From the positive grassmannian to gauge anomalies

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IAS

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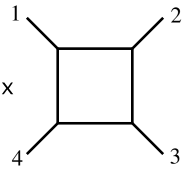
Prelude

In 1982, Green, Schwarz, Brink obtained $A_4^{\text{1-loop}}$ as the low-energy limit of the superstring scattering amplitude:

$st\,\mathcal{A}_4^{\text{tree}}\ \times$



$$\mathrm{I}_4 = -\frac{1}{\epsilon^2}\left[(-\frac{s}{\mu^2})^{-\epsilon} + (-\frac{t}{\mu^2})^{-\epsilon}\right] + \frac{1}{2}\log^2(\frac{s}{t}) + \frac{\pi}{2} + O(\epsilon)\,.$$

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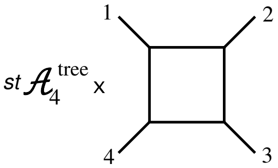

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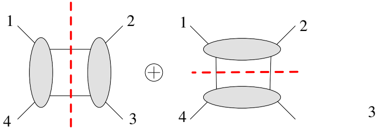

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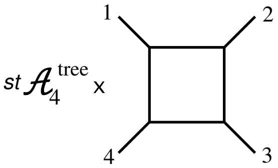

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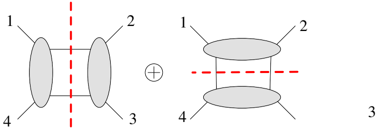

Birth of generalized unitarity

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- NLO high multiplicity QCD back grounds
- All loop planar integrand of $\mathcal{N} = 4$ SYM.N. Arkani-Hamed, J. Bourjaily, F. Cachazo, S. Caron-Huot, J. Trnka
- Three and four-loop six-point planar $\mathcal{N} = 4$ SYM result (without any integrals) L. Dixon, J. Drummond, M. v. Hippel, J. Pennington.

Modern on-shell approach:

On-shell elements $\rightarrow$ impose Locality: Singularities $\frac{1}{(p_i+p_j+\cdots+p_k)^2}$ Branch cuts $(p_i + p_j + \cdots + p_k)^2$

$\rightarrow$ impose Unitarity: residue factorized $\mathcal{A}_p \times \mathcal{A}_{n-p+2}$

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Prelude

- Can this construction "perturbatively" define general QFT ?
- If so, what can we learn for QFT in other dimensions ?
- What ever happened to gauge invariance?

D=3 Set up:

$$p_i^2 = 0 \to p_i^{ab} = \lambda^a \lambda^b$$

SL(2,R) Lorentz invariants: $\langle ij \rangle \equiv \lambda_i^a \lambda_j^b \epsilon_{ab}$

Supersymmetry$\rightarrow$ $\eta^A, A = 1, 2, \cdots, \mathcal{N}/2$

$$\Phi = X_4 + \eta_A \psi^A - \frac{1}{2}\epsilon^{ABC}\eta_A\eta_B X_C - \eta_1\eta_2\eta_3\psi^4 ,$$

$$\bar{\Psi} = \bar{\psi}_4 + \eta_A \bar{X}^A - \frac{1}{2}\epsilon^{ABC}\eta_A\eta_B \bar{\psi}_C - \eta_1\eta_2\eta_3 \bar{X}^4 .$$

We are interested in $\mathcal{A}_n(\Lambda_i)$ $\qquad \Lambda_i = (\lambda_i, \eta_i)$

D=3 The four-point amplitude

Impose symmetry:

$$Q^{aA} = \sum_i \lambda_i^a \eta_i^A, \qquad D = \sum_i \frac{1}{2}\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} + 1\right)$$

$$Q^{aA}\mathcal{A}_4 = D\mathcal{A}_4 = 0$$

- $\mathcal{N} = 8$:

$$\mathcal{A}_4 = \frac{\delta^3(P)\delta^8(Q^{Aa})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

  Completely antisymmetric structure constant $\rightarrow \quad f^{\alpha\beta\gamma\delta}$ (BLG)

- $\mathcal{N} = 6$:

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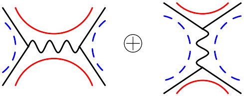
  Cyclic invariance by two-site, up to a sign under $i \rightarrow i+1$ ABJM



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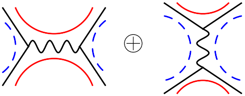

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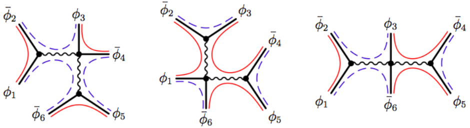
From Feyman diagrams: T. Bargheer, F. Loebbert, C. Meneghelli



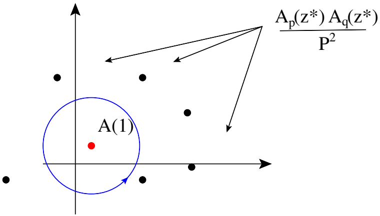
$$
\begin{aligned}
A_{6\phi}(1,\ldots,6) = C_6 \Bigg(& 4\frac{\langle 3|p_5|p_1|p_6|p_2|p_4|3\rangle + \langle 14\rangle^2\langle 2|p_3|p_6|p_5|2\rangle}{\langle 1|p_2|p_3|p_4|p_5|p_6|1\rangle} \\
+ \Big(2\frac{\frac{1}{3}\langle 16\rangle\langle 35\rangle\langle 24\rangle - \frac{1}{3}\langle 13\rangle\langle 56\rangle\langle 24\rangle + \langle 16\rangle\langle 23\rangle\langle 45\rangle}{\langle 12\rangle\langle 34\rangle\langle 56\rangle} & + 8\frac{\langle 5|p_1|6\rangle\langle 3|p_2|4\rangle}{\langle 34\rangle\langle 56\rangle p_{234}^2} + \{\text{shift by one}\}\Big) \\
& - 8\frac{\langle 26\rangle\langle 35\rangle(\langle 16\rangle^2\langle 34\rangle^2 + \langle 12\rangle^2\langle 45\rangle^2)}{\langle 2|p_1|6\rangle\langle 3|p_4|5\rangle p_{612}^2}\Bigg) + \{\text{two cyclic}\}\,. \qquad \text{(D.16)}
\end{aligned}
$$

