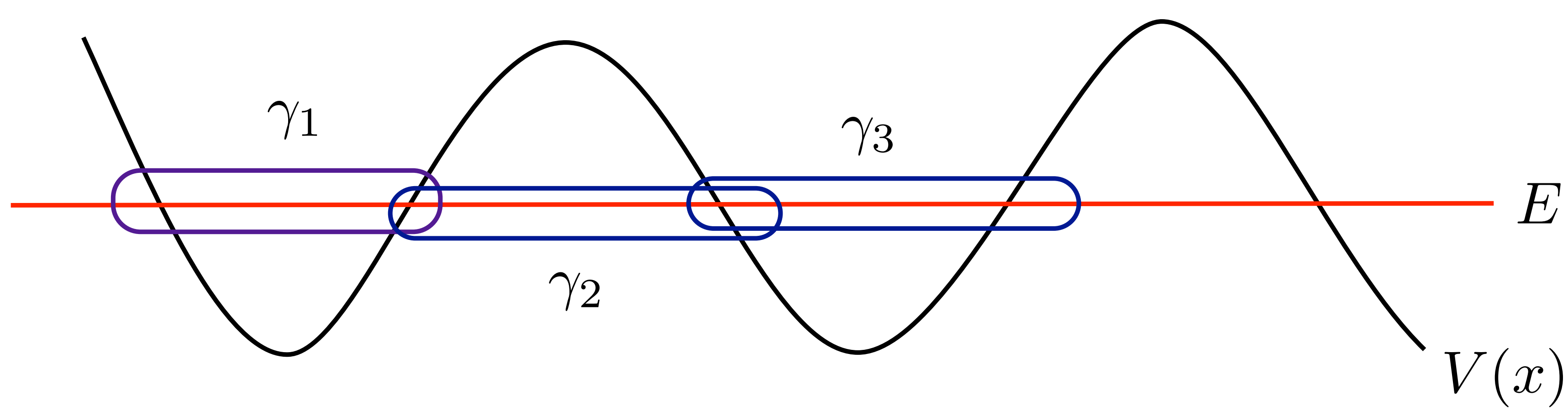
The analytic bootstrap is in fact a typical Riemann-Hilbert problem, of the type studied by GMN.

A solvable example

The analytic bootstrap can be solved with the tools of GMN in an important example: the exact WKB method in QM with polynomial potentials

$$V(x) = x^{r+1} - \sum_{i=1}^{r} u_i x^{r+1-i}$$

"minimal" chamber in moduli space



$$\Pi_1 \qquad \Pi_2 \qquad \ldots$$

The Stokes discontinuities in this case are given by the Delabaere-Pham formula:

$$s_+(\Pi_a) - s_-(\Pi_a) = -\mathrm{i}\hbar \log\left(1 + \mathrm{e}^{-s(\Pi_{a-1})}\right) - \mathrm{i}\hbar \log\left(1 + \mathrm{e}^{-s(\Pi_{a+1})}\right)$$

