IRC Safety
vs.
Calculability

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AJL, J.Thaler 1307.1699; AJL, I. Moult, D. Neill 1401.4458;
AJL, S. Marzani, G. Soyez, J.Thaler 1402.soon

Rutgers, February 11, 2014

Reminder: How to compute differential cross sections in perturbation theory

$$\frac{d\sigma}{d\mathcal{O}} = \sum_n \int d\Pi_n \, |\mathcal{M}_n|^2 \, \delta\left(\mathcal{O} - \hat{\mathcal{O}}(\Pi_n)\right)$$

n = number of external particles

O = observable

$$\mathcal{M}_n = \mathcal{M}_n^0 + \mathcal{M}_n^1 + \mathcal{M}_n^2 + \cdots$$

tree-level	one-loop	two-loop
g^{n-2}	g^{n-1}	g^{n}
Real	Virtual	

Infrared and Collinear Safety

The phase space constraints imposed by the observable are smooth through real and virtual contributions

Collinear safety

Linear in energy

Weighted by positive powers of angles

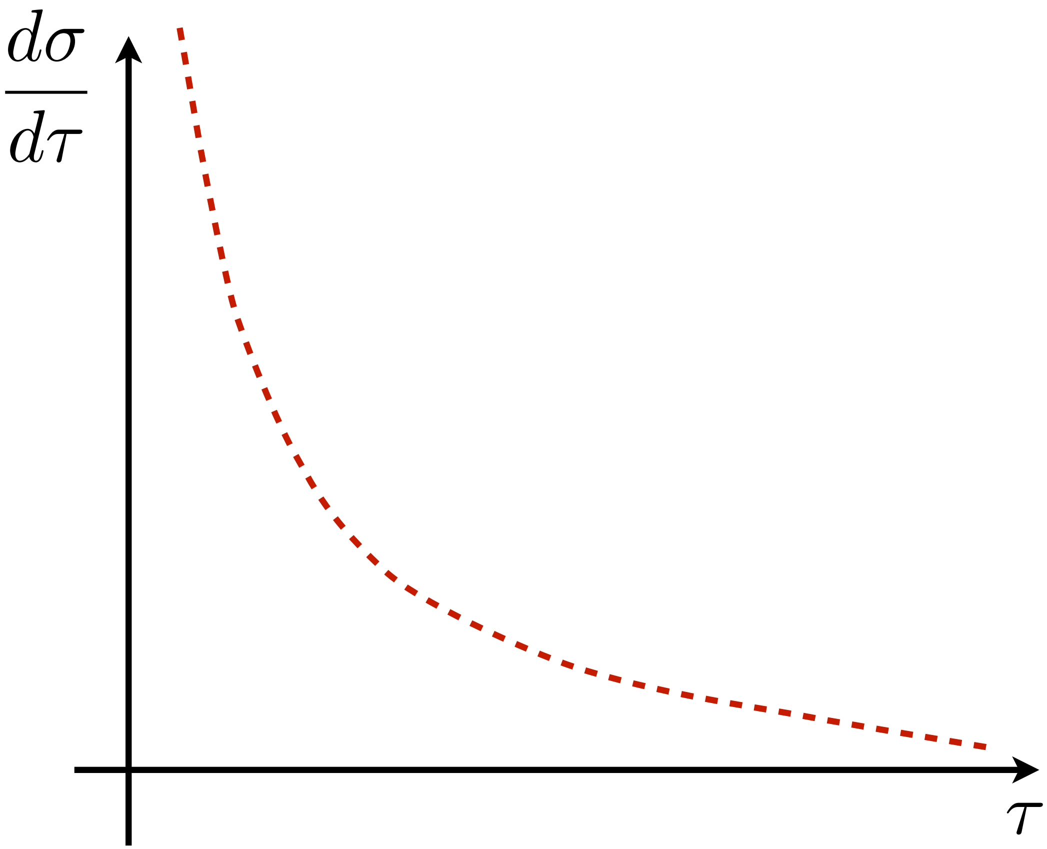
Infrared safety

Weighted by positive power of energy

Poster Child: Thrust

$$\tau \equiv \frac{1}{Q} \sum_i E_i \sin\theta_i \tan\frac{\theta_i}{2}$$

Reminder: How to compute differential cross sections in perturbation theory



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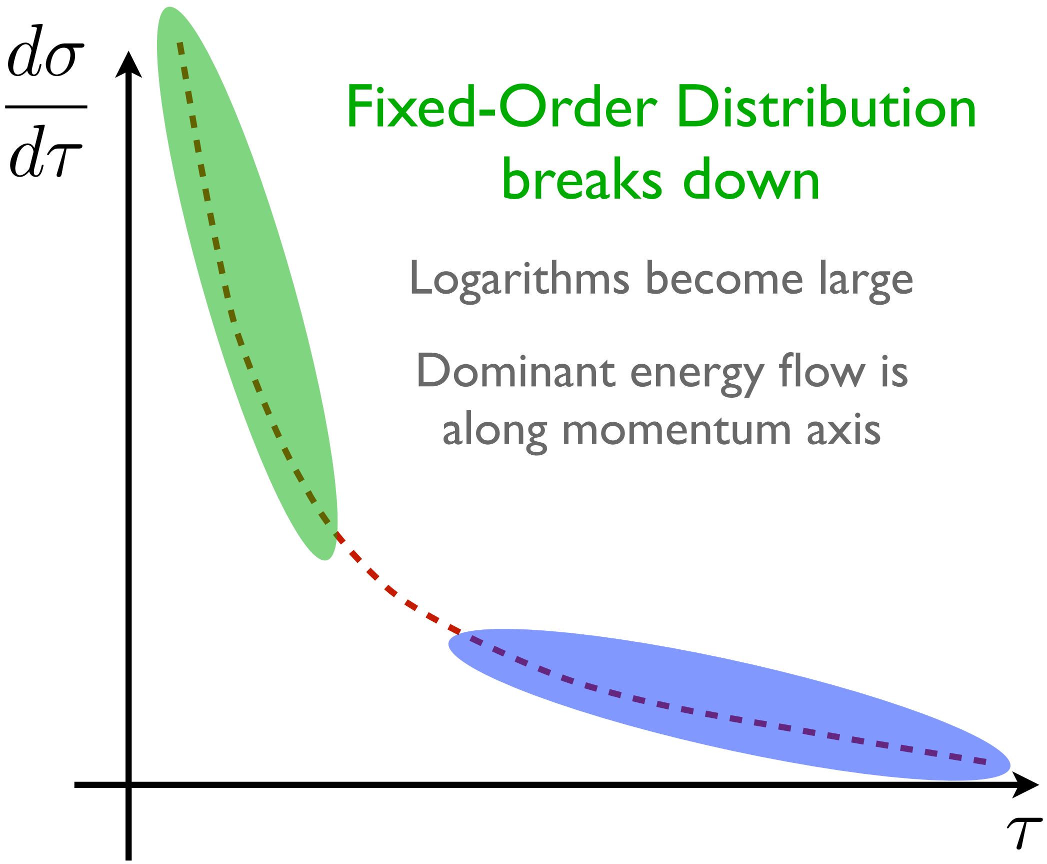


Fixed-Order Distribution breaks down

Logarithms become large

Dominant energy flow is along momentum axis

Must resum logarithms for reliable predictions

Soft-Collinear Effective Theory: Resummation by RG evolution

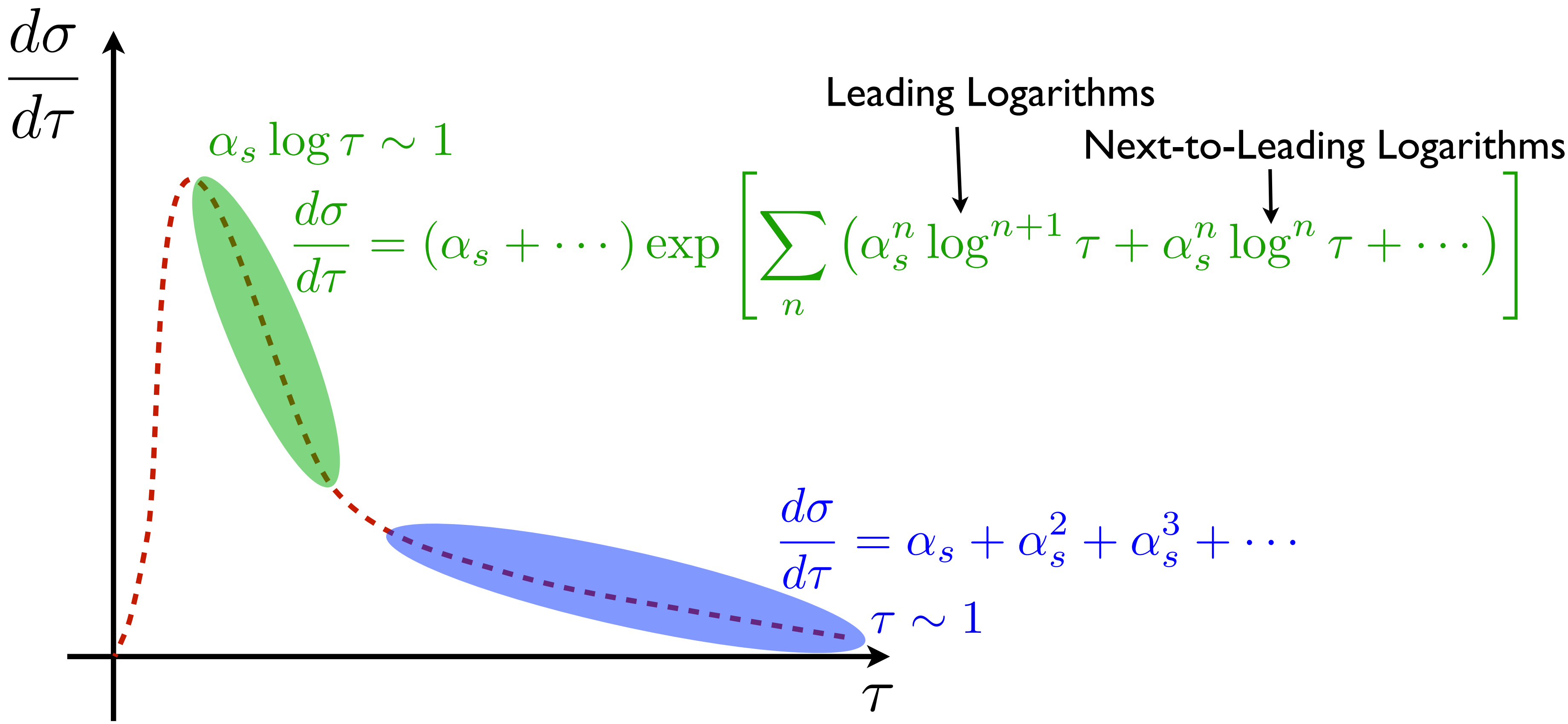
Explicit summation of singular approximation to matrix element