D=3 Recurssion

Impose locality and unitarity: D. Gang, S.Lee, A. Lipstein, E, Koh, Y-t

$$p_1, p_2 \to p_1(z), p_2(z)$$

$A \to A(z)$



$$(\lambda_1\lambda_1+\lambda_2\lambda_2)\to(\hat{\lambda}_1\hat{\lambda}_1+\hat{\lambda}_2\hat{\lambda}_2)\quad \begin{pmatrix}\hat{\lambda}_1\\ \hat{\lambda}_2\end{pmatrix}=R(z)\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}\quad R^TR=1$$

$$R(z)=\frac{1}{2}\begin{pmatrix} z+z^{-1} & i(z-z^{-1})\\ i(z^{-1}-z) & z+z^{-1}\end{pmatrix}\quad \mathcal{A}_n=\frac{1}{2\pi i}\oint_{z=1}\frac{\hat{\mathcal{A}}_n(z)}{z-1}=-\sum_{z*}\frac{h(z*)}{P^2}A_p(z*)A_q(z*)$$

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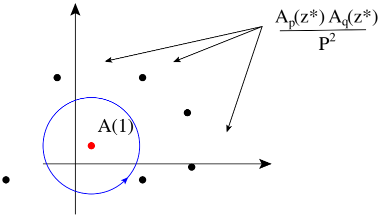

$$(\lambda_1\lambda_1 + \lambda_2\lambda_2) \to (\hat\lambda_1\hat\lambda_1 + \hat\lambda_2\hat\lambda_2) \quad \begin{pmatrix} \hat\lambda_1 \\ \hat\lambda_2 \end{pmatrix} = R(z)\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad R^T R = 1$$

$$R(z) = \frac{1}{2}\begin{pmatrix} z + z^{-1} & i(z - z^{-1}) \\ i(z^{-1} - z) & z + z^{-1} \end{pmatrix} \quad \mathcal{A}_n = \frac{1}{2\pi i}\oint_{z=1}\frac{\hat{\mathcal{A}}_n(z)}{z-1} = -\sum_{z*}\frac{h(z*)}{P^2}A_p(z*)A_q(z*)$$

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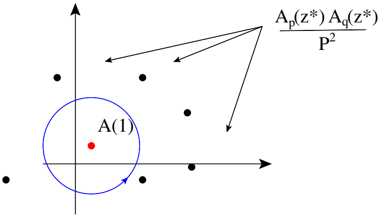

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The result:

$$
\begin{aligned}
\mathcal{A}_6 &= \pm\frac{\delta^6(Q)\delta^3\left\{\langle 24\rangle\eta_6^I+\langle 46\rangle\eta_2^I+\langle 62\rangle\eta_4^I\pm i(\langle 13\rangle\eta_5^I+\langle 35\rangle\eta_1^I+\langle 51\rangle\eta_3^I)\right\}}{(-\langle 2|p_{135}|5\rangle\pm i\langle 46\rangle\langle 31\rangle)(-\langle 4|p_{135}|1\rangle\pm i\langle 62\rangle\langle 53\rangle)(-\langle 6|p_{135}|3\rangle\pm i\langle 24\rangle\langle 15\rangle)}\\
&\equiv \frac{\delta^6(Q)\delta^3\left[\alpha^{I+}\right]}{c^+_{25}c^+_{41}c^+_{63}}-\frac{\delta^6(Q)\delta^3\left[\alpha^{I-}\right]}{c^-_{25}c^-_{41}c^-_{63}}
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$$\equiv \frac{\delta^6(Q)\delta^3\left[\alpha^{I+}\right]}{c_{25}^+ c_{41}^+ c_{63}^+} - \frac{\delta^6(Q)\delta^3\left[\alpha^{I-}\right]}{c_{25}^- c_{41}^- c_{63}^-} \quad (1)$$

Why is it so simple ?

$$c_{41}^- c_{41}^+ = p_{3,4,5}^2 p_{1,3,5}^2, \quad c_{63}^- c_{63}^+ = p_{5,6,1}^2 p_{1,3,5}^2, \quad c_{25}^- c_{25}^+ = p_{1,2,3}^2 p_{1,3,5}^2$$

The amplitude as a grassmannian integral

Consider a 3×6 matrix S,Lee; D. Gang, S.Lee, A. Lipstein, E, Koh, Y-t

$$C_{\alpha i} = \left(\begin{array}{c|c|c|c|c|c} c_{21} & 1 & c_{23} & 0 & c_{25} & 0 \\ c_{41} & 0 & c_{43} & 1 & c_{45} & 0 \\ c_{61} & 0 & c_{63} & 0 & c_{65} & 1 \end{array} \right)$$

$$\mathrm{A}_6 = \int \frac{d^6c}{(123)(234)(345)} \delta^6(C * C^T)\delta^{3\times 2}(C \cdot \lambda^a)\delta^{3\times 3}(C \cdot \eta^A)$$

