Non-relativistic AdS/CFT

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References

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Applications and Motivation

Fermionic Gases

Cold, dilute gases of ^{6}Li and ^{40}K atoms are controlled by three length scales:

- thermal de Broglie wavelength:

$$\lambda = \frac{h}{p} = \sqrt{\frac{2\pi\hbar^2}{mT}}$$

- interparticle spacing: $\ell \sim n^{-1/3}$
- scattering length: a

Quantum Regime

When the de Broglie wavelength becomes of the order of the interparticle spacing, quantum effects become important.

- For bosons, we find Bose-Einstein condensation and superfluidity at these temperatures

$$T_{BEC} \sim \frac{\hbar^2}{m} n^{2/3} \approx 3 \text{ K for liquid Helium}$$

- For fermions, such as electrons in metals, we get the Fermi sea. One reason the $T_F \gg T_{BEC}$ is that $m_e \ll m_{He}$.

Interactions

For these dilute fermionic gases, the interaction potential is van der Waals attraction at long distances and a hard core repulsion at short distances.

- The potential has some effective range r_{eff}.
- In our parameter regime, λ, $\ell \gg r_{eff} \Rightarrow kr_{eff} \ll 1$.
- s-wave scattering dominates over all other partial waves.

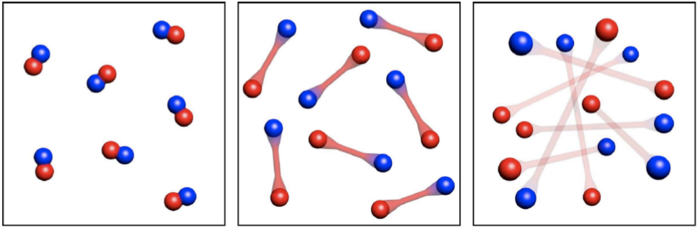
$$\psi(r) \sim e^{ik\cdot r} + f(k)\frac{e^{ikr}}{r}$$

where

$$f(k) = \frac{1}{-\frac{1}{a} + r_{eff}\frac{k^2}{2} - ik}$$

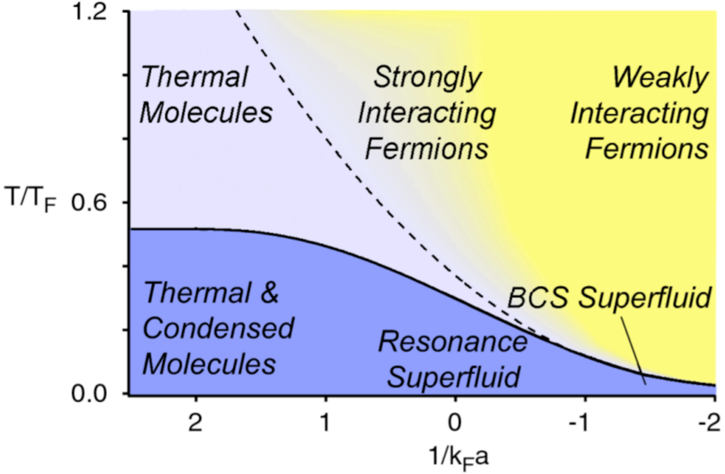
- The entire scattering process can be described by a single number, the scattering length a.

The three regimes



BEC of Molecules Crossover Superfluid BCS state

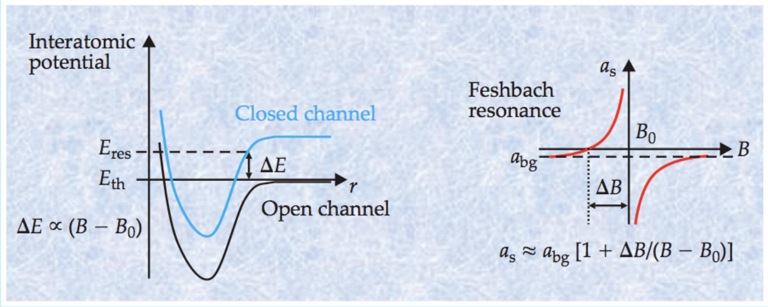
- $1/ka \to -\infty$: In this limit, the fermions experience a weak attractive interaction. We find a BCS state.
- $1/k|a| \to 0$: The unitarity limit. The fermions become very strongly interacting.
- $1/ka \to \infty$: A two-body bound state is available in the potential. The fermions form molecules.



