Perturbing the $U(1)$ Dirac Spin Liquid State in Spin-1/2 kagome

Raman scattering, magnetic field, and hole doping

Wing-Ho Ko

MIT

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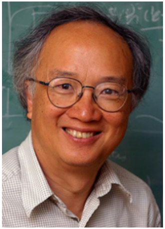
Acknowledgments

Xiao-Gang Wen



Patrick Lee

Tai-Kai Ng

Ying Ran

Zheng-Xin Liu

Outline

1. The Spin-1/2 Kagome Lattice
2. Derivation and Properties of the $U(1)$ DSL State
3. Raman Scattering
4. External Magnetic Field
5. Hole Doping

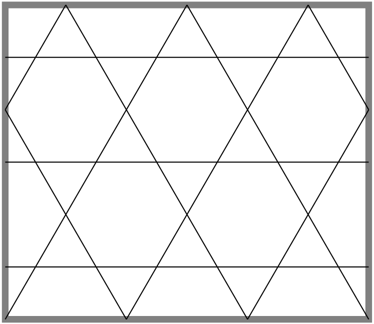
Reference: Ran, Ko, Lee, & Wen, PRL **102**, 047205 (2009)
Ko, Lee, & Wen, PRB **79**, 214502 (2009)
Ko, Liu, Ng, & Lee, PRB **81**, 024414 (2010)

Outline

The nearest-neighbor antiferromagnetic Heisenberg model is highly frustrated on the kagome lattice.

- kagome lattice = 2D lattice of corner-sharing triangles.
- Simple Néel order does not work well on the kagome lattice.
- Classical ground states: $\sum_{i \in \triangle} \mathbf{S}_i = 0$.
  - # classical ground states $\propto e^N$.

[Chalker *et al.*, PRL **68**, 855 (1992)].



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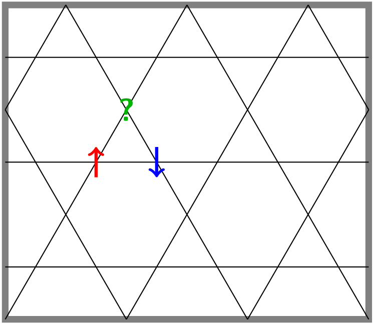
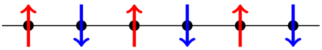


In the quantum case, singlet formation is possible and may be favored.

- 1D chain:

$$E_{\text{Néel}} = S^2 J \qquad \text{per bond}$$
$$E_{\text{singlet}} = \frac{1}{2} S(S+1) J \qquad \text{per bond}$$

- Higher dimensions $\Longrightarrow$ singlet less favorable.
- kagome is highly frustrated $\Longrightarrow$ rare opportunity of realizing a singlet ground state.



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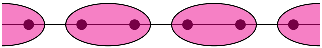

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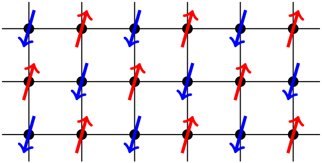

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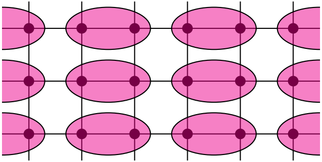

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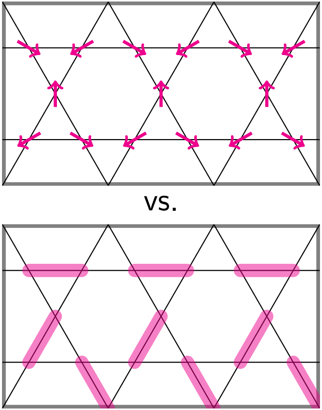
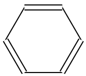


Two major classes of singlet states: valence bond solids (VBS) and spin liquid (SL).

- In a VBS, certain singlet bonds are preferred, which results in a symmetry-broken state.
- In a SL, different bond configurations superpose, which results in a state that breaks no lattice symmetry.
- A VBS state generally has a spin gap, while a SL state can be gapped or gapless.



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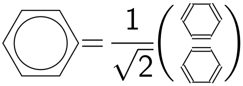
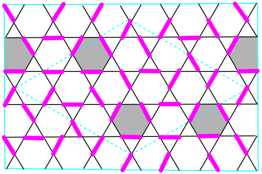


For $S = 1/2$ kagome, the leading proposals are the 36-site VBS and the U(1) Dirac spin liquid.

- The 36-site VBS pattern is found in series expansion [Singh and Huse, PRB **76**, 180407 (2007)] and entanglement renormalization [Evenbly and Vidal, arXiv:0904.3383].
- From VMC, the $U(1)$ Dirac spin liquid (DSL) state has the lowest energy among various SL states, and is stable against small VBS perturbations [e.g., Ran *et al.*, PRL **98**, 117205 (2007)].
- Exact diagonalization: initially found small ($\sim J/20$) spin gap; now leaning towards a gapless proposal [Waldtmann *et al.*, EPJB **2**, 501 (1998); arXiv:0907.4164].



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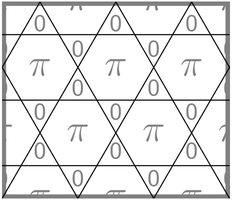
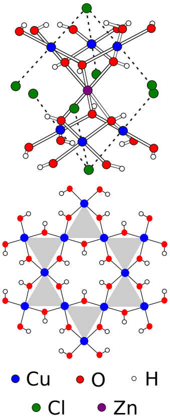


Experimental realization of $S = 1/2$ kagome: Herbertsmithite $ZnCu_3(OH)_6Cl_2$.

