From Navier-Stokes to Einstein

1006.1902, 1101.2451;
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History of Fluid/Gravity Duality: Membrane Paradigm

- Began with prescient thesis of Damour in 1978
- Consider fluctuations of a black hole horizon; these act like a viscous fluid
- Fluid viscosity is computed to be $\eta = \frac{1}{16\pi G}$
- Dividing by entropy density $s = \frac{1}{4G}$ gives $\frac{\eta}{s} = \frac{1}{4\pi}$
- Always considers fluctuations at the black hole horizon $r = r_h$ itself; produces Damour-Navier-Stokes equation

History of Fluid/Gravity Duality: AdS/CFT Method

- Policastro, Son, and Starinets (2001) considered the hydrodynamics of $\mathcal{N} = 4 \; U(N)$ SYM via AdS/CFT

- Again produces $\frac{\eta}{s} = \frac{1}{4\pi}$

- Performed at AdS spatial infinity $r = \infty$

- Result requires string theory, SUSY gauge theory, and AdS/CFT

- Initially $\frac{\eta}{s} = \frac{1}{4\pi}$ appeared to be a bound on the viscosity to entropy ratio; however higher derivative corrections actually break this bound (e.g. Kats, Petrov, Buchel, Myers, Sinha, Cremonini, Brigante, Liu, Shenker, Yaida, Cai, Nie, Ohta, Sun, Banerjee, Dutta, Paulos, Escobedo, Smolkin, Dasgupta, Mia, Gale, Jeon ...)

A "Wilsonian" Approach

We want to relate solutions of the incompressible Navier-Stokes equation

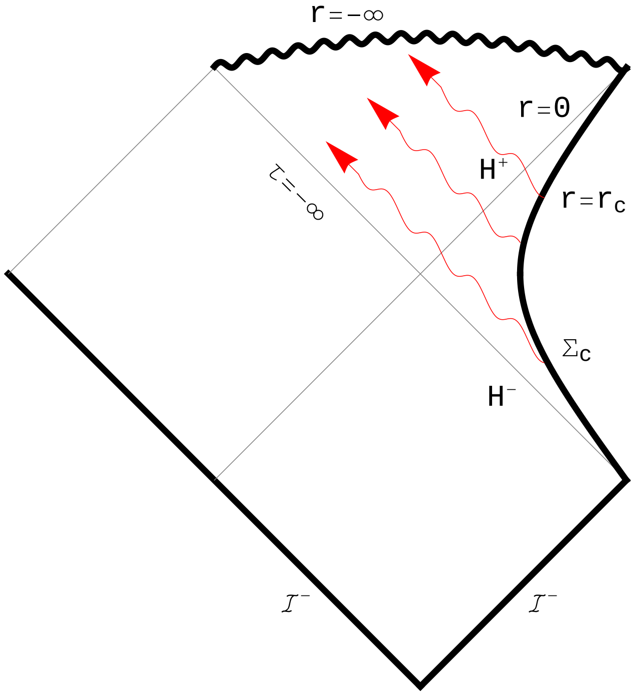
$$\partial^i v_i = 0, \qquad \partial_\tau v_i - \bar{\eta}\partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0$$

to those of the Einstein equation:

$$G_{\mu\nu} = 0,$$

We fix a cutoff surface at $r=r_c$, and consider experiments done there.

- induced metric at $r=r_c$ is Ricci flat
- waves are infalling at $r=r_h$
- extrinsic curvature at $r=r_c$ becomes fluid stress tensor



The Hydrodynamic Limit

Consider a solution of the incompressible Navier-Stokes equation $(v_i^\epsilon, P^\epsilon)$.

Now, rescale:

$$v_i^\epsilon(x^i, \tau) = \epsilon v_i(\epsilon x^i, \epsilon^2 \tau),$$

$$P_i^\epsilon(x^i, \tau) = \epsilon^2 P(\epsilon x^i, \epsilon^2 \tau).$$

These new quantities solve

$$\partial_\tau v_i^\epsilon - \overline{\eta} \partial^2 v_i^\epsilon + \partial_i P^\epsilon + v^{\epsilon j} \partial_j v_i^\epsilon = 0.$$

which is again just the N-S equation.

To produce the hydrodynamic limit, we take $\epsilon \to 0$.
This procedure will remove any corrections to N-S, as well as inducing incompressibility.