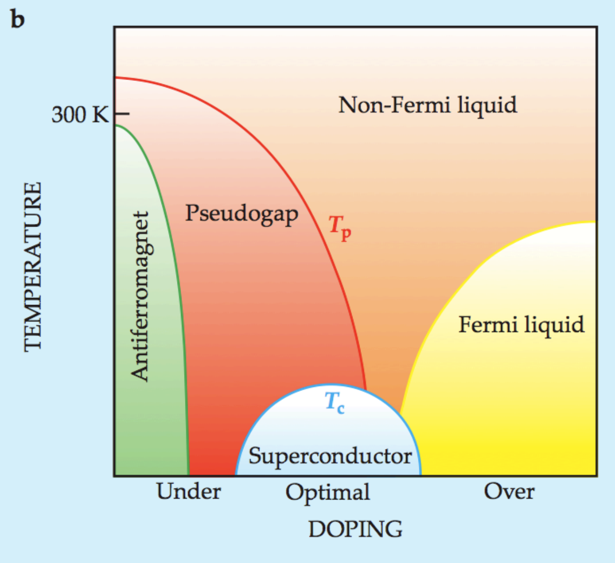
1.2
0.6
0.0
T/T_F
Thermal Molecules
Strongly Interacting Fermions
Weakly Interacting Fermions
Thermal & Condensed Molecules
Resonance Superfluid
BCS Superfluid
2
1
0
-1
-2
1/k_Fa

Feshbach resonance

- One cannot tune the interaction strength in neutron stars, nuclear matter, superconductors, or liquid ^{3}He.
- However, for ^{6}Li and ^{40}K, one can by tuning the magnetic field to a Feshbach resonance.



Can we learn anything about high-T_c superconductors from these systems?



What is the viscosity of fermions at unitarity?

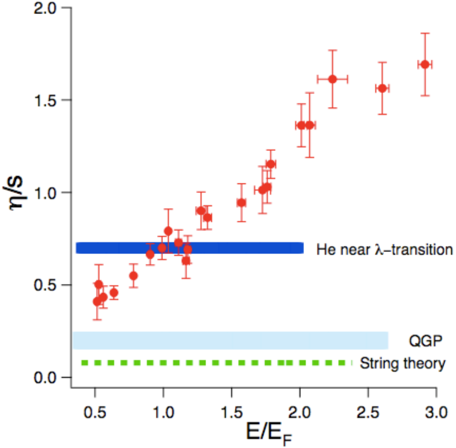


Fig. 5 Ratio of the shear viscosity η to the entropy density s for a strongly interacting Fermi gas as a function of energy E, red solid circles. The lower green dotted line shows the string theory prediction $1/(4\pi)$. The light blue bar shows the estimate for a quark-gluon plasma (QGP)[46], while the blue solid bar shows the estimate for ^{3}He and ^{4}He, near the λ- point.

arxiv:0707.2574, Turlapov, Kinast, Clancy, Luo, Joseph, Thomas

The Connection to String Theory

Symmetries

Fermions at unitarity depend on only two scales, the interparticle spacing ℓ and thermal de Broglie wavelength λ or equivalently the density n and the temperature T.

When $T = 0$ and $n = 0$, the theory in d spatial dimensions has an unbroken Schrödinger symmetry, Sch(d):

$$
\begin{aligned}
&[M^{ij}, M^{kl}] = i(\delta^{ik}M^{jl} + \delta^{jl}M^{ik} - \delta^{il}M^{jk} - \delta^{jk}M^{il}),\\
&[M^{ij}, P^{k}] = i(\delta^{ik}P^{j} - \delta^{jk}P^{i}), \quad [M^{ij}, K^{k}] = i(\delta^{ik}K^{j} - \delta^{jk}K^{i}),\\
&[D, P^{i}] = -iP^{i}, \quad [D, K^{i}] = iK^{i}, \quad [P^{i}, K^{j}] = -i\delta^{ij}M,\\
&[D, H] = -2iH, \quad [D, C] = 2iC, \quad [H, C] = iD.
\end{aligned}
$$

M^{ij} rotations, P^i spatial translations, K^i Galilean boosts, H time translation, D dilation ($x \to \lambda x$, $t \to \lambda^2 t$), C special conformal transformation, M mass operator.