Fixed-Order Distribution is Accurate

No large logarithms

Dominant emissions are away from soft/collinear region

Reminder: How to compute differential cross sections in perturbation theory



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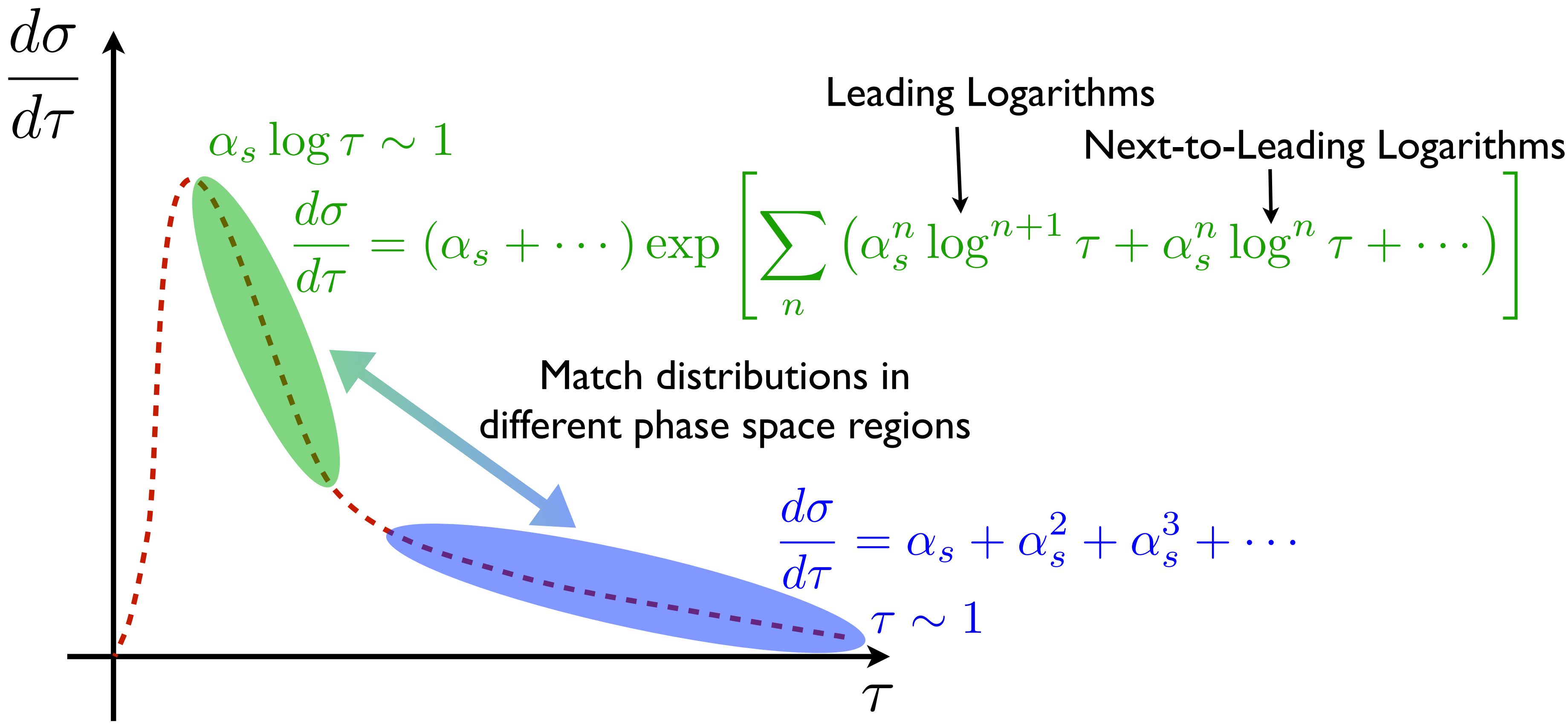

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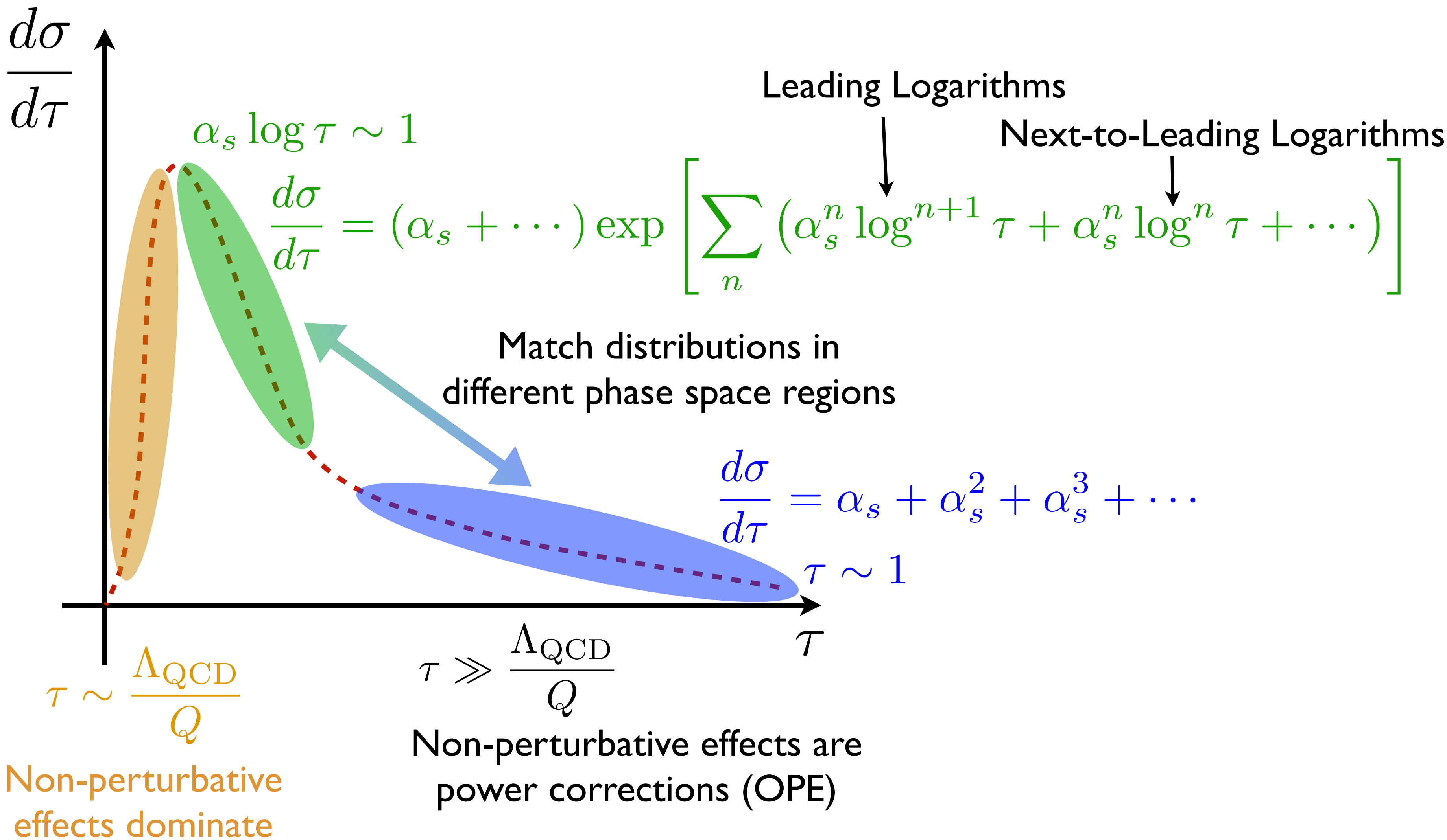
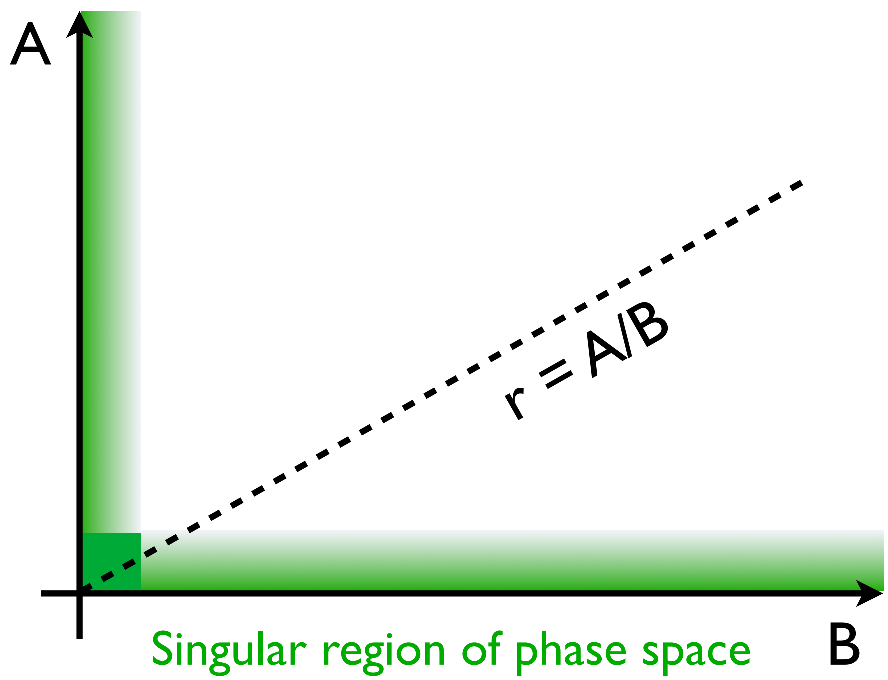