- Herbertsmithite: layered structure with Cu forming an AF kagome lattice.
- Caveats: Zn impurities and Dzyaloshinskii–Moriya interactions.
- Experimental Results [e.g., Helton *et al.*, PRL **98**, 107204 (2007); Bert *et al.*, JP:CS **145**, 012004 (2009)]:
  - Neutron scattering: no magnetic order down to 1.8 K.
  - μSR: no spin freezing down to 50 μK.
  - Heat capacity: vanishes as a power law as $T \to 0$.
  - Spin susceptibility: diverges as $T \to 0$.
  - NMR shift: power law as $T \to 0$.



Research motivation: Deriving further experimental consequences of the $U(1)$ DSL state.

- All experiments point to a state without magnetic order. But more data is needed to tell if it is a VBS state or a SL state, and which VBS/SL state it is.
- Without concrete theory, the experimental data are hard to interpret.
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Thus, our approach: Assume the DSL state and consider further experimental consequences.

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Deriving the $U(1)$ DSL state: Slave boson formulation

- Start with Heisenberg (or more generally t–J) model:

$$H_{tJ} = \sum_{\langle ij \rangle} J \left(\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j \right) - t \left(c^\dagger_{i\sigma} c_{j\sigma} + h.c. \right) \; ; \qquad \sum_\sigma c^\dagger_i c_i \leq 1$$

- Apply the slave boson decomposition [Lee *et al.*, RMP **78**, 17 (2006)]:

$$\mathbf{S}_i = \frac{1}{2} \sum_{\alpha,\beta} f^\dagger_{i\alpha} \boldsymbol{\tau}_{\alpha,\beta} f_{i\beta} \; ; \quad c^\dagger_{i\sigma} = f^\dagger_{i\sigma} h_i \; ; \quad f^\dagger_{i\uparrow} f_{i\uparrow} + f^\dagger_{i\downarrow} f_{i\downarrow} + h^\dagger_i h_i = 1$$

- Decouple four-operator terms by a Hubbard–Stratonovich transformation, with the following ansatz:

$$\chi_{ij} \equiv \sum_\sigma \langle f^\dagger_{i\sigma} f_{j\sigma} \rangle = \chi e^{i\alpha_{ij}} \; ; \quad \Delta_{ij} \equiv \langle f^\dagger_{i\uparrow} f_{j\downarrow} - f^\dagger_{i\downarrow} f_{j\uparrow} \rangle = 0$$

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$f_{i\beta}$: spinon; h_i: holon; Constraint enforced by Lagrange multiplier α^i_0

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Deriving the $U(1)$ DSL state: Emergent gauge field

$$H_{\mathrm{MF}} = \sum_{i\sigma} f^\dagger_{i\sigma}(i\alpha^i_0 - \mu_F) f_{i\sigma} - \frac{3\chi J}{8} \sum_{\langle ij \rangle,\sigma} (e^{i\alpha_{ij}} f^\dagger_{i\sigma} f_{j\sigma} + h.c.) + \sum_i h^\dagger_i (i\alpha^i_0 - \mu_B) h_i - t\chi \sum_{\langle ij \rangle} (e^{i\alpha_{ij}} h^\dagger_i h_j + h.c.)$$

- The α field is an emergent gauge field, corresponding to gauge symmetry $f^\dagger \mapsto e^{i\theta} f^\dagger$, $h \mapsto e^{-i\theta} h$.
- At the lattice level α is a *compact* gauge field (i.e., monopoles are allowed).
- But with Dirac fermions, the system can be in a deconfined phase (i.e., monopoles can be neglected) [Hermele *et al.*, PRB **70**, 214437 (2004)].

Deriving the $U(1)$ DSL state: Band Structure

$$H_{\mathrm{MF}} = \sum_{i\sigma} f^\dagger_{i\sigma}(i\alpha^i_0 - \mu_F) f_{i\sigma} - \frac{3\chi J}{8} \sum_{\langle ij\rangle,\sigma} (e^{i\alpha_{ij}} f^\dagger_{i\sigma} f_{j\sigma} + h.c.)$$
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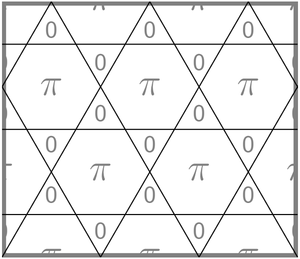
- Neglecting fluctuation of $\boldsymbol{\alpha}$, spinons and holons are decoupled.
- Mean-field ansatz for SL state can be specified by pattern of α flux.
- $U(1)$ Dirac spin liquid state: π flux per ⬡ and 0 flux per △.
- π flux $\Longrightarrow$ unit cell doubled in band structures.



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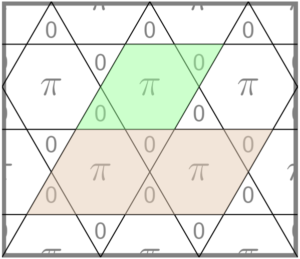

Enlarged Cell; Original Cell

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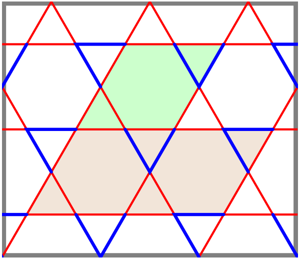
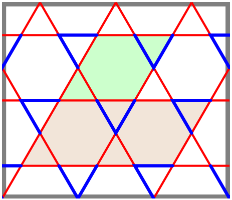


/ $t_{\rm eff} = +t$; / $t_{\rm eff} = -t$

Properties of the $U(1)$ DSL state: Band structure

Real space



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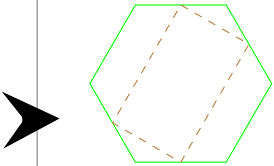
k-space



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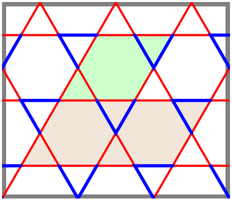