Conformal Symmetries

The group Sch(d) embeds into the conformal group O($d+2,2$). Time becomes light-like.

$$M = \tilde{P}^{+} = (\tilde{P}^{0} + \tilde{P}^{d+1})/\sqrt{2} \quad ; \quad H = \tilde{P}^{-} \quad ; \quad D = \tilde{D} + \tilde{M}^{+-}$$

We have a strongly interacting theory with a symmetry group that is a subgroup of the conformal group. Can we use AdS/CFT?

The embedding suggests that to describe this field theory in d spatial dimensions holographically, we will need something like AdS_{d+3}. Two extra dimensions.

The BMS proposal

Balasubramanian, McGreevy and Son noticed the metric

$$ds^2 = r^2\left(-2du\,dv - r^2du^2 + d\vec{x}^2\right) + \frac{dr^2}{r^2}$$

has the isometries of the full Sch(d) group.

For example

- $H: \; u \to u + a$
- $D: \; x^i \to a x^i, \; r \to r/a, \; u \to a^2 u, \; v \to v$
- $M: \; v \to v + a$
- $K^i: \; x^i \to x^i - a^i u, \; v \to v - a^i x^i$

The mass M is mapped onto the lightcone momentum P^v. To make this spectrum discrete, we could compactify v. Introduces some problems ....

A Phenomenological Action

This metric is a solution to the gravitational action with a massive vector field

$$S = \int d^{d+2}x \, dz \sqrt{-g} \left(R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A^\mu A_\mu \right)$$

with $F = dA$, $A^v = 1$,

$$m^2 = 2(d+2) \quad \text{and} \quad -2\Lambda = (d+1)(d+2) \; .$$

Embedding in string theory

Three claims

- Using a supergravity solution generating technique called the Null Melvin Twist, we can generate the BMS metric starting from $AdS_5 \times S^5$ in type IIB supergravity.
- We can generalize the BMS metric to nonzero temperature solutions by starting instead with the translationally invariant black D3-brane metric in type IIB supergravity.
- These new supergravity solutions can be reduced to 5d where they are also solutions to an effective 5d action similar to the action with massive gauge field above.

Starting with vanilla blackholes

Consider the planar Schwarzschild-AdS black hole (times S^5), a solution to the type IIB supergravity equations of motion:

$$ds^2 = r^2(-f(r)dt^2 + dy^2 + d\vec{x}^2) + \frac{dr^2}{r^2 f(r)} + (d\psi + A)^2 + d\Sigma_4^2$$

with

$$F_5 = 2(1+\star)d\psi \wedge J \wedge J = 4(1+\star)\mathrm{vol}(S^5)$$

and $f(r) = 1 - r_+^4/r^4$.

We are thinking of S^5 as a U(1) fibration over $\mathbb{CP}^2$ where J is the Kähler form and $dA = 2J$.

The Null Melvin Twist

1. Pick a translationally invariant direction (say y) and boost by amount γ along y.
2. T-dualize along y.
3. Twist some one-form (say $d\psi + A$)

$$\sigma : \sigma \to \sigma + \alpha\, dy \ .$$

4. T-dualize along y again.
5. Boost by $-\gamma$ along y.
6. Scale the boost and twist: $\gamma \to \infty$ and $\alpha \to 0$ keeping