Ratio Observables in Perturbation Theory

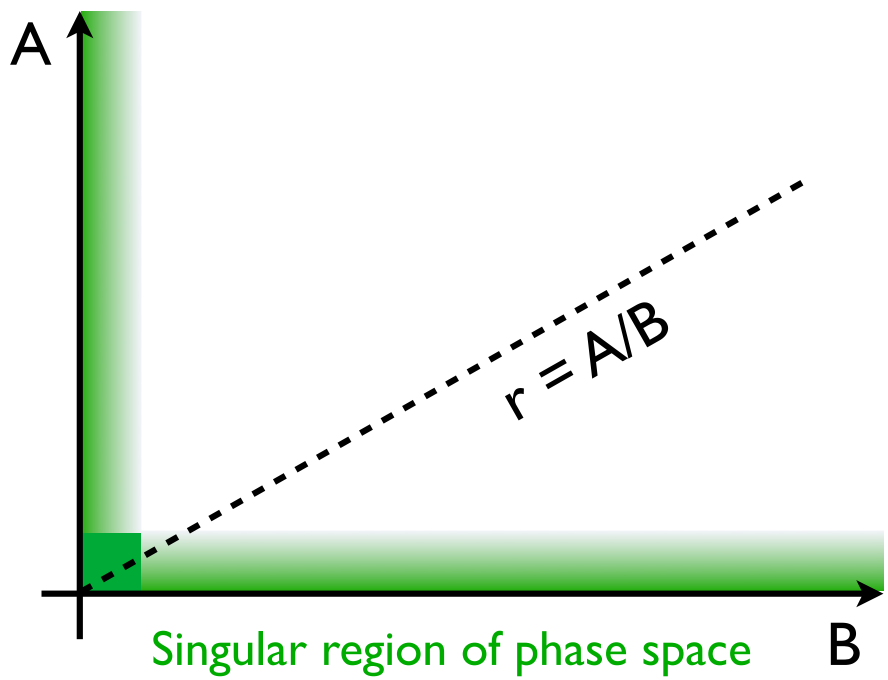


A, B: IRC safe observables

Real contribution:
divergent for all r

Virtual contribution:
divergent, proportional to $\delta(r)$

Ratio Observables in Perturbation Theory



A, B: IRC safe observables

Real contribution:
divergent for all r

Virtual contribution:
divergent, proportional to $\delta(r)$

IRC Unsafe!? Soyez, Salam, Kim, Dutta, Cacciari 2012

Standard computation methods are useless for these observables

A, B can be measured separately; why can't their ratio?

This is a major practical issue

Ratio Observables in Perturbation Theory

Example: *N*-subjettiness Thaler, van Tilburg 2010

$$\tau_N^{(\beta)} = \sum_{i\in J} p_{Ti} \min\{R_{i1}^\beta, R_{i2}^\beta, \ldots, R_{iN}^\beta\}$$

$$\tau_{2,1}^{(\beta)} = \frac{\tau_2^{(\beta)}}{\tau_1^{(\beta)}}$$

Powerful boosted *W* tagger

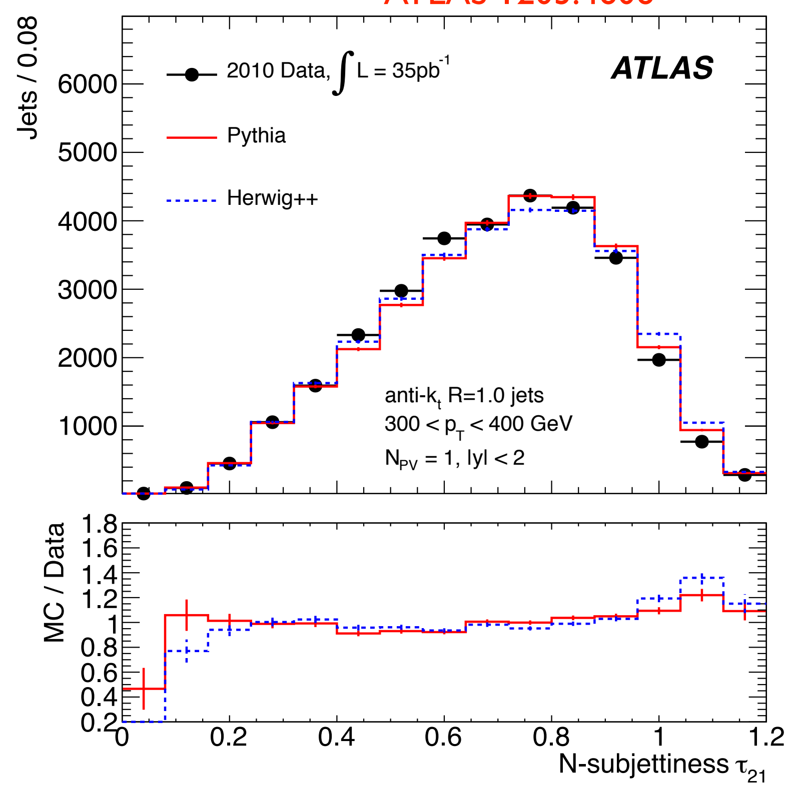
Selects for 2-subjet structures

$$\tau_{3,2}^{(\beta)} = \frac{\tau_3^{(\beta)}}{\tau_2^{(\beta)}}$$

Powerful boosted *t* tagger

Selects for 3-subjet structures

ATLAS 1203.4606



Other ratio observables used for jet substructure analysis:

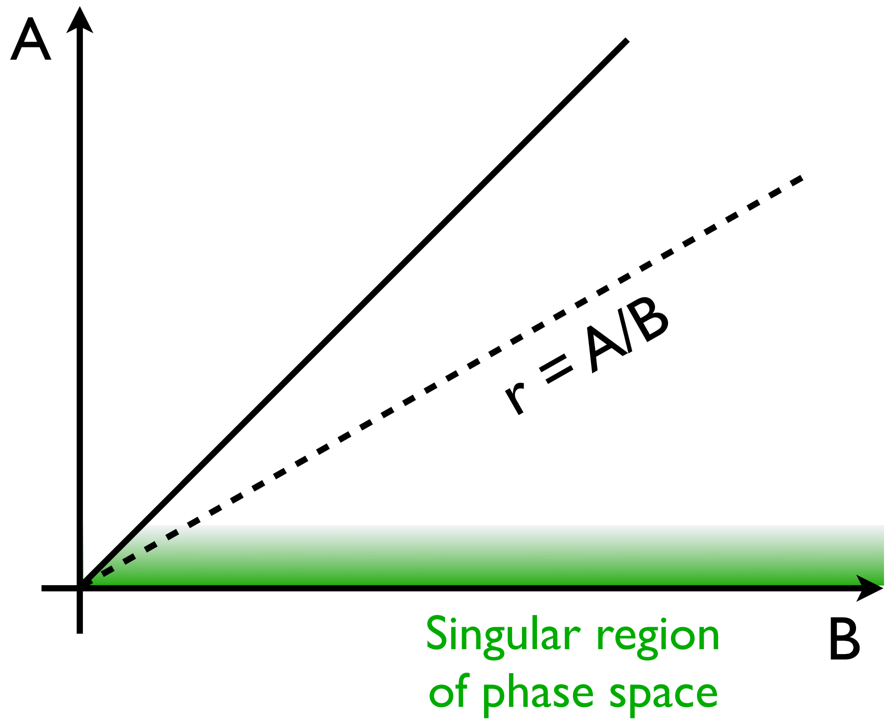
Energy correlation functions AJL, Salam, Thaler 2013

Angular correlation functions Jankowiak, AJL 2011

Planar flow Almeida, Lee, Perez, *et al.* 2008

Why does Monte Carlo model data so well?