The Nonlinear Metric in the Hydrodynamic Limit

We consider the metric

$$\begin{aligned} ds^2_{p+2} =& - r d\tau^2 + 2 d\tau dr + dx_i dx^i \\ & - 2\left(1 - \frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\ & + \left(1 - \frac{r}{r_c}\right)\left[(v^2 + 2P) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr \\ & - \frac{(r^2 - r_c^2)}{r_c} \partial^2 v_i dx^i d\tau + \quad \ldots \end{aligned}$$

with $v_i \sim \mathcal{O}(\epsilon), \quad P \sim \mathcal{O}(\epsilon^2), \quad \partial_i \sim \mathcal{O}(\epsilon), \quad \partial_\tau \sim \mathcal{O}(\epsilon^2)$.

Induced metric at cutoff is flat, and there are no singularities at r=0, so this satisfies our boundary conditions.

Does this solve the Einstein vacuum equations?

Satisfying the Einstein Constraints

The metric above satisfies the constraint equations through $\mathcal{O}(\epsilon^2)$ iff

$$\partial^i v_i = 0$$

The constraint equation at ϵ^3 becomes

$$\partial_\tau v_i - \bar{\eta}\partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0$$

with the specific value $\bar{\eta} = r_c$. for the viscosity.

We must also show that the solution can be evolved consistently towards the horizon. Direct computation shows

$$G_{ra}, G_{ab}, G_{rr} = \mathcal{O}(\epsilon^4)$$

and these components are nonsingular for finite r.

Towards a Near-Horizon Limit: Rescaling in λ

- hydrodynamic limit is related to a near-horizon limit
- Begin by rescaling coordinates:

$$x^i = \frac{r_c \hat{x}^i}{\epsilon}, \qquad \tau = \frac{r_c \hat{\tau}}{\epsilon^2}, \qquad r = r_c \hat{r}$$

- Additionally rescale metric overall: $d\hat{s}^2_{p+2} = \frac{\epsilon^2}{r_c^2} ds^2_{p+2}.$
- Define $\lambda \equiv \frac{\epsilon^2}{r_c}$

Now we obtain:

$$\begin{aligned} d\hat{s}^2_{p+2} = &- \frac{\hat{r}}{\lambda} d\hat{\tau}^2 \\ &+ \left[2 d\hat{\tau} d\hat{r} + d\hat{x}_i d\hat{x}^i - 2(1-\hat{r})\hat{v}_i d\hat{x}^i d\hat{\tau} + (1-\hat{r})(\hat{v}^2 + 2\hat{P}) d\hat{\tau}^2 \right] \\ &+ \lambda \left[(1-\hat{r})\hat{v}_i \hat{v}_j d\hat{x}^i d\hat{x}^j - 2\hat{v}_i d\hat{x}^i d\hat{r} + (\hat{v}^2 + 2\hat{P}) d\hat{\tau} d\hat{r} + (1-\hat{r}^2)\hat{\partial}^2 \hat{v}_i d\hat{x}^i d\hat{\tau} \right] \; + \ldots . \end{aligned}$$

Only λ is left!

The Near-Horizon Expansion

- Consider Rindler wedge with boundary at r=1 (rescaled as $r = \tilde{r}_c \hat{r},\ \tau = \frac{\hat{\tau}}{\tilde{r}_c}$):

$$ds^2_{p+2} = -\frac{\hat{r}}{\tilde{r}_c} d\hat{\tau}^2 + 2 d\hat{\tau} d\hat{r} + dx_i dx^i.$$

- For small v, we found the linearized solution in 1006.1902

- Extending to a nonlinear generalization we find:

$$\begin{aligned} d\hat{s}^2_{p+2} = & -\frac{\hat{r}}{\tilde{r}_c} d\hat{\tau}^2 \\ & + \left[2 d\hat{\tau} d\hat{r} + dx_i dx^i - 2(1-\hat{r}) v_i dx^i d\hat{\tau} + (1-\hat{r})(v^2 + 2\hat{P}) d\hat{\tau}^2 \right] \\ & + \tilde{r}_c \left[(1-\hat{r}) v_i v_j dx^i dx^j - 2 v_i dx^i d\hat{r} + (v^2 + 2\hat{P}) d\hat{\tau} d\hat{r} + (1-\hat{r}^2) \partial^2 v_i dx^i d\hat{\tau} - 2(1-\hat{r}) \hat{q}_i(\hat{\tau}, \hat{r}, x) dx^i d\hat{\tau} \right] \end{aligned}$$

plus higher order corrections.