$$\beta = \frac{1}{2}\alpha e^{\gamma} = \text{fixed} \ .$$

The BMS solution in string theory

$$
\begin{aligned}
ds^2_{str} &= r^2\left(-\frac{\beta^2\, r^2\, f(r)}{k(r)}\,(dt+dy)^2 - \frac{f(r)}{k(r)}\,dt^2 + \frac{dy^2}{k(r)} + d\mathbf{x}^2\right)\\
&\quad + \frac{dr^2}{r^2\, f(r)} + \frac{(d\psi + A)^2}{k(r)} + d\Sigma_4^2,\\
e^{\varphi} &= \frac{1}{\sqrt{k(r)}},\\
F_{(5)} &= dC_{(4)} = 2\,(1+\star)\,d\psi\wedge J\wedge J,\\
B_{(2)} &= \frac{r^2\,\beta}{k(r)}\,(f(r)\,dt + dy)\wedge(d\psi + A),
\end{aligned}
$$

with

$$
f(r) = 1 - \frac{r_+^4}{r^4}\,, \qquad k(r) = 1 + \beta^2\, r^2\,(1 - f(r)) = 1 + \frac{\beta^2\, r_+^4}{r^2}.
$$

A 5d reduction

$$
\begin{aligned}
ds_E^2 &= r^2\, k(r)^{-\frac{2}{3}} \left(\left[\frac{1-f(r)}{4\beta^2} - r^2 f(r) \right] du^2 + \frac{\beta^2 r_+^4}{r^4}\, dv^2 \right. \\
&\qquad \left. - [1+f(r)]\, du\, dv \right) + k(r)^{\frac{1}{3}} \left(r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2 f(r)} \right),
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{r^2 \beta}{k(r)} (f(r)\, dt + dy) = \frac{r^2}{k(r)} \left(\frac{1+f(r)}{2}\, du - \frac{\beta^2 r_+^4}{r^4}\, dv \right), \\
e^{\phi} &= \frac{1}{\sqrt{k(r)}},
\end{aligned}
$$

Claim: This solution approaches the BMS solution as $r_+/r \to 0$.
Worries: As $r_+/r \to 0$, the size of the v circle gets smaller and smaller.

A 5d effective action

$$\mathcal{S} \;=\; \frac{1}{16\pi\, G_5}\int d^5x\sqrt{-g}\Big(R-\frac{4}{3}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu} \\ -4\,A_\mu A^\mu - V(\phi)\Big),$$

where the scalar potential is

$$V(\phi) = 4\,e^{2\phi/3}(e^{2\phi}-4)\ .$$

Claim: The 5d metric, gauge field, and dilaton of the previous slide are a solution to the equations of motion that follow from this 5d effective action.

A Gauge Theory Interpretation

- The field theory is a DLCQ of $\mathcal{N} = 4$ super Yang-Mills with a light-like twist in the R-symmetry directions.
- The mode of the NS-NS two-form that we turned on is part of a massive vector transforming in the **15** of $SO(6)$.
- The corresponding operator in the CFT has dimension 5:

$$\mathcal{O}^{IJ}_{\mu} = \mathrm{Tr}\left(F_{\mu}{}^{\nu}\, \Phi^{[I}\, D_{\nu}\Phi^{J]} + \sum_{K} D_{\mu}\, \Phi^{K}\, \Phi^{[K}\Phi^{I}\Phi^{J]} \right) + \mathrm{fermions}$$

- We break Lorentz invariance by giving $\langle \mathcal{O}^{IJ}_{u} \rangle$ a nonzero expectation value.

Thermodynamics

Entropy, temperature, and chemical potential

The thermodynamics of this non-relativistic field theory is inherited from $\mathcal{N} = 4$ SYM. However, various quantities get re-interpreted.

- The entropy (i.e. the horizon area) is invariant under NMT: $S = r_+^3 \Delta y \Delta x_1 \Delta x_2 / 4G_5$.
- The Hawking temperature is unchanged as well $T_H = r_+/\pi$.
- However, the Killing generator of the event horizon is ∂_t while boundary time is ∂_u:

$$\xi = \beta^{-1}\partial_t = \partial_u + (2\beta^2)^{-1}\partial_v \ .$$

- Recalling the Boltzmann factor $e^{-E/T+\mu Q/T}$, we conclude that

$$T = r_+/\pi\beta \quad \text{and} \quad \mu = -1/2\beta^2 \ .$$

The free energy

To make sure we know what we are doing, we should check that the free energy obtained from the on-shell gravity action agrees with the prescription for entropy and temperature above.