Ratio Observables in Perturbation Theory

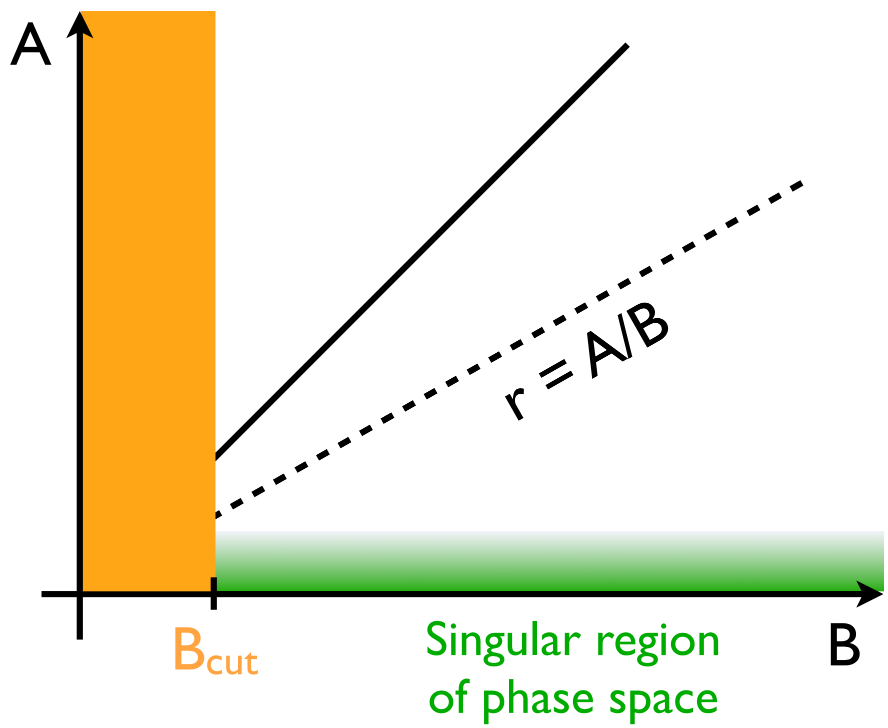


Assume $A \leq B$

$0 \leq r \leq 1$

Need to regulate the singular region of phase space for calculability

Ratio Observables in Perturbation Theory



Assume $A \leq B$

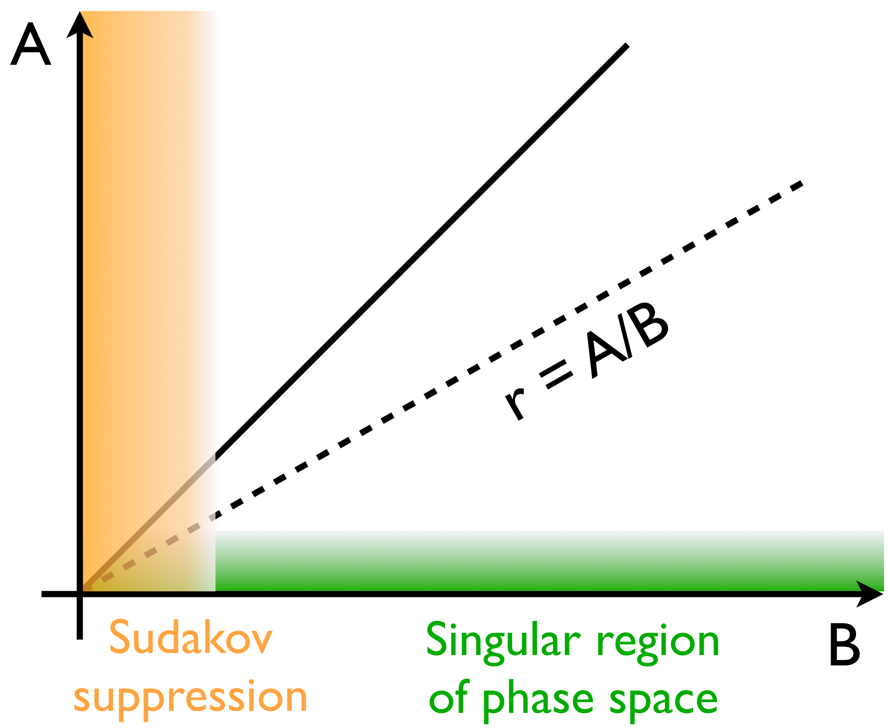
$0 \leq r \leq 1$

Need to regulate the singular region of phase space for calculability

1) Explicit cut on denominator observable B

May introduce undesired logarithmic sensitivity to B_{cut}

Ratio Observables in Perturbation Theory



Assume $A \leq B$

$0 \leq r \leq 1$

Need to regulate the singular region of phase space for calculability

1) Explicit cut on denominator observable B

May introduce undesired logarithmic sensitivity to B_{cut}

2) Include emissions to all-orders in perturbation theory

Exponentially suppresses singular region organically

Ratio Observables in Perturbation Theory

Definition: $$\frac{d\sigma}{dr} = \int dA\, dB\, \frac{d^2\sigma}{dA\, dB}\, \delta\left(r - \frac{A}{B}\right)$$

$\dfrac{d^2\sigma}{dA\, dB}$ is the fundamental object

Well-defined order-by-order in perturbation theory

Follows from IRC safety of A and B

To all-orders, singular region is exponentially suppressed by perturbative Sudakov factor

Marginalization is well-defined

Ratio observable is “Sudakov safe”

Outline

Example: Ratio of Angularities

IRC Unsafety at fixed order
Sudakov Safety at all-orders
Controlled non-perturbative sensitivity

Looking Forward

Higher-order effects
Other examples of Sudakov Safe observables

Conclusions

Ex: Ratio of Angularities

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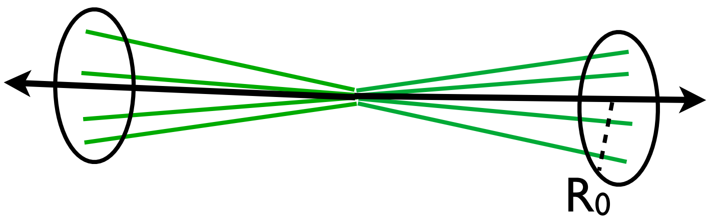
Angularities Measured on Jets



Recoil-free jet axis
(broadening, winner-take-all)

AJL, Neill, Thaler 2014

$$e_\alpha = \frac{1}{E_{\rm jet}} \sum_{i\in {\rm jet}} E_i \left(\frac{\theta_i}{R_0}\right)^\alpha$$

angle measured wrt jet axis

Familiar angularities:

$\alpha = 2$, thrust

$\alpha = 1$, broadening/width/girth

Want to measure:

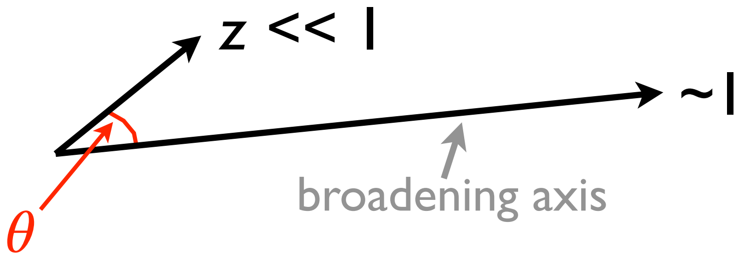
$$\frac{d\sigma}{dr} = \int de_\alpha\, de_\beta\, \frac{d^2\sigma}{de_\alpha\, de_\beta}\, \delta\left(r - \frac{e_\alpha}{e_\beta}\right)$$

We take $\alpha > \beta$ so $e_\alpha < e_\beta$

$0 \leq r \leq 1$

Compute double differential cross section to different accuracies

Fixed-Order Distribution



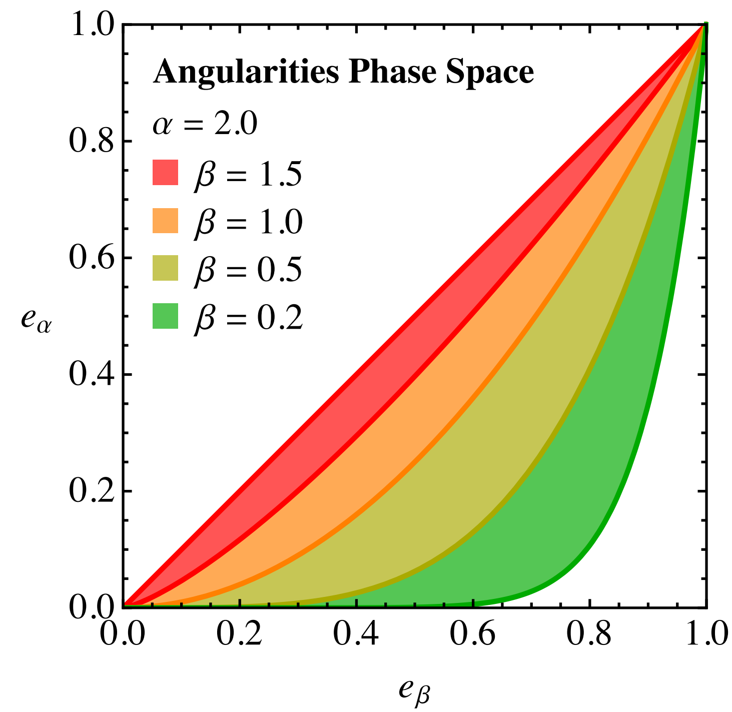
$$e_\alpha = z\left(\frac{\theta}{R_0}\right)^\alpha \qquad e_\beta = z\left(\frac{\theta}{R_0}\right)^\beta$$

$$z = e_\alpha^{-\frac{\beta}{\alpha-\beta}} e_\beta^{\frac{\alpha}{\alpha-\beta}} \qquad \frac{\theta}{R_0} = e_\alpha^{\frac{1}{\alpha-\beta}} e_\beta^{-\frac{1}{\alpha-\beta}}$$

Phase space constraints:

Energy conservation: $z < 1 \quad \rightarrow \quad e_\beta^\alpha < e_\alpha^\beta$

Emission within jet: $\theta < R_0 \quad \rightarrow \quad e_\alpha < e_\beta$



Singular matrix element:

$$S(z,\theta)dz\,d\theta = 2\frac{\alpha_s}{\pi}C_F\frac{d\theta}{\theta}\frac{dz}{z}$$

Double Differential Cross Section:

$$\frac{1}{\sigma}\frac{d^2\sigma}{de_\alpha\,de_\beta} = 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\frac{1}{e_\alpha\,e_\beta}\Theta\left(e_\alpha^\beta - e_\beta^\alpha\right)\Theta\left(e_\beta - e_\alpha\right)$$

Fixed-Order Distribution

$$\frac{1}{\sigma}\frac{d^2\sigma}{de_\alpha\, de_\beta} = 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\frac{1}{e_\alpha\, e_\beta}\Theta\left(e_\alpha^\beta - e_\beta^\alpha\right)\Theta\left(e_\beta - e_\alpha\right)$$

$$r = \frac{e_\alpha}{e_\beta}$$

$$e_\beta^\alpha < e_\alpha^\beta \quad \to \quad e_\beta < r^{\frac{\beta}{\alpha-\beta}}$$

$$e_\alpha < e_\beta \quad \to \quad r < 1$$

Measuring *r* does not regulate e_β singularity!

$$\frac{1}{\sigma}\frac{d\sigma}{dr} = 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\frac{1}{r}\int_0^{r^{\frac{\beta}{\alpha-\beta}}}\frac{de_\beta}{e_\beta}$$