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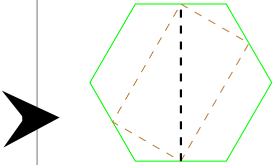

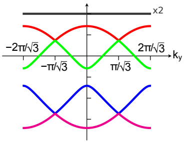

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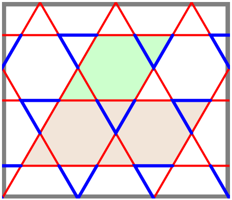

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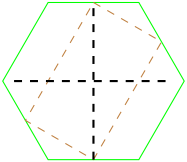

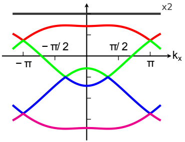

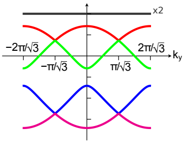

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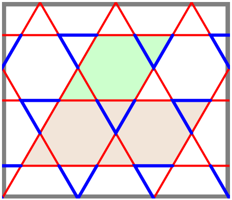

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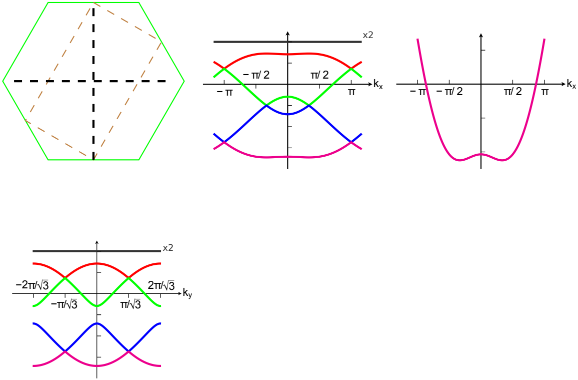
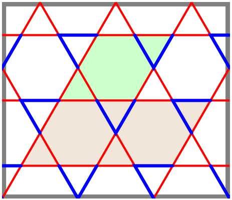


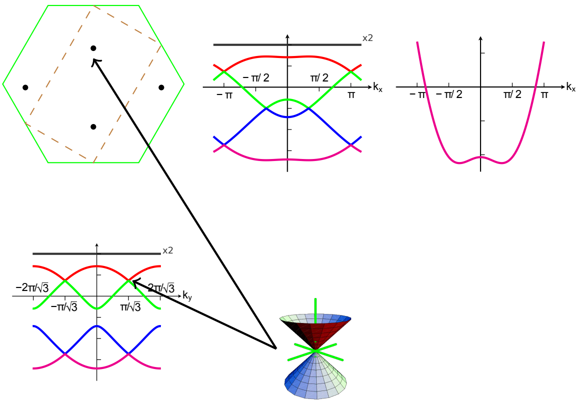
Properties of the $U(1)$ DSL state: Band structure

Real space



$t_{\text{eff}} = +t$; $t_{\text{eff}} = -t$

k-space



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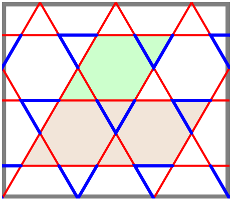

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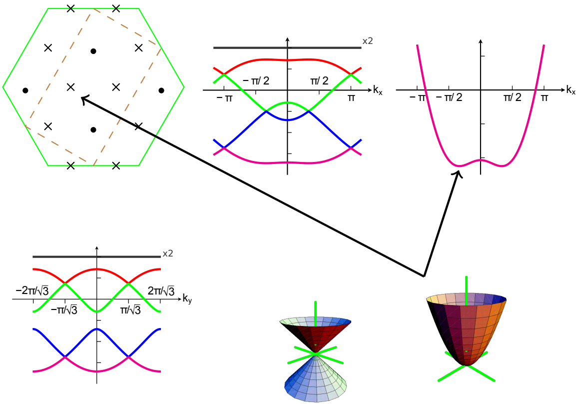
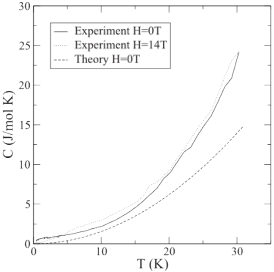


Properties of the $U(1)$ DSL state: Thermodynamics and correlation

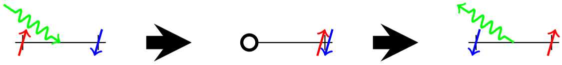
- At low energy, the $U(1)$ DSL state is described by QED_3.
  - i.e., gauge field coupled to Dirac fermions in (2+1)-D.
- Thermodynamics of the $U(1)$ DSL state is dominated by the spinon Fermi surface.
  - Zero-field spin susceptibility: $\chi(T) \sim T$.
  - Heat capacity: $C_V(T) \sim T^2$.
- $U(1)$ DSL state is "quantum critical"—many correlations decay algebraically [Hermele *et al.*, PRB **77**, 224413 (2008)].
  - Emergent SU(4) symmetry among Dirac nodes $\Longrightarrow$ different correlations can have the same scaling dimension.



Outline

Raman scattering in Mott-Hubbard system: the Shastry–Shraiman formulation

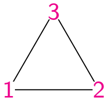
- Raman scattering = inelastic scattering of photon.
  - Good for studying excitations of the system.
  - Probe only excitations with $\mathbf{q} \approx 0$.
- We are concerned with a half-filled Hubbard system, in the regime where $|\omega_i - \omega_f| \ll U$ and $\omega_i \approx U$.
  - Both initial and final states are spin states.
    $\Longrightarrow$ T-matrix can be written in terms of spin operators.
  - Intermediate states are dominated by the sector where $\sum_i n_{i\uparrow} n_{i\downarrow} = 1$.
  - The T-matrix can be organized as an expansion in $t/(U - \omega_i)$ [Shastry & Shraiman, IJMPB **5**, 365 (1991)].