The amplitude is given by an integral over a orthogonal Grassmannian manifold!

$$A_{2k} \in \int \frac{d^{k\times 2k}C}{GL(k)} \frac{1}{M_1 \cdots M_k} \delta^{k(k+1)/2}(C * C^T)\delta^{k\times 2}(C \cdot \lambda^a)\delta^{k\times 3}(C \cdot \eta^A)$$

The origin of this formula:

- The existence of an infinite dimensional Yangian symmetry: T. Bargheer, F. Loebbert, C. Meneghelli
- Equivalent to the presence of a dual superconformal symmetry: A. Lipstein, Y-t

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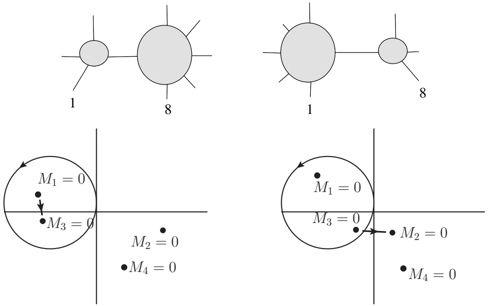
The dimension

$$2k \times k - k^2 - k(k+1)/2 - 2k + 3 = (k-2)(k-3)/2$$

for $k > 3$



Spurious Physical

The amplitude as a grassmannian integral

Remarkable resemblance:

$$\mathcal{N}=4\ \text{SYM}: \quad \mathcal{A}_{k,n} = \sum_{i} \text{res}_i \in \int \frac{dC}{M_1 \cdots M_n} \delta^{4k|4k}(C \cdot Z)$$

$$\text{ABJM}: \quad \mathcal{A}_{k,2k} = \sum_{i} \text{res}_i \in \int \frac{dC}{M_1 \cdots M_k} \delta(C \cdot C^T) \delta^{2k|3k}(C \cdot \Lambda)$$

Why?

The amplitude as a grassmannian integral

The information of the scattering amplitude

$\mathrm{res_i} = C^* \quad \rightarrow \delta(C^* \cdot C^{*T}) = \delta^{k\times 2}(C^* \cdot \lambda^a) = M_i = 0$

What is special about C^*?

$$C = \left(\begin{array}{c|c|c|c} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array} \right)$$

$$\mathrm{r(M_1) = 2,\ r(M_2) = 2,\ r(M_3) = 2,\ r(M_4) = 2}$$

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Linear interdependency of consecutive columns of the Grassmannian is termed "Positroid Stratification" Postnikov

Each cell C^* appears to be 1-1 correspondence with $\mathrm{r_i}$

The recursion is building a particular stratification! *what is it?*

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On-shell diagrams

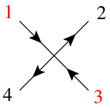
The fundamental 4-pt amp:

$$\mathcal{A}_4 = \int \frac{d^4C}{M_1M_2}\delta^3(C\cdot C^T)\delta^{4|6}(C\cdot\Lambda)$$

$$= \int \frac{d\theta}{\sin\theta\cos\theta}\delta^{4|6}(C\cdot\Lambda) \quad C = \left(\begin{array}{c|c|c|c} 1 & i\cos\theta & 0 & i\sin\theta \\ 0 & -i\sin\theta & 1 & i\cos\theta \end{array}\right)$$

$$\Lambda_1 + i\cos\theta\Lambda_2 + i\sin\theta\Lambda_4 = 0, \quad \Lambda_3 - i\sin\theta\Lambda_2 + i\cos\theta\Lambda_4 = 0$$

Graphical representation:



On-shell diagrams

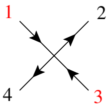
The fundamental 4-pt amp:

$$\mathcal{A}_4 = \int \frac{d^4C}{M_1 M_2} \delta^3(C \cdot C^T) \delta^{4|6}(C \cdot \Lambda)$$

$$= \int \frac{d\theta}{\sin\theta \cos\theta} \delta^{4|6}(C \cdot \Lambda) \quad C = \left(\begin{array}{c|c|c|c} 1 & i\cos\theta & 0 & i\sin\theta \\ 0 & -i\sin\theta & 1 & i\cos\theta \end{array} \right)$$

$$\Lambda_1 + i\cos\theta\Lambda_2 + i\sin\theta\Lambda_4 = 0, \quad \Lambda_3 - i\sin\theta\Lambda_2 + i\cos\theta\Lambda_4 = 0$$

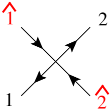
Graphical representation:



On-shell diagrams

$$\Lambda_{\hat{1}} + i\cos\theta\Lambda_2 + i\sin\theta\Lambda_1 = 0, \quad \Lambda_{\hat{2}} - i\sin\theta\Lambda_2 + i\cos\theta\Lambda_1 = 0$$

Graphical representation:



Recall the BCFW shift

$$(\lambda_1\lambda_1 + \lambda_2\lambda_2) \to (\hat{\lambda}_1\hat{\lambda}_1 + \hat{\lambda}_2\hat{\lambda}_2) \quad \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{pmatrix} = R(z) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad R^T R = 1$$

$$R(z) = \frac{1}{2} \begin{pmatrix} z + z^{-1} & i(z - z^{-1}) \\ i(z^{-1} - z) & z + z^{-1} \end{pmatrix}$$