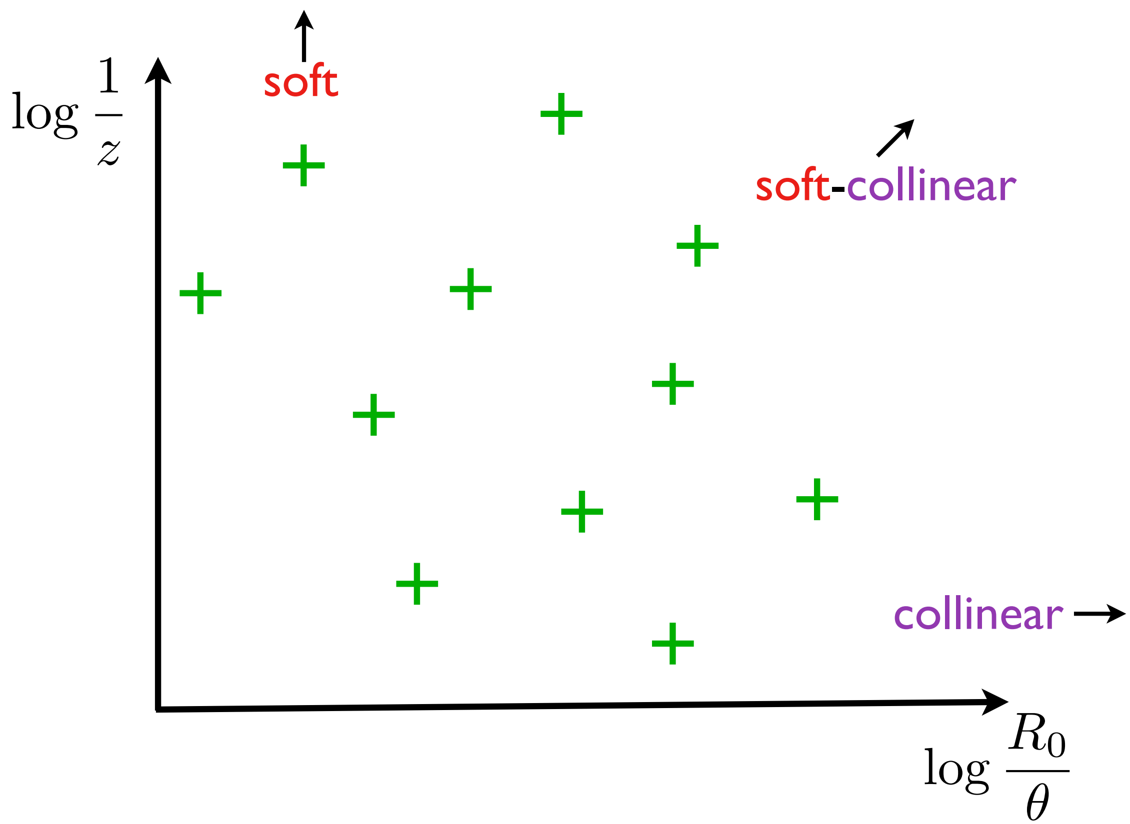
Logarithmic sensitivity to any lower bound

Ratio observable cross section undefined at fixed order:
IRC unsafe

Double-Logarithmic Distribution

Probability for an emission: $2\frac{\alpha_s}{\pi}C_F\frac{d\theta}{\theta}\frac{dz}{z} = 2\frac{\alpha_s}{\pi}C_F\,d\log\frac{1}{\theta}\,d\log\frac{1}{z}$

Emissions are uniformly distributed in the plane

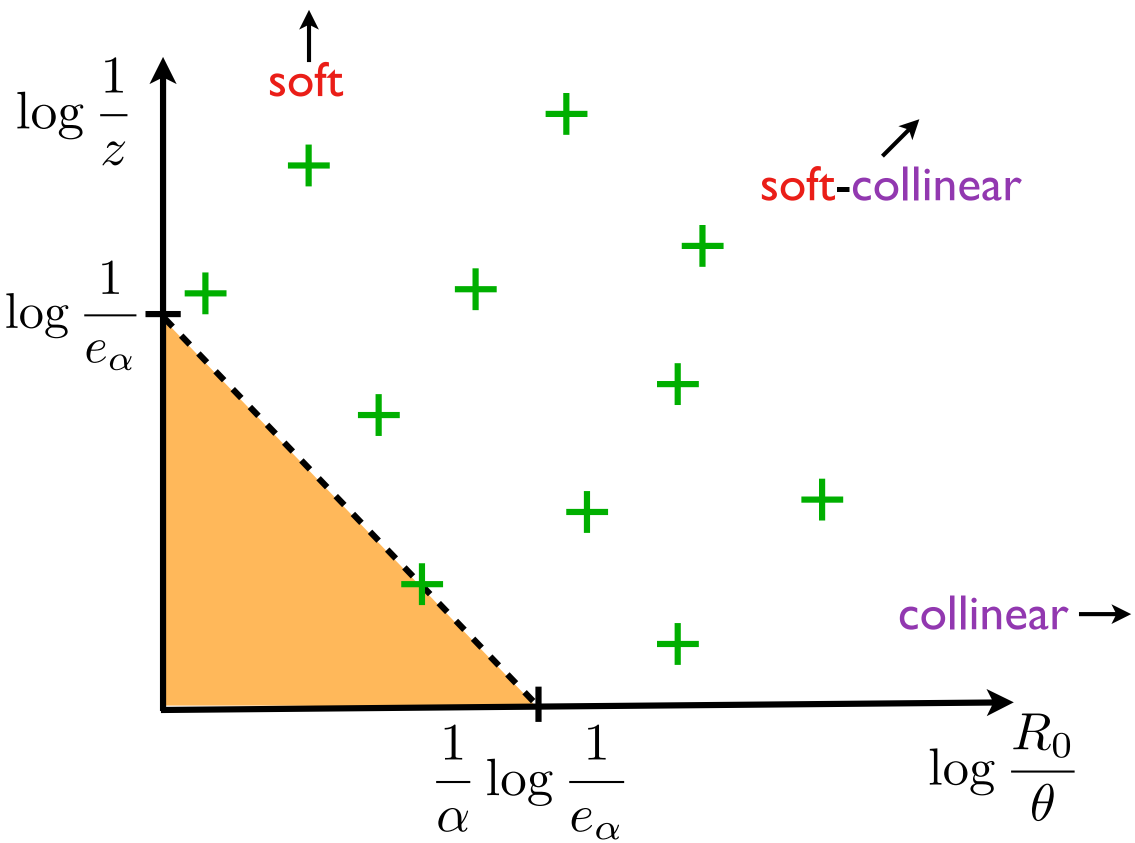


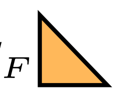
Double-Logarithmic Distribution

Probability for an emission: $2\frac{\alpha_s}{\pi}C_F\frac{d\theta}{\theta}\frac{dz}{z} = 2\frac{\alpha_s}{\pi}C_F\,d\log\frac{1}{\theta}\,d\log\frac{1}{z}$

Emissions are uniformly distributed in the plane

$$\log\frac{1}{e_\alpha} = \log\frac{1}{z} + \alpha\log\frac{R_0}{\theta}$$



$$P(x < e_\alpha) = e^{-2\frac{\alpha_s}{\pi}C_F}$$ 

$$\Sigma(e_\alpha) = e^{-\frac{\alpha_s}{\pi}\frac{C_F}{\alpha}\log^2 e_\alpha}$$

$$\frac{1}{\sigma}\frac{d\sigma}{de_\alpha} = \frac{\partial}{\partial e_\alpha}\Sigma(e_\alpha)$$

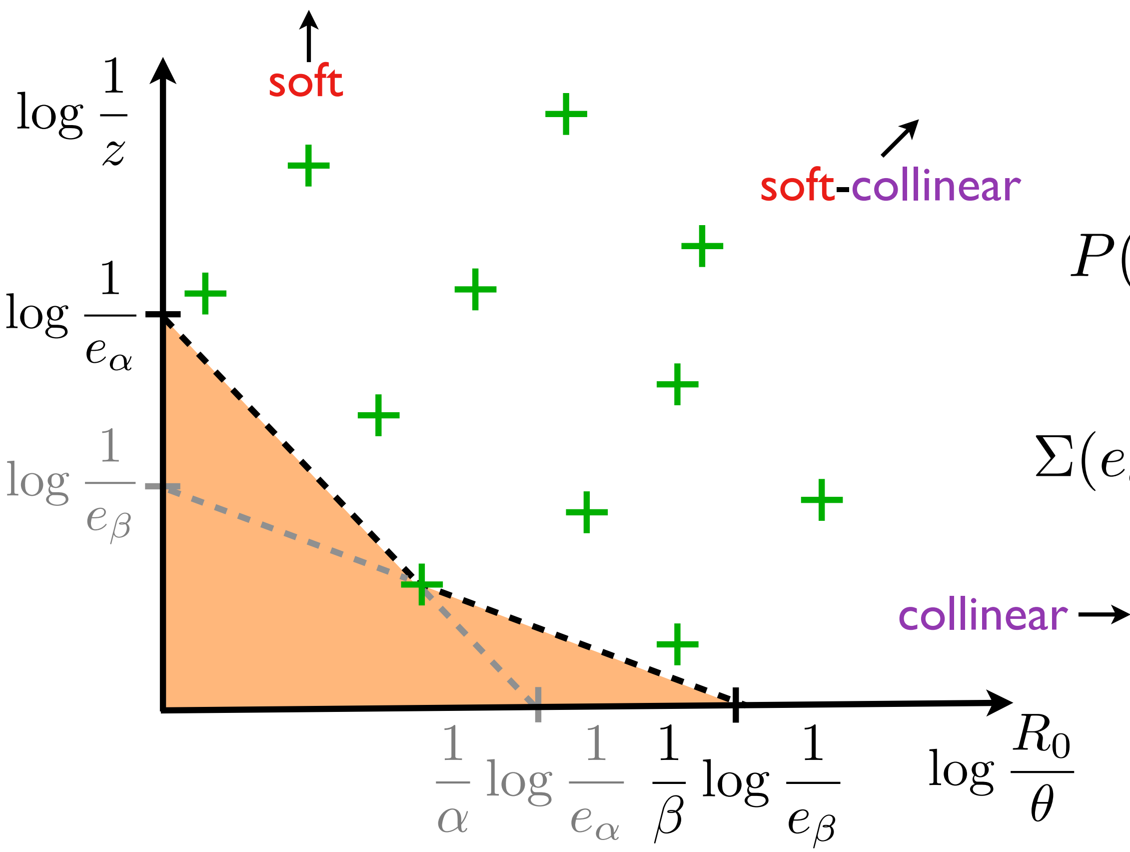
Double-Logarithmic Distribution

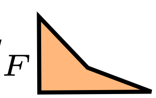
Probability for an emission: $2\frac{\alpha_s}{\pi}C_F\frac{d\theta}{\theta}\frac{dz}{z} = 2\frac{\alpha_s}{\pi}C_F\, d\log\frac{1}{\theta}\, d\log\frac{1}{z}$