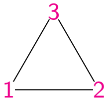


$$T^{(2)} \sim \frac{t^2}{U-\omega_i} \mathbf{S}_i \cdot \mathbf{S}_j + \dots$$

Spin-chirality terms in the Shastry–Shraiman formulation

- Because of holon-doublon symmetry, there is no t^3 order contribution.





- For the square and triangular lattice, there is no t^4 order contribution to spin-chirality because of a non-trivial cancellation between 3-site and 4-site pathways.

- But such cancellation is absent in the kagome lattice.

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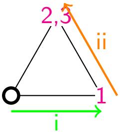

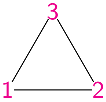

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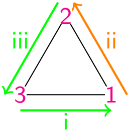

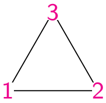

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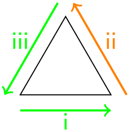

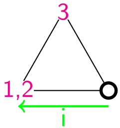

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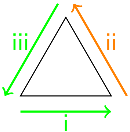

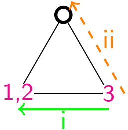

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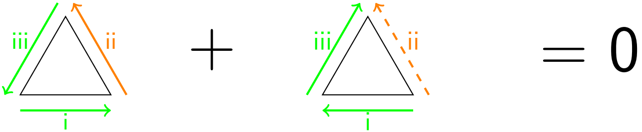

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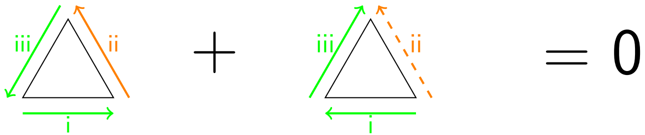

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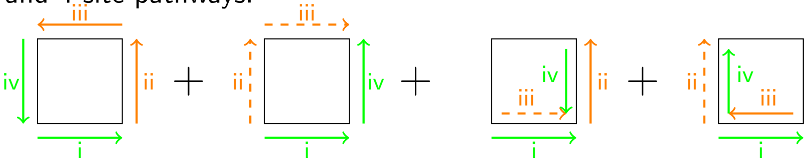

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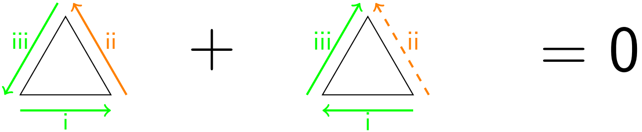

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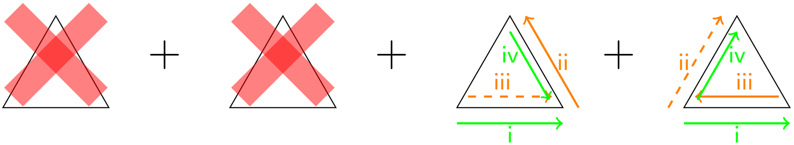

$$\sim \frac{t^4}{(U-\omega_i)^3} \mathbf{S}_i \cdot \mathbf{S}_j \times \mathbf{S}_k + \ldots$$

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Raman T-matrix for the kagome geometry

- For kagome geometry, the Raman T-matrix decomposes into 3 irreps:

$$T = T(A_{1g})(\bar{x}x+\bar{y}y)+T(E_g^{(1)})(\bar{x}x-\bar{y}y)+T(E_g^{(2)})(\bar{x}y+\bar{y}x)+T(A_{2g})(\bar{x}y-\bar{y}x)$$

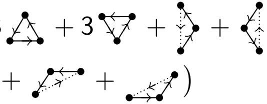
- To lowest order in inelastic terms,

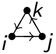
$$T(E_g^{(1)}) = \frac{4t^2}{\omega_i-U}\frac{1}{4}\left(\sum_{\langle ij\rangle,/}+\sum_{\langle ij\rangle,\backslash}-2\sum_{\langle ij\rangle,-}\right)\mathbf{S}_i\cdot\mathbf{S}_j$$

$$T(E_g^{(2)}) = \frac{4t^2}{\omega_i-U}\frac{\sqrt{3}}{4}\left(\sum_{\langle ij\rangle,/}-\sum_{\langle ij\rangle,\backslash}\right)\mathbf{S}_i\cdot\mathbf{S}_j$$

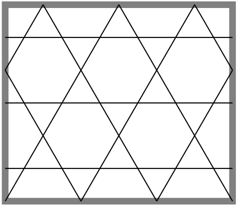
$$T(A_{1g}) = \frac{-t^4}{(\omega_i-U)^3}\left(2\sum_{\langle\langle\langle ij\rangle\rangle\rangle}+\sum_{\langle\langle ij\rangle\rangle}\right)\mathbf{S}_i\cdot\mathbf{S}_j$$

$$T(A_{2g}) = \frac{2\sqrt{3}it^4}{(\omega_i-U)^3}\sum_R \Big(3\ \cdots + 3\ \cdots + \cdots + \cdots + \cdots + \cdots + \cdots + \cdots\Big)$$



(

$= \mathbf{S}_i\cdot\mathbf{S}_j\times\mathbf{S}_k$, etc.)



Raman signals: spinon-antispinon pairs and gauge mode

- $\mathbf{S}_i \cdot \mathbf{S}_j \sim f^\dagger f \, f^\dagger f$ and $\mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k) \sim f^\dagger f \, f^\dagger f \, f^\dagger f$

  $\Longrightarrow$ contributions from spinon-antispinon pairs.