On-shell diagrams

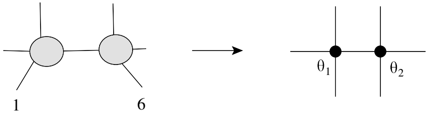
The fundamental 4-pt amp:

$$\mathcal{A}_4 = \int \frac{d\theta}{\sin\theta\cos\theta}\,\delta^4\left(C(\theta)\cdot\lambda\right)\delta^3\left(C(\theta)\cdot\eta\right)$$

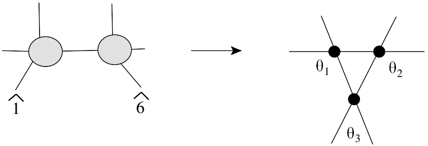
$$C(\theta) = \left(\begin{array}{c|c|c|c} 1 & i\cos\theta & 0 & i\sin\theta \\ 0 & -i\sin\theta & 1 & i\cos\theta \end{array}\right)$$

Graphical representation:

$\mathcal{A}_4 \times \mathcal{A}_4$

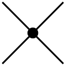


$\frac{h(\theta_3^*)}{P^2}\mathcal{A}_4(\theta_3^*) \times \mathcal{A}_4(\theta_3^*)$



On-shell diagrams

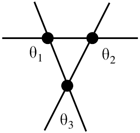
The fundamental 4-pt amp:



$$\mathcal{A}_4 = \int \mathrm{d}\log\tan\theta\ \delta^4\left(\mathrm{C}(\theta)\cdot\lambda\right)\delta^3\left(\mathrm{C}(\theta)\cdot\eta\right)$$

$$C(\theta) = \left(\begin{array}{c|c|c|c} 1 & \mathrm{ic} & 0 & \mathrm{is} \\ 0 & -\mathrm{is} & 1 & \mathrm{ic} \end{array}\right)$$

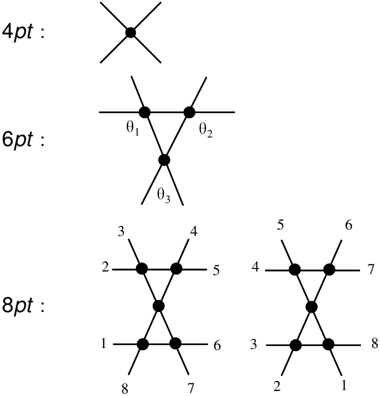
The 6-pt amp:



$$\mathcal{A}_6 = \int \mathrm{d}\log\tan_1 \mathrm{d}\log\tan_2 \mathrm{d}\log\tan_3\ (1+\mathrm{s}_1\mathrm{s}_2\mathrm{s}_3)\ \delta^6\left(\mathrm{C}(\theta_i)\cdot\lambda\right)\delta^9\left(\mathrm{C}(\theta_i)\cdot\eta\right)$$

$$\left(\begin{array}{c|c|c|c|c|c} 1 & \mathrm{ic}_1\mathrm{c}_2/\mathrm{J} & 0 & -\mathrm{ic}_1\mathrm{s}_2\mathrm{c}_3/\mathrm{J} & 0 & \mathrm{i}(\mathrm{s}_1+\mathrm{s}_2\mathrm{s}_3)/\mathrm{J} \\ -\mathrm{i}(\mathrm{s}_2+\mathrm{s}_1\mathrm{s}_3)/\mathrm{J} & 0 & 1 & \mathrm{c}_2\mathrm{c}_3/\mathrm{J} & 0 & \mathrm{ic}_2\mathrm{s}_3\mathrm{c}_1/\mathrm{J} \\ 0 & \mathrm{ic}_3\mathrm{s}_1\mathrm{c}_2/\mathrm{J} & 0 & \mathrm{i}(\mathrm{s}_3+\mathrm{s}_1\mathrm{s}_2)/\mathrm{J} & 1 & -\mathrm{ic}_3\mathrm{c}_1/\mathrm{J} \end{array}\right)$$

On-shell diagrams



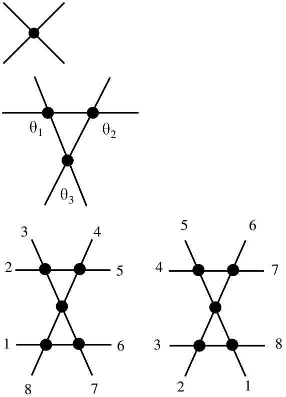
- Each term in the recursion correspond to a particular on-shell diagram
- Each diagram encodes a particular C^* (stratification)
- What is special about C^*?

On-shell diagrams

$4pt$:

$6pt$:

$8pt$:



- Each term in the recursion correspond to a particular on-shell diagram
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On-shell diagrams

What is special about C^*?

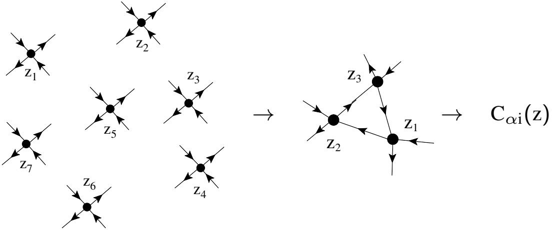
Analytic continue the signature $(+,-,+,\cdots,-)$

$$k=2 \quad \left(\begin{array}{c|c|c|c} 1 & ic & 0 & is \\ 0 & -is & 1 & ic \end{array}\right) \rightarrow \left(\begin{array}{c|c|c|c} 1 & c & 0 & -s \\ 0 & s & 1 & c \end{array}\right)$$

$$k=3 \quad \left(\begin{array}{c|c|c|c|c|c} 1 & \frac{c_1+c_2c_3}{1+c_1c_2c_3} & 0 & \frac{s_1c_3s_2}{1+c_1c_2c_3} & 0 & \frac{s_1s_3}{1+c_1c_2c_3} \\ 0 & \frac{s_2s_1}{1+c_1c_2c_3} & 1 & \frac{c_2+c_1c_3}{1+c_1c_2c_3} & 0 & \frac{s_3c_1s_2}{1+c_1c_2c_3} \\ 0 & \frac{s_3c_2s_1}{1+c_1c_2c_3} & 0 & \frac{s_2s_3}{1+c_1c_2c_3} & 1 & \frac{c_3+c_1c_2}{1+c_1c_2c_3} \end{array}\right)$$

All ordered minors are strictly positive for $0 \leq \theta \leq \pi/2$! Amplitude is constructed from cells of the positive orthogonal grassmannian!