- Indeed we find the $\mathcal{N} = 4$ result for the Euclidean action:

$$I = -\frac{1}{16\pi G_5} \int d^4\xi \, r_+^4 = -\frac{\beta}{16 G_5} r_+^3 V \Delta v$$

- As follows from thermodynamic identities, we can check

$$\mathcal{S} = -\left(T \frac{\partial}{\partial T} + 1 \right) I$$

- As expected given the scaling symmetry $x \to \lambda x$ and $t \to \lambda^2 t$, we find the equation of state

$$2E = PV \cdot (\# \text{ of spatial dimensions})$$

Regulating the Action

We regulated the 5d action with the boundary terms:

$$
\begin{aligned}
\mathcal{S} \;=\; & \frac{1}{16\pi\, G_5} \int d^5x \sqrt{-g} \left(R - \frac{4}{3}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{4}\, e^{-8\phi/3}\, F_{\mu\nu} F^{\mu\nu} \right. \\
& \qquad\qquad \left. -4\, A_\mu A^\mu - V(\phi) \right) \\
& + \frac{1}{16\pi\, G_5} \int d^4\xi \sqrt{-h}\, \big(2K - 2\, c_0 + c_1\, \phi + c_2\, \phi^2 + c_3\, A_\alpha A^\alpha \\
& \qquad\qquad + c_4\, A_\alpha A^\alpha \phi + c_5\, (A_\alpha A^\alpha)^2 \big) \;\; .
\end{aligned}
$$

We insist that $\delta \mathcal{S} = 0$ for a class of variations that includes our solution.

A Few More Details

Consider the near boundary expansion of the metric component g_{uu}:

$$g_{uu} = -r^4 + \frac{2}{3}\gamma^2 r^2 + \mathcal{O}(1) \ ,$$

where $\gamma = \beta^2 r_+^4$.

- The r^4 behavior is a boundary condition and is not varied in calculating $\delta\mathcal{S}$.
- The leading γ^2 term is varied in a correlated way:

$$\delta g_{ab} = \frac{dg_{ab}}{d\gamma^2}\delta a \ , \ \ \delta A_a = \frac{dA_a}{d\gamma^2}\delta a \ , \ \ \delta\phi = \frac{d\phi}{d\gamma^2}\delta a \ ,$$

- The variation subleading to the γ^2 term ($\mathcal{O}(1)$ here) is allowed to be completely arbitrary.

The Viscosity

- The viscosity to entropy density ratio in this theory is the same as for $\mathcal{N} = 4$ super Yang-Mills

$$\eta/s = \hbar/4\pi k_B \ .$$

- We calculate η from a two point function of the spatial stress tensor component T^{12}:

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im}(G_{12,12}(\omega)) \ .$$

- The field dual to T^{12} in the bulk is δg_{12} which decouples from the rest of the fluctuations to linear order.
- Calculating the viscosity boils down to solving the equation of motion for a minimally coupled scalar in the bulk.

What's troubling me

- The compactification of v that becomes light-like at the boundary.
- Understanding one point functions of the stress-tensor that involve a u index.
- Understanding how to compute two point functions that involve u indices.
- What does it mean that our theory is a DLCQ of $\mathcal{N} = 4$ SYM twisted by the R-symmetry?
- To what extent are we learning anything about theories with Schrödinger symmetry that are not related to $\mathcal{N} = 4$ SYM?

Summary and Outlook

- We found a string theory embedding for the BMS construction.
- We found a nonzero temperature generalization of the BMS construction.
- We were able to verify the first law of thermodynamics and calculate a viscosity for this field theory.
- Can we find a string theory embedding for the BMS construction for a 3+1 dimensional field theory?
- Can we find string theory embeddings with different dynamical exponent z, $x \to \lambda x$, $t \to \lambda^z t$?
- Can the link to fermions at unitarity be made more precise?

Extra slides

Scalars in the BMS background

Consider a minimally coupled scalar

$$S = -\int d^{d+3}x \sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + m^2\phi^*\phi\right) \ .$$