  $\Longrightarrow$ continuum of signal $I_\alpha(\Delta\omega) = \overline{\left|\langle f|O_\alpha|i\rangle\right|^2}\,\mathrm{DOS}(\Delta\omega)$.
  - At low energy, one-pair states dominates.
  - For Dirac node, $\mathrm{DOS}_{1pair} \sim \mathcal{E}$, and matrix element is suppressed in E_g and A_{1g}, but not in A_{2g}.
    $\Longrightarrow$ Spinon-antispinon pairs contribute $I_{A_{2g}}(\Delta\omega) \sim \mathcal{E}$ and $I_{E_g/A_{1g}}(\Delta\omega) \sim \mathcal{E}^3$ at low energy.

- However, an additional collective excitation is available in A_{2g}:

  $\mathbf{S}_i \cdot \mathbf{S}_j \times \mathbf{S}_k \sim i\chi^3 \exp(i \oint_\triangle \alpha \cdot d\ell) + h.c. \sim \chi^3 \iint_\triangle b\, d^2\mathbf{x}$

  $\Longrightarrow$ $I_{A_{2g}} \sim \langle \Phi_b \Phi_b \rangle + \ldots \sim q^2 \langle \alpha\alpha \rangle + \ldots$

  (Recall that $\langle f_i^\dagger f_j \rangle \sim \chi \exp(i\alpha_{ij})$)
  - In our case (QED$_3$ with Dirac fermions), turns out that $\langle \alpha\alpha \rangle \sim 1/\omega$ when $\mathbf{q} \approx 0$ [Ioffe & Larkin, PRB **39**, 8988 (1989)].

- Analogy: plasmon mode vs. particle-hole continuum in normal metal.

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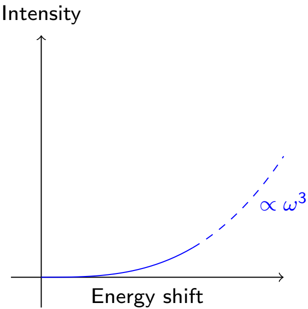
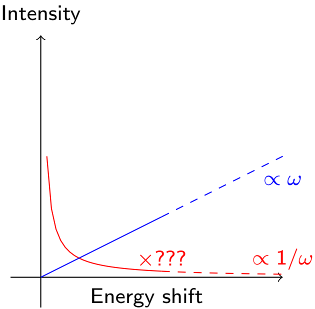
A_{1g}, E_g

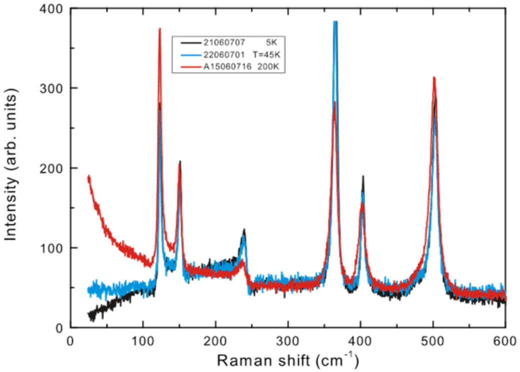


A_{2g}



Experimental Comparisons: Some qualitative agreements

- Wulferding and Lemmens [unpublished] have obtained Raman signal on herbertsmithite.
  - At low T, data shows a broad background with a near-linear piece at low-energy.
  - Roughly agree with the theoretical picture presented previously.



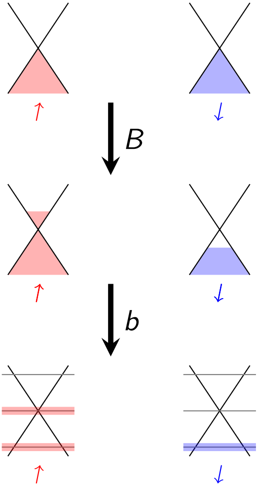
Outline

External magnetic field and the formation of Landau levels

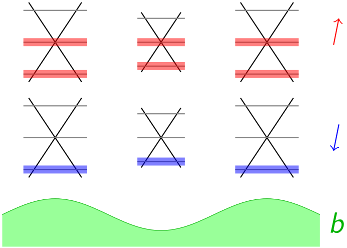
- In Mott systems, B-field causes only Zeeman splitting.
  - This induces spinon and antispinon pockets near the Dirac node.
- However, with the emergent gauge field α, Landau levels can form spontaneously.
- From VMC calculations,

$$\Delta e_{FP}^{Prj} \approx 0.33(2)B^{3/2} + 0.00(4)B^2$$
$$\Delta e_{LL}^{Prj} \approx 0.223(6)B^{3/2} + 0.03(1)B^2$$

$\Delta e_{MF}^{LL} < \Delta e_{MF}^{FP}$ to leading order in Δn
$\implies$ Landau level state is favored.



S_z density fluctuation as gapless mode



- b is an emergent gauge field $\implies$ its strength can fluctuate in space.
- The fluctuation of b is tied to the fluctuation of S_z density.
- In long-wavelength limit, energy cost of b fluctuation $\to 0$ $\implies$ The system has a gapless mode!
  - Derivative expansion $\implies$ linear dispersion.
- Other density fluctuations and quasiparticle excitations are gapped $\implies$ gapless mode is unique.
- Mathematical description given by Chern–Simons theory.