On-shell diagrams in Orthogonal Grasmmannian

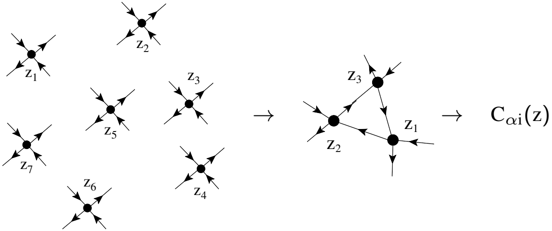


$$\left(\prod_{i\in n_v}\int d\log\tan_i\right) J\,\delta^{3k|2k}(C\cdot\Lambda)$$

$$\int d\log\tan\theta$$

Logarithmic singularity at the boundary of positive OG_k, $\theta = 0,\ \pi/2$

On-shell diagrams in Orthogonal Grasmmannian



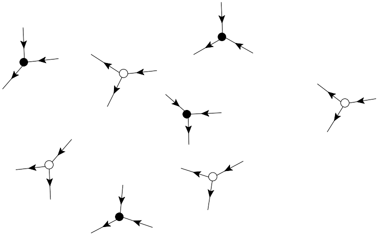
$$\left(\prod_{i\in n_v}\int \mathrm{d}\log\tan_i\right)\; J\,\delta^{3k|2k}(C\cdot\Lambda)$$

$$\int d\log\tan\theta$$

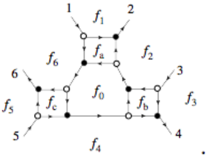
Logarithmic singularity at the boundary of positive $\mathrm{OG_k}$, $\theta = 0,\ \pi/2$

On-shell Diagrams

Planar $\mathcal{N}=4$ SYM $\in \mathrm{Gr}(\mathrm{k},\mathrm{n})_+$ Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Goncharov, A. Postnikov, J. Trnka



$\rightarrow$



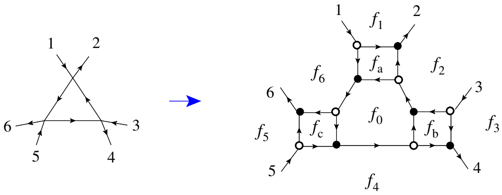
$\rightarrow \quad \mathrm{C}_{\alpha\mathrm{i}}(\mathrm{f})$

$$C_{\alpha i}(f)=\begin{pmatrix} 1 & \frac{1}{f_1}+\frac{1}{f_1 f_a(1+f_0)} & 0 & \frac{f_4 f_5 f_6 f_c}{1+1/f_0} & 0 & \frac{f_6}{1+1/f_0} \\ 0 & \frac{f_2}{1+1/f_0} & 1 & \frac{1}{f_3}+\frac{1}{f_3 f_b(1+f_0)}, & 0 & \frac{f_1 f_2 f_6 f_a}{1+1/f_0} \\ 0 & \frac{f_3 f_4 f_2 f_b}{1+1/f_0} & 0 & \frac{f_4}{1+1/f_0} & 1 & \frac{1}{f_5}+\frac{1}{f_5 f_c(1+f_0)} \end{pmatrix}$$

$$0 \leq f_i \leq \infty$$

$$\mathcal{A}_n=\left(\sum_{\mathrm{dia}}\int\prod_i \log \mathrm{f_i}\right)\delta^{4k|4k}\left(C\cdot\mathcal{W}\right)$$

On-shell diagrams in Orthogonal Grasmmannian

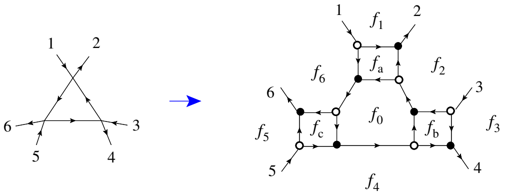


$$(f_a, f_b, f_c) = (c_1^2/s_1^2, c_2^2/s_2^2, c_3^2/s_3^2),\ f_0 = \frac{1}{c_1 c_2 c_3}$$

$$f_1 = \frac{1}{c_1},\ f_2 = s_1 s_2,\ f_3 = \frac{1}{c_3},\ f_4 = s_2 s_3,\ f_5 = \frac{1}{c_3},\ f_6 = s_1 s_3$$

- On-shell diagrams of ABJM construct cells in positive OG_k
- Each cell in positive OG_k has an image in positive $G_{k,2k}$

On-shell diagrams in Orthogonal Grasmmannian



$$(\mathrm{f_a}, \mathrm{f_b}, \mathrm{f_c}) = (c_1^2/s_1^2, c_2^2/s_2^2, c_3^2/s_3^2),\ \mathrm{f_0} = \frac{1}{c_1 c_2 c_3}$$

$$\mathrm{f_1} = \frac{1}{c_1},\ \mathrm{f_2} = s_1 s_2,\ \mathrm{f_3} = \frac{1}{c_3},\ \mathrm{f_4} = s_2 s_3,\ \mathrm{f_5} = \frac{1}{c_3},\ \mathrm{f_6} = s_1 s_3$$