Emissions are uniformly distributed in the plane

$$\log\frac{1}{e_\alpha} = \log\frac{1}{z} + \alpha\log\frac{R_0}{\theta}$$

$$\log\frac{1}{e_\beta} = \log\frac{1}{z} + \beta\log\frac{R_0}{\theta}$$



$$P(x < e_\alpha, y < e_\beta) = e^{-2\frac{\alpha_s}{\pi}C_F}$$ 

$$\Sigma(e_\alpha, e_\beta) = e^{-\frac{\alpha_s}{\pi}C_F\left(\frac{\log^2 e_\beta}{\beta} + \frac{\log^2\frac{e_\alpha}{e_\beta}}{\alpha-\beta}\right)}$$

$$\frac{1}{\sigma}\frac{d^2\sigma}{de_\alpha\, de_\beta} = \frac{\partial^2}{\partial e_\alpha\,\partial e_\beta}\Sigma(e_\alpha, e_\beta)$$

Double-Logarithmic Distribution

$$\Sigma(e_\alpha, e_\beta) = e^{-\frac{\alpha_s}{\pi} C_F \left(\frac{\log^2 e_\beta}{\beta} + \frac{\log^2 \frac{e_\alpha}{e_\beta}}{\alpha - \beta} \right)}$$

(phase space constraints suppressed)

$$\frac{1}{\sigma}\frac{d\sigma}{dr} = \int de_\alpha \, de_\beta \, \frac{\partial^2}{\partial e_\alpha \, \partial e_\beta} \Sigma(e_\alpha, e_\beta) \, \delta\left(r - \frac{e_\alpha}{e_\beta} \right)$$

$$= \frac{\sqrt{\alpha_s C_F \beta}}{\alpha - \beta} \frac{1}{r} \left(1 - 2 \frac{\alpha_s}{\pi} \frac{C_F}{\alpha - \beta} \log^2 r \right) \left(\text{erf}\left[\frac{\sqrt{\alpha_s C_F \beta}}{\sqrt{\pi}(\alpha - \beta)} \log r \right] + 1 \right) e^{-\frac{\alpha_s}{\pi} \frac{C_F}{\alpha - \beta} \log^2 r}$$

$$- 2 \frac{\alpha_s}{\pi} \frac{C_F}{\alpha - \beta} \frac{\log r}{r} e^{-\frac{\alpha_s}{\pi} C_F \frac{\alpha}{(\alpha - \beta)^2} \log^2 r}$$

Expanding in α_s:

$$\frac{1}{\sigma}\frac{d\sigma}{dr} = \sqrt{\alpha_s} \frac{\sqrt{C_F \beta}}{\alpha - \beta} \frac{1}{r} + \mathcal{O}\left((\sqrt{\alpha_s})^2 \right)$$

No Taylor expansion about $\alpha_s = 0$!

Double-Logarithmic Distribution

$$\frac{1}{\sigma}\frac{d\sigma}{dr} = \sqrt{\alpha_s}\frac{\sqrt{C_F \beta}}{\alpha - \beta}\frac{1}{r} + \mathcal{O}\left(\left(\sqrt{\alpha_s}\right)^2\right)$$

Consequences:

IRC unsafe ⇔ No Taylor series about $\alpha_s = 0$

"Sudakov safe": finite cross section with all-orders included

Observations:

Taylor series expansion about $\alpha_s \neq 0$

Can this cross section be computed with CFT techniques?

Connections:

QCD "Pink Book"

Anomalous dimension of fragmentation function moments

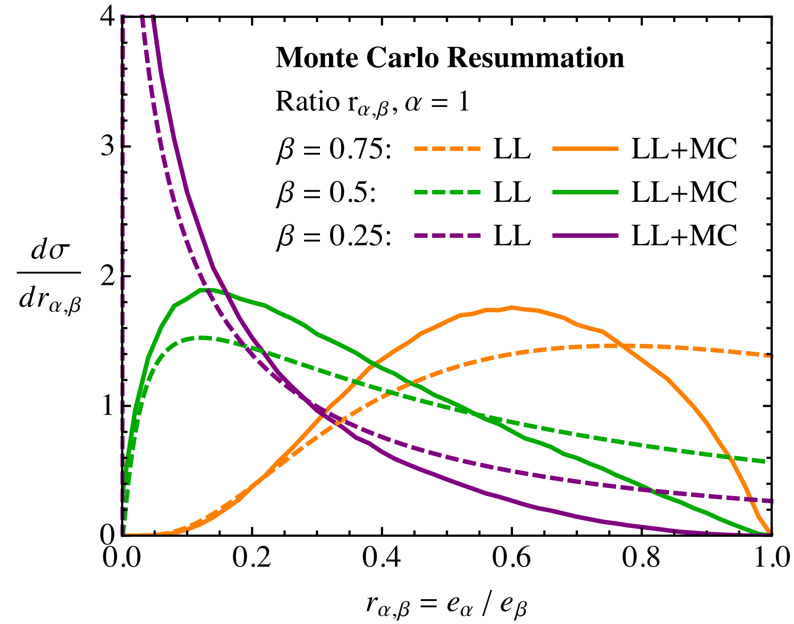
$j \neq 1$: $\gamma\left(j, \alpha_s\right) = \frac{\alpha_s C_A}{\pi}\frac{1}{j-1} + \mathcal{O}(\alpha_s^2)$ $j = 1$: $\gamma\left(j = 1, \alpha_s\right) = \sqrt{\frac{\alpha_s C_A}{2\pi}}$

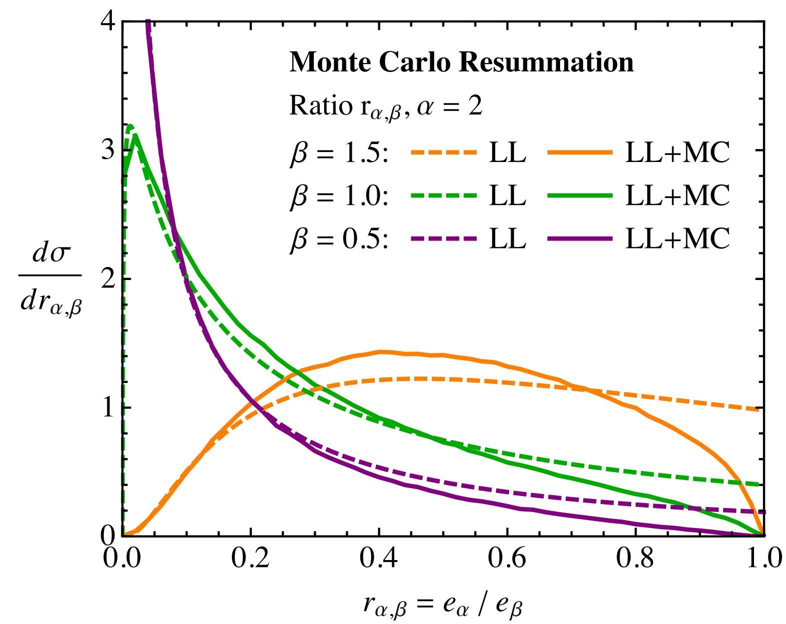
Practical Consequences of Calculability: Monte Carlos and the ratio observable

Monte Carlos approximate all-orders exclusive cross sections

Should accurately reproduce the cross section for the ratio near r = 0

Deviation for r ~ 1 where multiple emissions become important

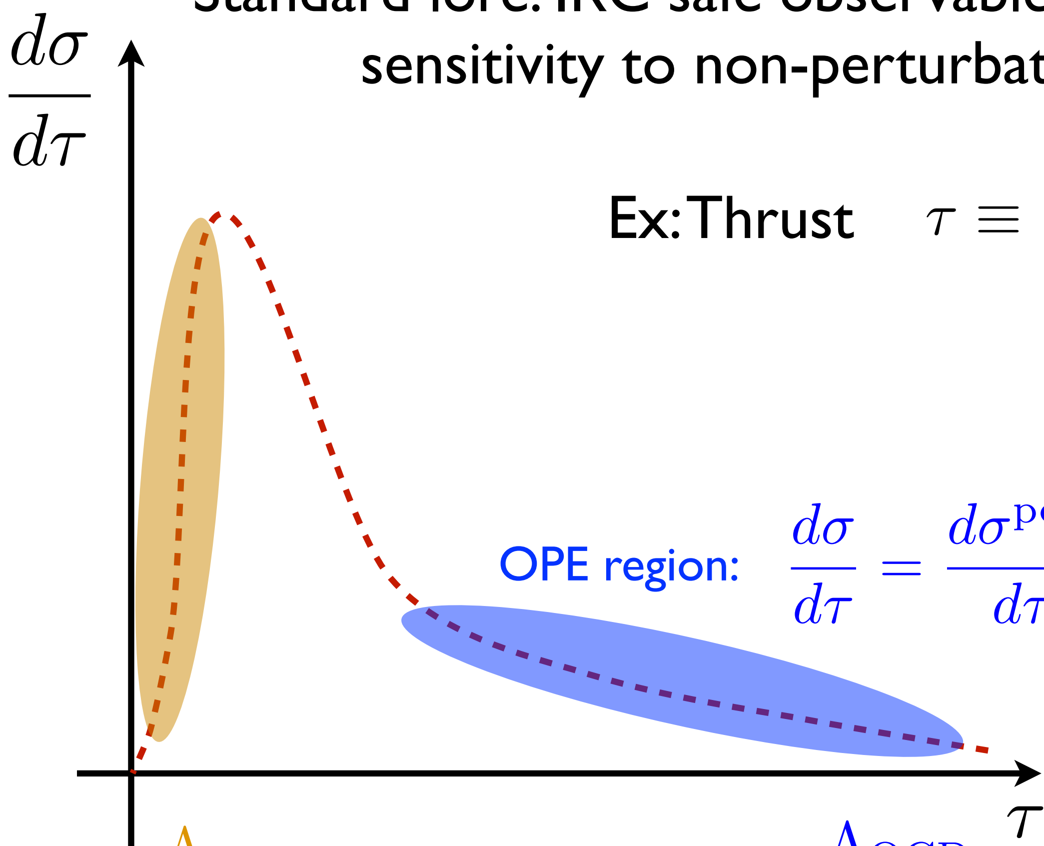




Practical Consequences of Calculability: Non-perturbative Corrections

Standard lore: IRC safe observables have controlled sensitivity to non-perturbative physics