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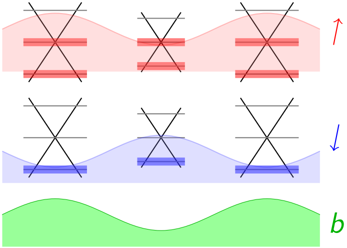

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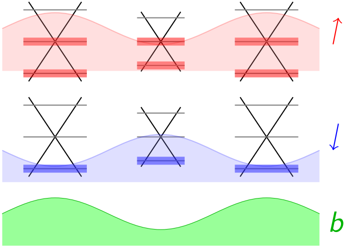

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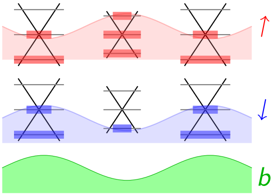

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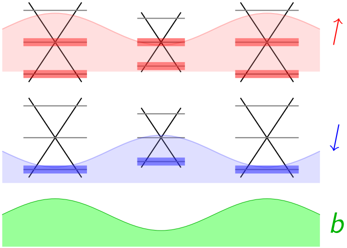

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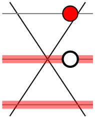
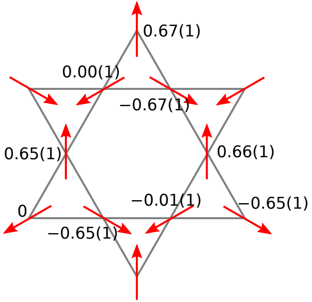


Gapless mode and XY-ordering

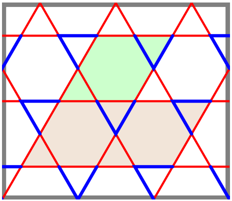
- Recap: we found a single linearly-dispersing gapless mode, which looks like...
- A Goldstone boson! And indeed it is.
  - Corresponding to this Goldstone mode is a spontaneously broken XY order.
- Analogy:
  - Superfluid: $\hat{\psi} = \sqrt{\hat{\rho}} e^{-i\hat{\theta}}$, $[\hat{\rho}, \hat{\theta}] = i$, gapless ρ fluctuation $\implies$ ordered (SF) phase;
  - XY model: $\hat{S^+} = e^{i\hat{\theta}}$, $[\hat{S}_z, \hat{\theta}] = i$, gapless S_z fluctuation $\implies$ XY ordered phase.
- VMC found the $\mathbf{q} = \mathbf{0}$ order.
- S^+ in XY model $\sim V^\dagger$ in QED$_3$ $\implies$ in-plane magnetization $M \sim B^\gamma$.
  ($V^\dagger$ monopole operator, γ its scaling dimension)



Outline

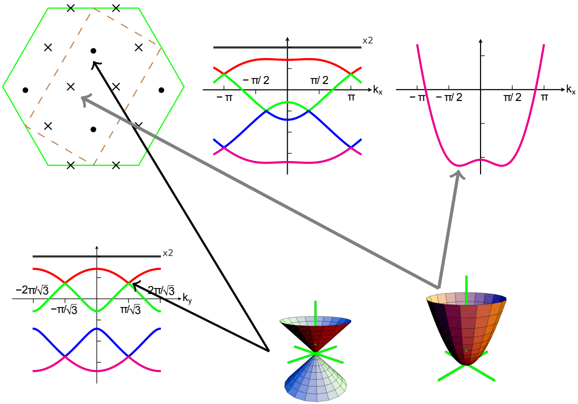
Recap: Band structure

Real space



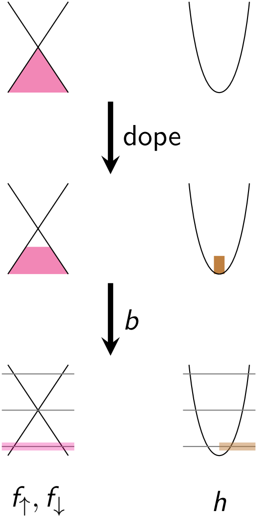
$t_{\text{eff}} = +t$; $t_{\text{eff}} = -t$

k-space



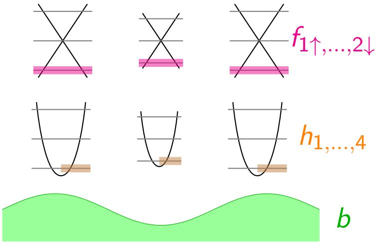
Hole doping and formation of Landau levels

- Doping can in principle be achieved by substituting Cl with S.
- In slave-boson picture, hole doping introduces holons and antispinons.
- As before, an emergent b field can open up Landau levels in the spinon and holon bands.
  - At mean-field, $\Delta E_{\text{spinon}} \sim -B^{3/2}$ while $\Delta E_{\text{holon}} \sim B^2$
    $\Longrightarrow$ LL state favored.
  - At mean-field, b optimal when antispinons form $\nu = -1$ LL state
    $\Longrightarrow$ holons form $\nu = 1/2$ Laughlin state.



Charge fluctuation, Goldstone mode, and superconductivity

- b fluctuation $\sim$ holon density fluctuation $\sim$ charge density fluctuation.
- Long-wavelength b fluctuation cost $\mathcal{E} \searrow 0$ if real EM-field is "turned off."
  $\implies$ a single linearly-dispersing mode $\sim$ Goldstone boson.
- This time the Goldstone boson is eaten up by the EM-field to produce a superconductor.
  - This superconducting state breaks T-invariance.
- Intuitively, four species of holon binded together $\implies$ expects $\Phi_{\mathrm{EM}} = hc/4e$ for a minimal vortex.
  - Confirmed by Chern–Simons formulation.



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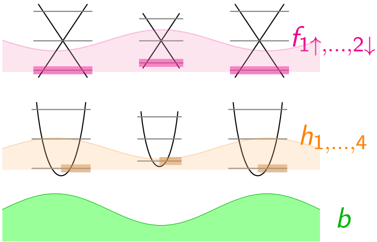
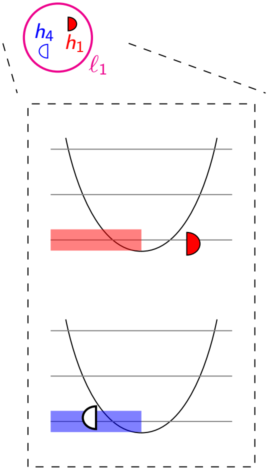


Quasiparticle Statistics—Intuitions

- Quasiparticles are bound states of semions in holon sectors and/or fermions in spinon sector.
  - For finite energy, bound states must be neutral w.r.t. the gapless mode.
- There are two types of "elementary" quasiparticles:
  1. semion-antisemion pair in holon sector;
  2. spinon-holon pair ($\sim$ Bogoliubov q.p. in conventional SC).
- All other quasiparticles can be built from these elementary ones.
- Statistics can be derived by treating different species as uncorrelated.
  - Semions from holon sector $\Longrightarrow$ existence of semionic (mutual) statistics.