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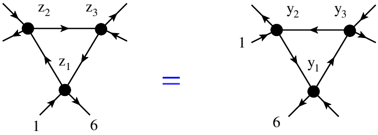
- Each cell in positive $\mathrm{OG_k}$ has an image in positive $\mathrm{G_{k,2k}}$

What is special about this embedding ?

$$\mathcal{A}_6 = \sum_{\text{branch}} \int \mathrm{d}\log \tan_1 \mathrm{d}\log \tan_2 \mathrm{d}\log \tan_3 \, (1 + \sin_1 \sin_2 \sin_3) \delta^{2k}(C\cdot\lambda)\delta^{3k}(C\cdot\eta)$$



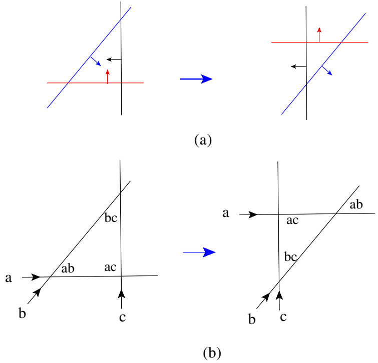
$$\mathcal{A}_6 = \sum_{\text{branch}} \int \mathrm{d}\log \tan_1 \mathrm{d}\log \tan_2 \mathrm{d}\log \tan_3 \, (1 + \cos_1 \cos_2 \cos_3) \delta^{2k}(C\cdot\lambda)\delta^{3k}(C\cdot\eta)$$

$\mathcal{J}$

$$R: \quad (y_1, y_2, y_3) \rightarrow (x_1, x_2, x_3)$$

On-shell diagrams in Orthogonal Grasmmannian

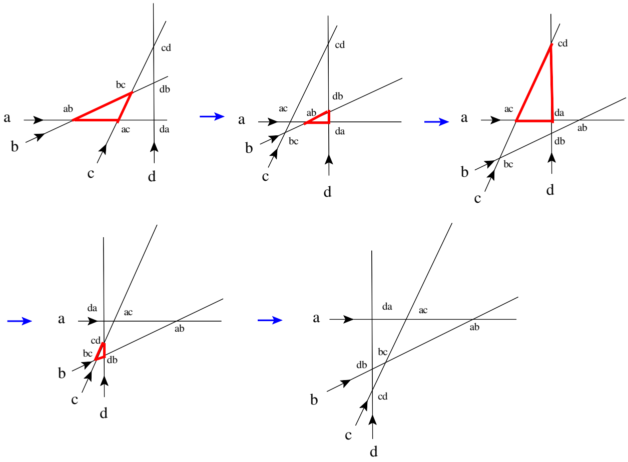
What is special about this embedding ?



$$|\theta'_{ab}, \theta'_{ac}, \theta'_{bc}\rangle = R_{abc}(\theta_{ab}, \theta_{ac}, \theta_{bc})|\theta_{ab}, \theta_{ac}, \theta_{bc}\rangle$$

On-shell diagrams in Orthogonal Grasmmannian

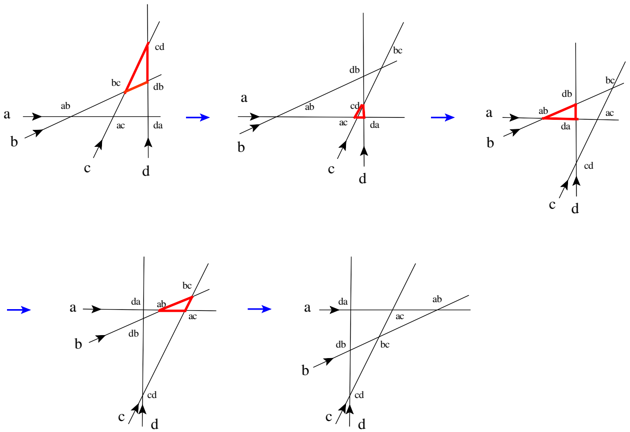
What is special about this embedding ?



$$R_{acb}R_{abd}R_{acd}R_{bcd}$$

On-shell diagrams in Orthogonal Grasmmannian

What is special about this embedding ?



$$R_{bcd}R_{acd}R_{abd}R_{acb}$$

On-shell diagrams in Orthogonal Grasmmannian

What is special about this embedding ?

A solution to the Tetrahedron equation:

$$R_{acb}R_{abd}R_{acd}R_{bcd} = R_{bcd}R_{acd}R_{abd}R_{acb}$$

$$R: \quad (\theta_1, \theta_2, \theta_3) \;\rightarrow\; \left(\frac{\theta_1\theta_2}{\theta_1+\theta_3-\theta_1\theta_2\theta_3}, \theta_1+\theta_3-\theta_1\theta_2\theta_3, \frac{\theta_2\theta_3}{\theta_1+\theta_3-\theta_1\theta_2\theta_3}\right) .$$

On-shell diagrams

The fundamental 4-pt amp:

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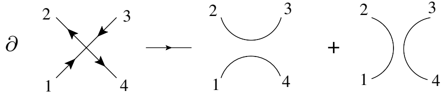
$$\Lambda_1 + \cos\theta\Lambda_2 - \sin\theta\Lambda_4 = 0, \quad \Lambda_3 + \sin\theta\Lambda_2 + \cos\theta\Lambda_4 = 0$$