Ex: Thrust $\tau \equiv \frac{1}{Q} \sum_i E_i \sin\theta_i \tan\frac{\theta_i}{2}$



OPE region: $\frac{d\sigma}{d\tau} = \frac{d\sigma^{\mathrm{pert}}}{d\tau} + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right)$

$\tau \sim \frac{\Lambda_{\mathrm{QCD}}}{Q}$

$\tau \gg \frac{\Lambda_{\mathrm{QCD}}}{Q}$

Dominated by non-perturbative physics

Practical Consequences of Calculability: Non-perturbative Corrections

Assume: $$\frac{d^2\sigma}{de_\alpha\, de_\beta} = \frac{d^2\sigma^{\text{pert}}}{de_\alpha\, de_\beta} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right)$$

$$e_\alpha, e_\beta \gg \frac{\Lambda_{\text{QCD}}}{Q}$$

$$\Lambda_{\text{QCD}} \ll \eta \ll Q$$

$$\frac{d\sigma}{dr} = \int_{\eta/Q} de_\alpha\, de_\beta \left[\frac{d^2\sigma^{\text{pert}}}{de_\alpha\, de_\beta} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right)\right] \delta\left(r - \frac{e_\alpha}{e_\beta}\right)$$

← NP effects are power corrections

$$+ \int^{\eta/Q} de_\alpha\, de_\beta\, \frac{d^2\sigma}{de_\alpha\, de_\beta}\, \delta\left(r - \frac{e_\alpha}{e_\beta}\right)$$

← Dominated by NP effects

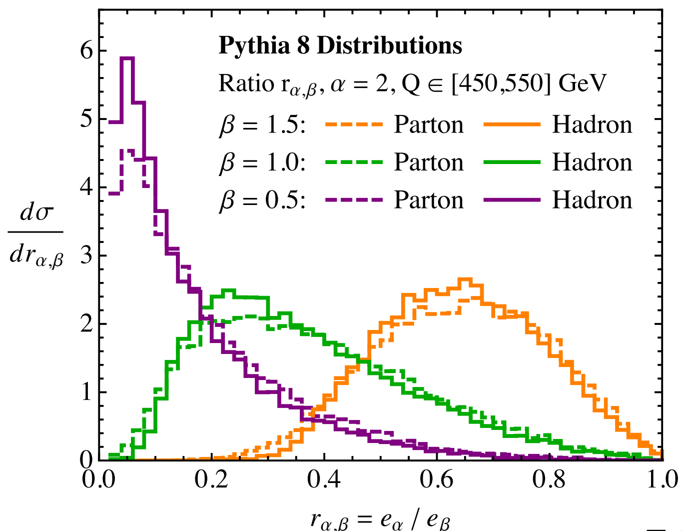
Direct non-perturbative contribution is small for: $$Q \gg \eta\, e^{\frac{\pi\sqrt{\beta}}{2\sqrt{\alpha_s}\sqrt{C_F}}} \equiv Q_{\text{Sud}}$$

Requires finite α_s

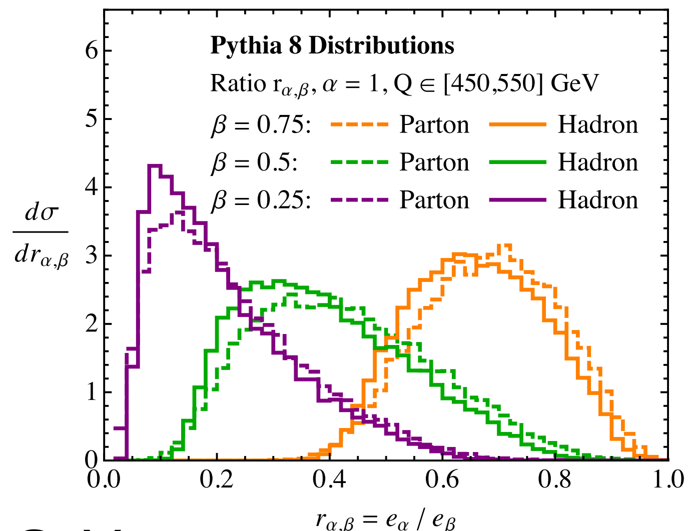
NP effects are power-suppressed at large Q for r ~ 1

For reasonable values of the parameters, Q_{Sud} ~ 250 GeV

Practical Consequences of Calculability: Non-perturbative Corrections

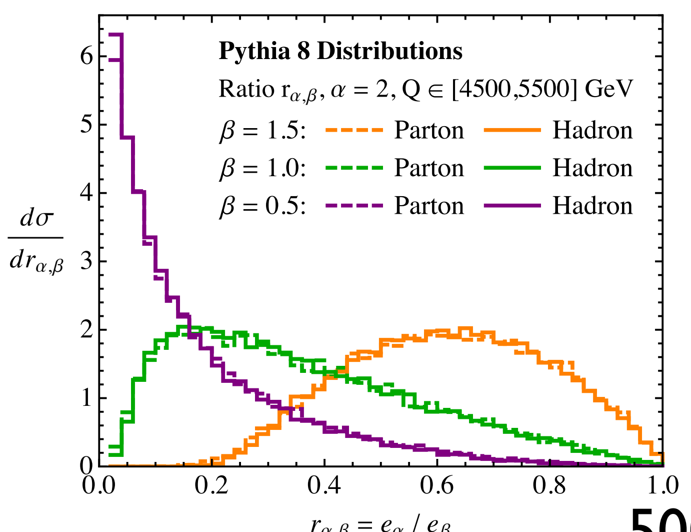


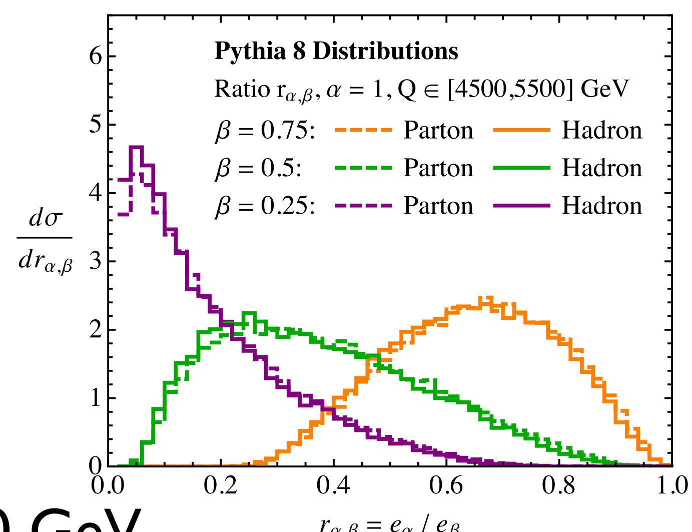
(a)



(b)

500 GeV





5000 GeV

Power-suppressed non-perturbative corrections!

Review

Ratio observables are IRC unsafe but calculable: "Sudakov safe"

Cross section series in $\alpha_s^{1/2}$

Monte Carlo should describe these observables accurately

Includes all-orders approximation to matrix element

Non-perturbative corrections are power-suppressed

Similar behavior as with IRC safe observables

Going Further

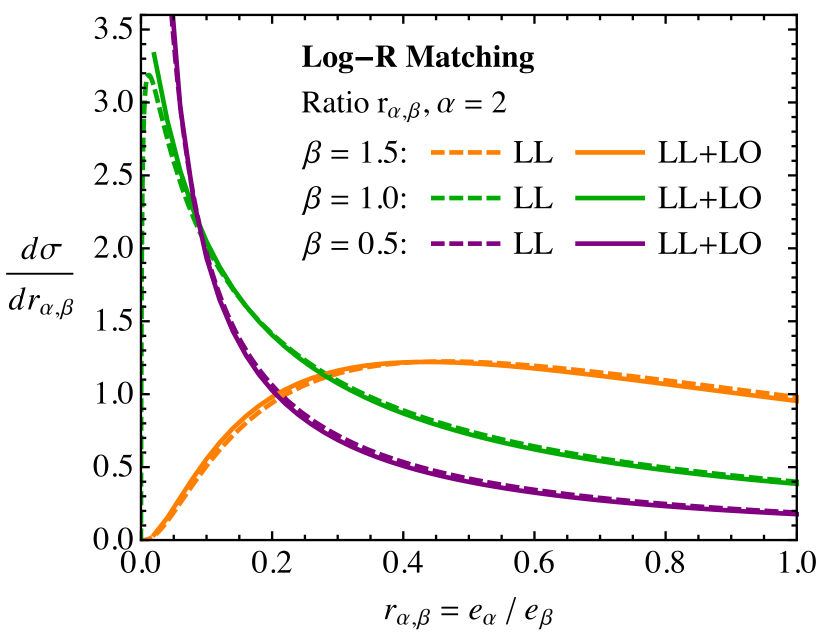
Calculating double differential cross section to higher accuracy

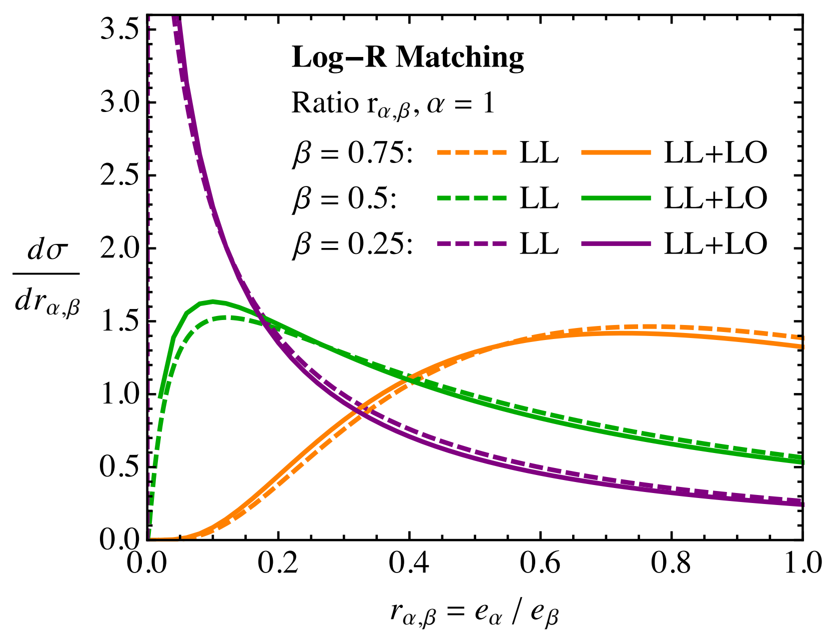
Fixed-Order Corrections

Schematic form: $$\frac{d^2\sigma^{\text{match}}}{de_\alpha\, de_\beta} = \frac{d^2\sigma^{\text{resum}}}{de_\alpha\, de_\beta} + \frac{d^2\sigma^{\text{FO}}}{de_\alpha\, de_\beta} - \frac{d^2\sigma^{\text{sing}}}{de_\alpha\, de_\beta}$$