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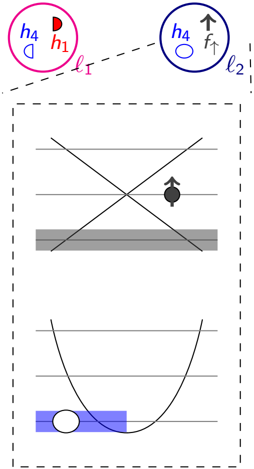

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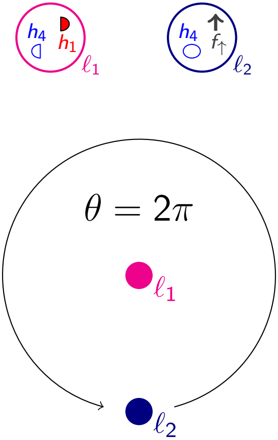

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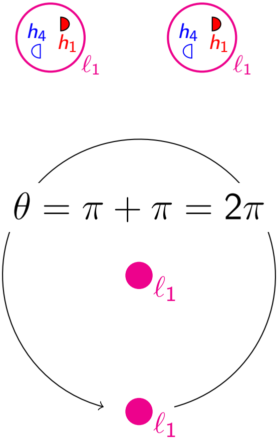

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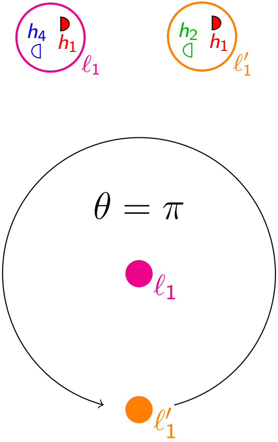
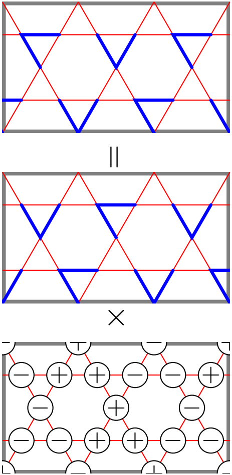


Crystal momenta of quasiparticle—projective symmetry group study

- For spinon-holon pair, **k** is well-defined on the original Brillouin zone.
  - These can be recovered using Projective symmetry group (PSG).

- In contrast, the semion is fractionalized from a holon.
  $\Longrightarrow$ semion-antisemion pair may not have well-defined **k**.



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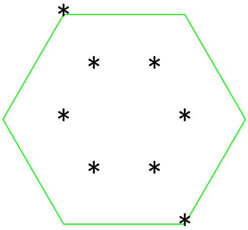


$*$ = **k** of spinon-holon pairs

Conclusions

- The $U(1)$ Dirac spin-liquid state possess many unusual properties and may be experimentally realized in herbertsmithite.
- The Raman signal of the DSL state has a broad background (contributed by spinon-antispinon continuum) and a $1/\omega$ singularity (contributed by collective [gauge] excitations).
- Under external magnetic field, the DSL state forms Landau levels, which corresponds to a XY symmetry broken state with Goldstone boson corresponding to S_z density fluctuation.
- When the DSL state is doped, an analogous mechanism give rise to an Anderson–Higgs scenario and hence superconductivity.
  - But minimal vortices carry $hc/4e$ flux and the system contains exotic quasiparticle having semionic mutual statistics.

Reference: Ran, Ko, Lee, & Wen, PRL **102**, 047205 (2009)
Ko, Lee, & Wen, PRB **79**, 214502 (2009)
Ko, Liu, Ng, & Lee, PRB **81**, 024414 (2010)

Appendix

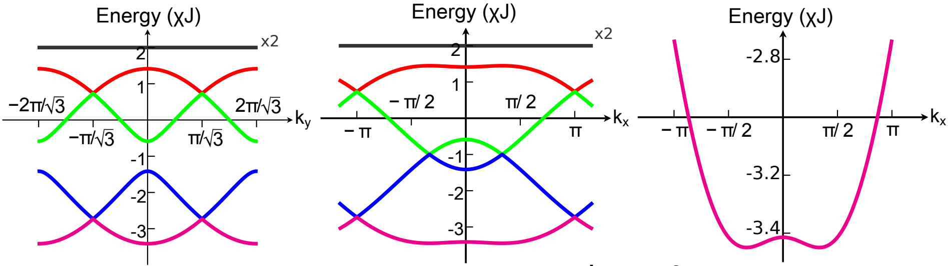
(a.k.a. hip pocket slides)

Comparison of ground-state energy estimate

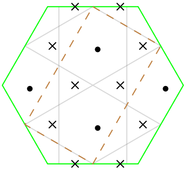
Method	Max. Size	Energy	State	Ref.
Exact Diag.	36	-0.43	—	[1]
DMRG	192	$-0.4366(7)$	SL	[2]
VMC	432	$-0.42863(2)$	U(1) Dirac SL	[3]
Series Expan.	—	$-0.433(1)$	36-site VBS	[4]
Entang. Renorm.	—	-0.4316	36-site VBS	[5]