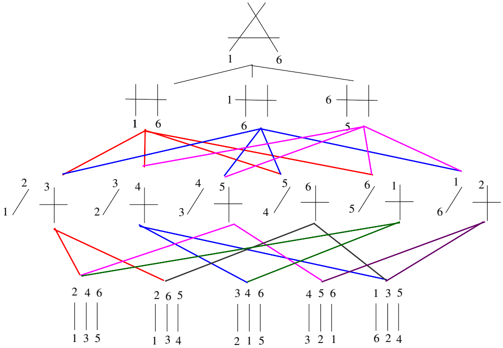
The boundary $\theta = \pi/2$, $\theta = 0$



On-shell diagrams

Each cell is combinatoricaly a polytope

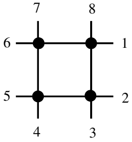
$$\sum_i (-1)^{d_i} n_i$$



$$-1 + 3 - 6 + 5 = 1$$

On-shell diagrams

Each cell is combinatoricaly a polytope



dimensions	cell	multiplicity
4	(16)(24)(38)(47)	1
3	(23)(47)(58)(16), (45)(16)(27)(38), (18)(25)(36)(47), (67)(38)(14)(25) (28)(16)(47)(35), (24)(38)(16)(57), (17)(38)(25)(46), (13)(25)(47)(68)	8
2	(23)(48)(57)(16), (23)(68)(47)(15), (23)(17)(58)(46), (45)(26)(38)(17) (45)(13)(27)(68), (45)(16)(28)(37), (18)(26)(47)(35), (18)(57)(36)(24) (18)(46)(25)(37), (67)(25)(48)(13), (67)(35)(14)(28), (67)(24)(38)(15) (12)(35)(47)(68), (28)(17)(35)(46), (34)(28)(16)(57), (13)(24)(57)(68) (56)(24)(38)(17), (78)(46)(25)(13)	18
1	(35)(18)(46)(57), (23)(14)(57)(68), (23)(56)(17)(48), (23)(67)(48)(15) (45)(23)(17)(68), (45)(36)(28)(17), (45)(78)(26)(13), (45)(18)(26)(37) (18)(67)(24)(35), (18)(27)(35)(46), (18)(34)(28)(57), (67)(58)(13)(24) (67)(45)(28)(13), (67)(12)(35)(48), (23)(78)(46)(15), (45)(12)(37)(68) (18)(56)(24)(37), (12)(78)(35)(46), (12)(34)(68)(57), (34)(56)(17)(28) (34)(67)(15)(28), (56)(78)(13)(24)	22
0	(23)(18)(45)(67), (23)(18)(56)(47), (23)(14)(56)(78), (23)(14)(67)(58) (45)(23)(78)(16), (45)(36)(78)(12), (45)(36)(18)(27), (18)(67)(34)(25) (18)(27)(34)(56), (67)(58)(12)(34), (67)(45)(12)(38), (12)(34)(56)(78)	12

$$1 - 8 + 18 - 22 + 12 = 1$$

- Efficiently imposing locality and unitarity we've uncovered new symmetries.
- The amplitude of ABJM = cells of positive $\mathrm{OG_k}$
- cells of positive $\mathrm{OG_k}$ is a sub manifold of positive $\mathrm{G_{k,2k}}$: satisfies tetrahedron equation
- The elementary building blocks are simply polytopes in nature.
- Amplitude is simply a polytope with physical boundaries.

summary

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Gauge anomalies

If one can perturbatively define a theory using only on-shell elements, then gauge symmetry is truly a figment of our imagination

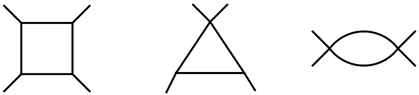
How do we see chiral theories are sick?

Consider 1-loop 4-pt

$$\mathcal{A}_4 = Tr(1234)A(1234) + Tr(1342)A(1342) + Tr(1423)A(1423) + \text{flip}$$

The color-ordered amplitude can be conveniently written as:

$$A(1234) = C_4 I_4 + C_{3s} I_{3s} + C_{3t} I_{3t} + C_{2s} I_{2s} + C_{2t} I_{2t} + R$$



Unitarity: C_i

Locality: R

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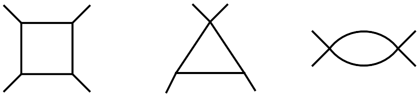

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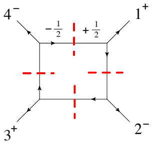
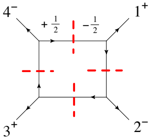
The devil is rational

Four-dimensional prelude

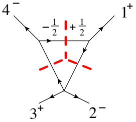
$$-\frac{t^4 s^2}{u^4}$$



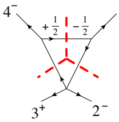
$$-\frac{s^4 t^2}{u^4}$$



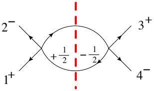
$$\frac{t^4 s}{u^4}$$



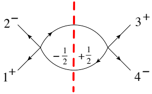
$$\frac{t^2 s^3}{u^4}$$



$$\frac{t(su - 6st - 2ut)}{6u^3}$$



$$\frac{t(4s^2 + 2t^2 - 7su)}{6u^3}$$



Four-dimensional prelude

Parity-even:

$$\frac{A^{\text{even}}(1^+2^-3^+4^-)}{A^{\text{tree}}} = -\frac{st(s^2+t^2)}{2u^4}\left(\log\left(\frac{t}{s}\right)^2+\pi^2\right)$$
$$+\left[\left(\frac{s-t}{3u}-\frac{st(s-t)}{u^3}\right)\right]\log\left(\frac{s}{t}\right)-\frac{(-s)^{-\epsilon}+(-t)^{-\epsilon}}{3\epsilon}$$

$$u=(p_1+p_3)^2$$

Locality requires these spurious poles to be absent.