+classical limit $\Pi_a(\hbar) \sim \Pi_a^{(0)}, \qquad \hbar \to 0$

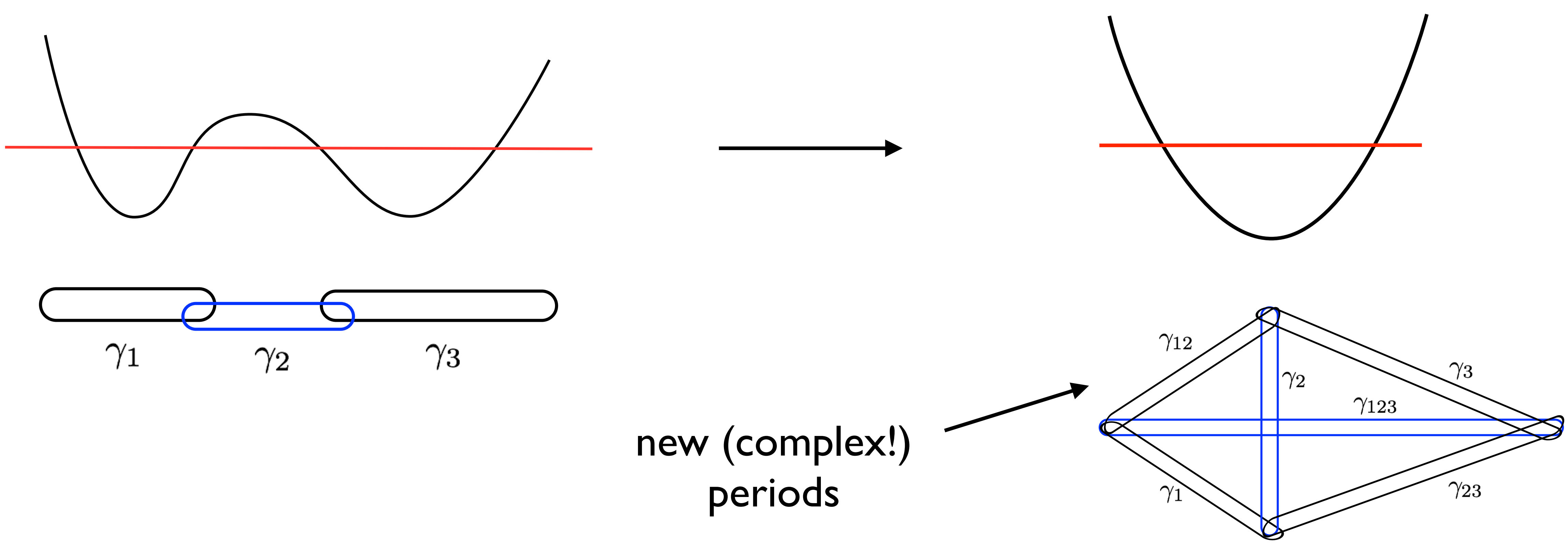
As in GMN, one can solve this Riemann-Hilbert problem in terms of TBA-like equations [Ito-M.M.-Shu]

$$\epsilon_a(\theta) = \Pi_a^{(0)} \mathrm{e}^{\theta} - \int_{\mathbb{R}} \frac{L_{a-1}(\theta')}{\cosh(\theta - \theta')} \frac{\mathrm{d}\theta'}{2\pi} - \int_{\mathbb{R}} \frac{L_{a+1}(\theta')}{\cosh(\theta - \theta')} \frac{\mathrm{d}\theta'}{2\pi}$$

$$\hbar = \mathrm{e}^{-\theta} \qquad \epsilon_a(\theta) = \frac{1}{\hbar} s(\Pi_a)(\hbar) \qquad L_a(\theta) = \log\left(1 + \mathrm{e}^{-\epsilon_a(\theta)}\right)$$

This provides a "resurgent" derivation and generalization of a conjecture by Gaiotto. It extends the **ODE/IM correspondence** of Dorey-Tateo (which was derived for monic potentials) to **arbitrary polynomial potentials**

As we move in moduli space to different "chambers", one has to consider additional quantum periods and include them in the TBA equations. This is the well-known wall-crossing phenomenon.



This picture can be extended to the quantum versions of general SW curves: the resurgent properties of the corresponding quantum periods can be deduced from the BPS spectrum and its wall-crossing

[Gaiotto, Grassi-Gu-M.M.]

Conclusions and outlook

• WKB is alive and well. Renewed interest in the theory of resurgence, and recent developments in string theory and gauge theory, have provided new insights and fresh solutions of old problems in the theory

• Many open problems! We are still lacking e.g. an exact WKB method for local mirror curves (difference equations). This would be potentially very useful to understand topological strings and BPS states on local Calabi-Yau threefolds

•GMN-like arguments give us the exact resummed quantum periods, but not the quantization conditions. Is there a natural meaning for these in the framework of GMN?

More conceptually, we need a deeper understanding of why many problems in gauge/string theory can be solved by quantizing the underlying curve

Thank you for your attention!