[1] Waldtmann *et al.*, EPJB **2**, 501 (1998)
[2] Jiang *et al.*, PRL **101**, 117203 (2008)
[3] Ran *et al.*, PRL **98**, 117205 (2007)
[4] Singh and Huse, PRB **76**, 180407 (2007)
[5] Evenbly and Vidal, arXiv:0904.3383

Magnified band structure of DSL state, with scales

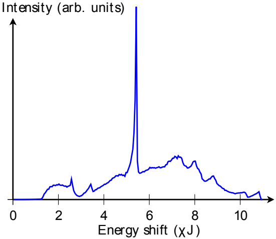


$k_x = 0$ $k_y = 0$

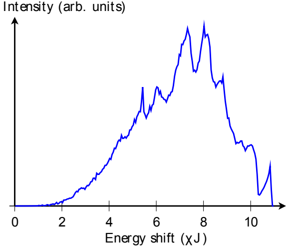


Brillouin zone

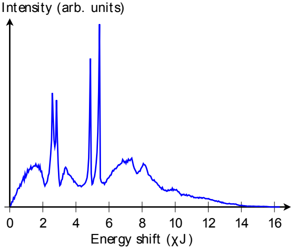
Raman signals contributed by spinon-antispinon: full scale



E_g



A_{1g}



A_{2g}

$(\chi J \approx 56 \text{ cm}^{-1})$

Chern–Simons description: B-field case

- For the B-field case, introduce two species of gauge fields to describe the current of up/down spins:

$$J_{\pm}^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_{\nu}a_{\pm,\lambda}$$

- The Lagrangian in terms of α and $a_{\pm}$:

$$\mathcal{L} = \pm\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_{\pm,\mu}\partial_{\nu}a_{\pm,\lambda} + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\alpha_{\mu}\partial_{\nu}a_{\pm,\lambda} + \dots$$

  - higher derivative terms (e.g., Maxwell term $\sim \partial a \partial a$ for $a_{\pm}$) omitted
  - $\mathcal{L}$ yields the correct equation of motion $J_{\pm}^{\mu} = \mp\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_{\nu}\alpha_{\lambda}$
- Let $\mathbf{c} = (\alpha, a_+, a_-)^T$, can rewrite $\mathcal{L}$ as

$$\mathcal{L} = -\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\mathbf{c}_{\mu}^{T}K\partial_{\nu}\mathbf{c}_{\lambda} + \dots.$$

  - K has one null vector $c_0 \sim$ gapless mode argued previously.
  - Dynamics of c_0 is driven by Maxwell term $\Longrightarrow$ linearly dispersing.

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where $\mathbf{c} = [\alpha; a_1, \dots a_4, a_5, a_6; b_1, \dots b_4]$

spinon spinon* holon

  - K describes the self-dynamics of the system; has null vector $\mathbf{p}_0 = [2; -2, \dots, -2, 2, 2; 1, \dots, 1]$ corresponding to gapless mode c_0.
  - $\mathbf{q} = [0; 0 \dots 0, 0, 0; 1, \dots, 1]$ is the "charge vector."
  - ℓ is an integer vector with 0 α-component and characterizes vortices.
- Varying $\mathcal{L}$ w.r.t. c_0 gives $B = -\frac{2\pi}{e}\frac{\ell\cdot\mathbf{p}_0}{\mathbf{q}\cdot\mathbf{p}_0}j_V^0 = -\frac{2n\pi}{4e}j_V^0$

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Quasiparticles and their statistics

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- Vortices with $\boldsymbol{\ell}\cdot\mathbf{p}_0 = 0$ does not couple to c_0
  $\implies$ They can exist when $B = 0$ and corresponds to quasiparticles.
- When particle $\boldsymbol{\ell}_1$ winds around another particle $\boldsymbol{\ell}_2$, the statistical phase $\theta = 2\pi\boldsymbol{\ell}_1^T K_\perp^{-1}\boldsymbol{\ell}_2$
  - $K_\perp$ is part of K that's $\perp$ $\mathbf{p}_0$.
  - Derived by integrating out all gauge fields having non-zero Chern–Simons term.
- Taking $\begin{cases} \boldsymbol{\ell}_1 & = [0;0,\dots,0,0,0;1,0,0,-1] \\ \boldsymbol{\ell}_2 & = [0;0,\dots,0,0,0;1,0,-1,0] \end{cases}$, found:

$$\theta_{11} = \theta_{22} = 2\pi\ , \qquad \theta_{12} = \pi$$

$\implies$ Fermions with semionic statistics!

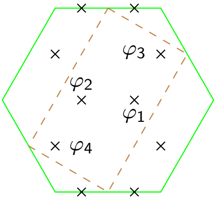
The full form of K-matrix for doped case

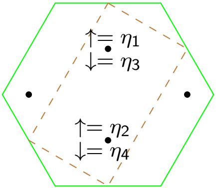
$$K = \left(\begin{array}{c|cccc|cc|cccc}
0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ \hline
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{array}\right)$$

(Recall $\mathbf{c} = [\alpha; a_1, \ldots a_4, a_5, a_6; b_1, \ldots b_4]$)

spinon spinon* holon

Spinon and holon PSG





$$T_x[\varphi_1^2] = \varphi_4^2$$
$$T_x[\varphi_2^2] = \varphi_3^2$$
$$T_x[\varphi_3^2] = e^{-\frac{i\pi}{3}}\varphi_2^2$$
$$T_x[\varphi_4^2] = e^{\frac{i\pi}{3}}\varphi_1^2$$

$$T_x[\eta_1] = e^{\frac{i\pi}{12}}\eta_2$$
$$T_x[\eta_2] = e^{\frac{11i\pi}{12}}\eta_1$$
$$T_x[\eta_3] = e^{\frac{i\pi}{12}}\eta_4$$
$$T_x[\eta_4] = e^{\frac{11i\pi}{12}}\eta_3$$