$$\left.\frac{A^{\text{even}}}{A^{\text{tree}}}\right|_{u\to 0} = -\frac{s^2}{u^2}-\frac{s}{u}+\mathcal{O}(u^0)\,.$$

$$R^{\text{even}}(1,2,3,4) = -\frac{st}{u^2}$$

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$$\frac{A^{\rm odd}}{A^{\rm tree}}|_{u\to 0} = -\frac{s}{u}+\mathcal{O}(u^0).$$

Locality again requires such spurious poles to cancel against that from $R^{\rm odd}$

$$A^{\rm tree}R^{\rm odd}(1,2,3,4) = A^{\rm tree}\frac{s-t}{2u} = \langle 24\rangle^2[13]^2\frac{s-t}{2stu}.$$

Locality forces us to have a new factorization channel $\to$ dimension counting and helicity weight fixes the residue to by that of spin-1

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Parity-odd:

$$\mathcal{R} = \frac{\langle 24\rangle^2[13]^2}{2stu}[(s-t)Tr(1234)+(u-s)Tr(1342)+(t-u)Tr(1423)$$
$$+(s-u)Tr(1243)+(u-t)Tr(1324)+(t-s)Tr(1432)]\,.$$

Let us consider the residue for the $s\to 0$

$$\frac{\langle 24\rangle^2[13]^2}{2su}[-Tr(1234)-Tr(1342)+Tr(1243)+Tr(1432)]$$

$$Tr(1432)-Tr(1234)+(1\leftrightarrow 2)=d^{1a4}f^{23}{}_a+d^{13a}f^{24}{}_a+d^{1a2}f^{34}{}_a+(1\leftrightarrow 2)=0\,.$$

$$d^{abc}f^{de}{}_a=0\,.$$

The non-abelian box-anomaly

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The non-abelian box-anomaly

Six-dimensions

Parity-odd:

$$C_4 = \frac{(s-t)}{6u^2}F^4, \quad C_{3s} = -\frac{(s-t)}{6tu^2}F^4,$$
$$C_{3t} = -\frac{(s-t)}{6su^2}F^4, \quad C_{2s} = \frac{F^4}{stu}, \quad C_{2t} = -\frac{F^4}{stu},$$

The function F^4 is explicitly given as:

$$F^4 \equiv \langle 4_d|p_2p_3|4_{\dot{d}}]F^3_{(123)} + (\sigma_i)\text{cyclic},$$

$$\mathcal{A}_4 \xrightarrow{u=0} -\frac{F^4}{18ut}str(T^4) + \mathcal{O}(u^0), \quad \mathcal{A}_4 \xrightarrow{s=0} -\frac{F^4}{18su}str(T^4) + \mathcal{O}(s^0)$$
$$\mathcal{A}_4 \xrightarrow{t=0} -\frac{F^4}{18ts}str(T^4) + \mathcal{O}(t^0), \tag{2}$$

$$str(T^4) = a\, str(t^4) + b\, tr(t^2)tr(t^2),$$

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We are not done yet

$$\mathcal{A}_4 \xrightarrow{u=0} -\frac{F^4}{18ut}\left(tr(t_1t_2)(t_3t_4)+tr(t_1t_3)(t_2t_4)+tr(t_1t_4)(t_3t_2)\right)+\mathcal{O}(u^0)\,.$$

Clearly only the group theory factor $tr(t_1t_3)(t_2t_4)$ makes any sense as a factorization channel for the u channel pole

$$\mathcal{R}=F^4 tr\left(tr(t_1t_2)(t_3t_4)\frac{u-t}{18stu}+tr(t_1t_3)(t_2t_4)\frac{t-s}{18stu}+tr(t_1t_4)(t_3t_2)\frac{s-u}{18stu}\right)\,.$$

With the above rational term, we now find:

$$\mathcal{A}_4 \xrightarrow{u=0} -\frac{F^4}{6ut}tr(t_1t_3)(t_2t_4)+\mathcal{O}(u^0),\quad \mathcal{A}_4 \xrightarrow{s=0} -\frac{F^4}{6su}tr(t_1t_2)(t_3t_4)+\mathcal{O}(s^0)$$

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Where did this factor come from?

$$R^{anom} = -\frac{1}{18}\left[\left(\frac{(\epsilon_1\cdot k_2)}{s} + \frac{(\epsilon_1\cdot k_3)}{u} + \frac{(\epsilon_1\cdot k_4)}{t}\right) F_2\wedge F_3\wedge F_4 + (\text{cyclic})\right]$$

$$\begin{aligned} R^{anom} + GS &= \frac{-1}{18}\left[\left(\frac{(\epsilon_1\cdot k_2)}{s} - 2\frac{(\epsilon_1\cdot k_3)}{u} + \frac{(\epsilon_1\cdot k_4)}{t}\right) F_2\wedge F_3\wedge F_4 + (\text{cyclic})\right] \\ &= F^4\frac{t-s}{18stu} \end{aligned}$$

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Summary

There is no gauge anomaly per se..

- Rational terms holds locality for ransom
- Anomaly cancellation + GS mechanism $\rightarrow$ off-shell way of obtaining *R*
- Rational terms occur only for $\mathrm{D} = \mathrm{even}, \mathrm{n} = \mathrm{D}/2 + 1.$
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