Relabel $v \to \hat{v},\ x^i \to \hat{x}^i \text{ and } \tilde{r}_c \to \lambda$ and we recover the λ-dependent metric!

Highlights of Cutoff Approach

- does not require AdS (or any other asymptotics)
- in long wavelength limit, works at any radius r_c.
- Equivalently one can take a near horizon limit r_c ->0
- physical quantities such as diffusion constant have a local definition on r_c.
- spacetime is Petrov type II: there exists a real null vector k^μ which satisfies

$$W_{\mu\nu\rho[\sigma}k_{\lambda]}k^\nu k^\rho = 0.$$

Linearized Results

• linearized limit (with uniqueness) done in 1006.1902; nonlinear embedding in 1101.2451

• Radial evolution of $S = \beta E + \beta \mathcal{P} V_p - \beta \mu Q$ is just a component of Einstein equation

• Diffusion constant must decrease with increasing radius assuming null energy condition:

$$\partial_{r_c} \bar{D}_c \leq 0$$

where we define the diffusion constant by

$$i\omega_c = \bar{D}_c k_c^2.$$

• Forcing solutions...

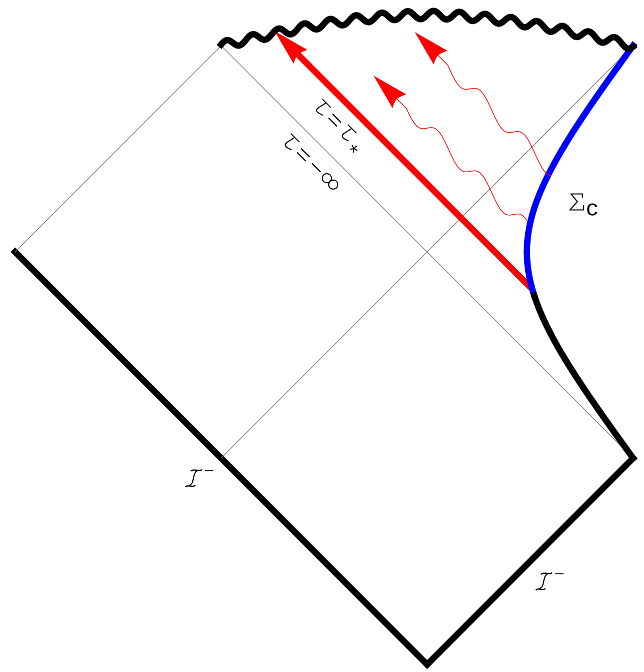
Forcing the fluid

Consider the metric

$$
\begin{aligned}
&- r d\tau^2 + 2 d\tau dr + dx_i dx^i \\
&- \left[2(1 - r/r_c) v_i dx^i d\tau + (1 - r/r_c)\left(\partial_j v_i + \partial_i v_j\right) dx^i dx^j - 2\left(r - \frac{r^2}{2r_c} - r_c/2\right)\partial^2 v_i dx^i d\tau \right] \\
&- \delta(\tau - \tau_*)\left[\left(4(1 - r/r_c) F_i + \frac{2}{r_c}\alpha_i\right) dx^i d\tau - \frac{2}{r_c}\beta_{ij} dx^i dx^j \right] + \ldots
\end{aligned}
$$

where β_{ij} and α_i are functions of only x and satisfy

$$
\partial^j \partial_j \alpha_i = F_i, \qquad \beta_{ij} = \partial_i \alpha_j + \partial_j \alpha_i.
$$



We find

$$
\partial_\tau v^i - \tilde{\eta}\partial^2 v^i = F^i(x)\delta(\tau - \tau_*).
$$

so the velocity field is zero for $\tau < \tau_*$
and jumps to $F_i(x^i)$ at $\tau = \tau_*$.

Conclusions and etc.

• We can embed any solution of Navier-Stokes as data on a flat hypersurface in a solution of Einstein equations

• The solutions of Einstein equations combined with the boundary conditions we impose correspond one-to-one with solutions of incompressible Navier-Stokes

• Our near-horizon limit provides a precise mathematical sense in which horizons are incompressible fluids

• This approach is asymptotics-agnostic

Future possibilities:
• Other geometries (e.g. Schwarzschild)
• Further understanding of Petrov type (e.g. higher dimensions)
• Uniqueness, existence of both GR and NS solutions
• Understanding gravity geometry of specific solutions (e.g. vortex)
• Forcing at the nonlinear level