Bulk contribution to ratio cross section

Small (power-suppressed) corrections

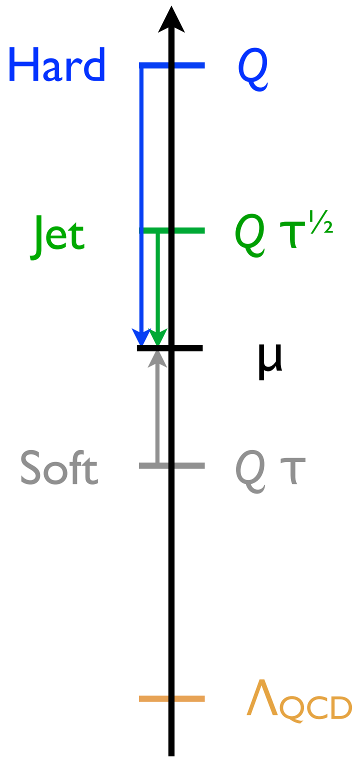




Going Further

Calculating double differential cross section to higher accuracy

Higher-order Resummation AL, Moult, Neill 2014



Factorization theorem:

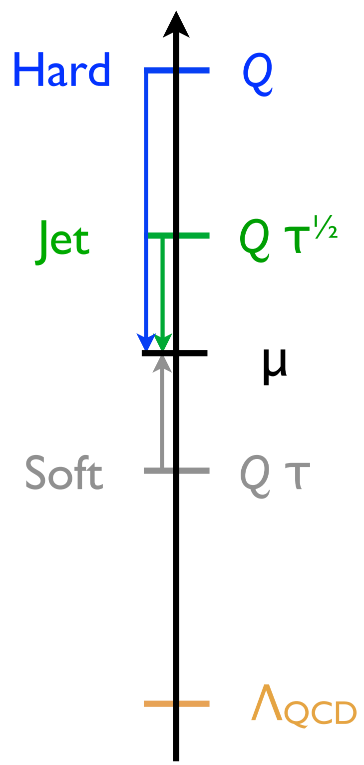
$$\frac{d\sigma}{d\tau} \simeq H(\mu) \times J(\tau,\mu) \otimes S(\tau,\mu)$$

Going Further

Calculating double differential cross section to higher accuracy

Higher-order Resummation

AL, Moult, Neill 2014



Constraints:

$$\Sigma(e_\alpha, e_\beta = e_\alpha^{\beta/\alpha}) = \Sigma(e_\alpha)$$

$$\Sigma(e_\alpha = e_\beta, e_\beta) = \Sigma(e_\beta)$$

$$\frac{\partial}{\partial e_\alpha}\Sigma(e_\alpha, e_\beta)\bigg|_{e_\beta = e_\alpha^{\beta/\alpha}} = \frac{d\sigma}{de_\alpha}$$

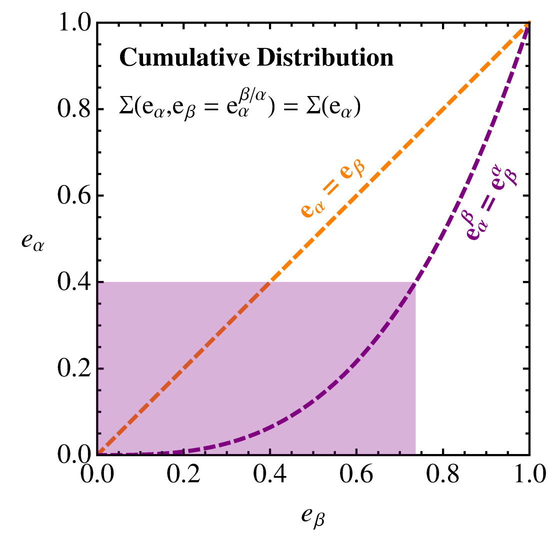
$$\frac{\partial}{\partial e_\beta}\Sigma(e_\alpha, e_\beta)\bigg|_{e_\alpha = e_\beta} = \frac{d\sigma}{de_\beta}$$

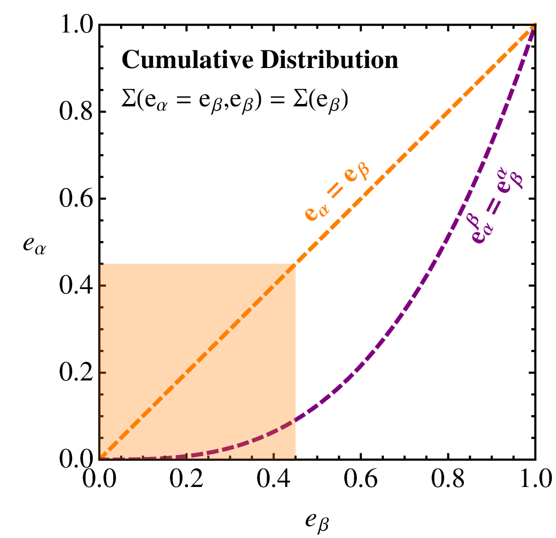
$$\Sigma(e_\alpha, e_\beta)$$

Defined by interpolating function that satisfies boundary conditions!

Unique up to $O(\alpha_s^4)$

"Holographic Factorization Theorem"





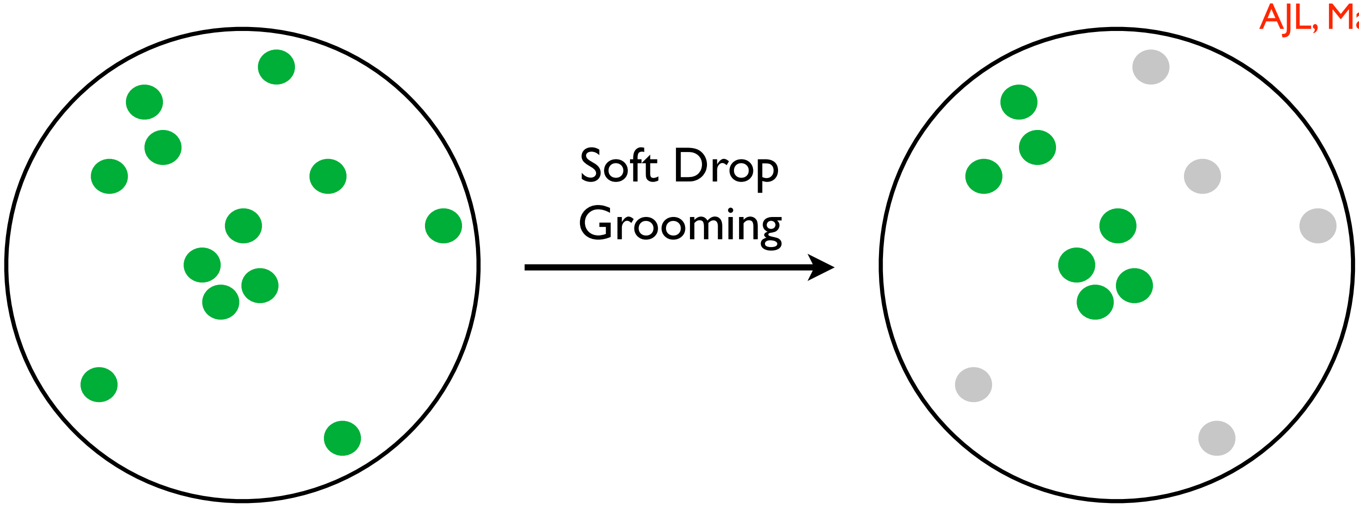
Factorization theorem:

$$\frac{d\sigma}{d\tau} \simeq H(\mu) \times J(\tau, \mu) \otimes S(\tau, \mu)$$

Going Further

Other Examples of Sudakov Safety: Groomed Energy Loss

AJL, Marzani, Soyez, Thaler 1402.soon



Original jet: E_0 Groomed jet: E_g

$$\Delta_E \equiv \frac{E_0 - E_g}{E_0}$$

$$\Sigma^{\text{energy-drop}}(\Delta_E) = \frac{\log z_{\text{cut}} - B_i}{\log \Delta_E - B_i} + \frac{\pi\beta}{2C_i\alpha_s(\log\Delta_E - B_i)^2}\left(1 - e^{-2\frac{\alpha_s}{\pi}\frac{C_i}{\beta}\log\frac{z_{\text{cut}}}{\Delta_E}\left(\log\frac{1}{\Delta_E}+B_i\right)}\right)$$

α_s expansion: $$\Sigma^{\text{energy-drop}}(\Delta_E) = 1 - \frac{\alpha_s}{\pi}\frac{C_i}{\beta}\log^2\frac{z_{\text{cut}}}{\Delta_E} + \mathcal{O}\left(\left(\frac{\alpha_s}{\beta}\right)^2\right)$$

$\beta = 0$: $$\Sigma^{\text{energy-drop}}(\Delta_E)_{\beta=0} = \frac{\log z_{\text{cut}} - B_i}{\log\Delta_E - B_i}$$

independent of α_s!?

Conclusions

What other examples of Sudakov safe observables are there?

Can these techniques be applied to observables like *N*-subjettiness?

Do we need a new definition of IRC safety/calculability in perturbation theory?

Can techniques from CFTs help in understanding these observables?

Is QCD better approximated by a free theory, a weakly coupled CFT or something else?