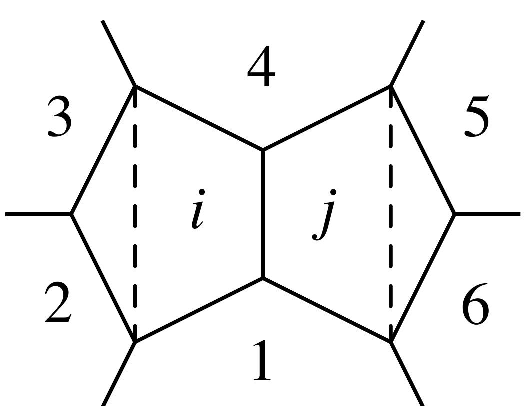
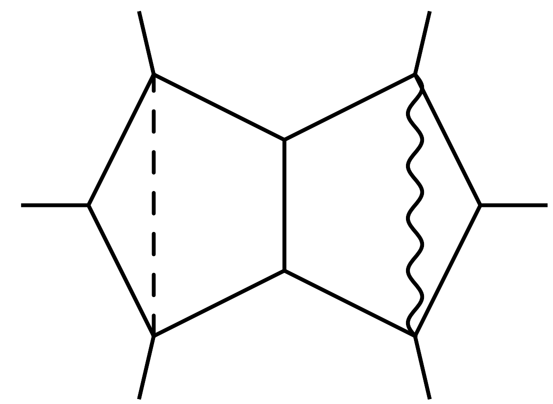
Analytic results for scattering amplitudes in N=4 super Yang-Mills





Johannes M. Henn, IAS

with James Drummond and Lance Dixon

arXiv:1108.4461 [hep-th], arXiv:1111.1704 [hep-th]

What is N=4 SYM?

- SU(N) Yang - Mills theory

  fermions and scalars, all in adjoint representation

- conformal field theory (CFT)

  no intrinsic ultraviolet (UV) divergences,

  beta function vanishes

=> perturbatively, very similar to QCD

- relation to string theory:

  AdS/CFT correspondence

Motivation: why N=4 SYM?

Discover fascinating new QFT structures

- New dualities

Wilson loops   Scattering amplitudes

Correlation functions   (Form factors)

- Hidden symmetries & integrability

Dual conformal / Yangian symmetry

AdS/CFT correspondence

Why scattering amplitudes? (I)

supersymmetric Yang-Mills as a tool for QCD

- tree-level amplitudes known analytically — Drummond, J.M.H. (2008)
- massless QCD trees from N=4 SYM — Dixon, J.M.H., Plefka, Schuster (2010)

  being used in Blackhat for phenomenology
- N=4 SYM suggests good loop-level integral basis — Arkani-Hamed et al. (2010)

new methods and insights:

- unitarity-based methods — ``The analytic S-matrix''; Bern, Dixon, Dunbar, Kosower, ...
- better understanding of IR divergences — Becher, Neubert; Dixon, Gardi, Magnea
- OPE constraints from Wilson loops — Alday, Gaiotto, Maldacena, Sever, Vieira (2010)
- recursion relations for planar loop integrands — Arkani-Hamed et al. (2010), Caron-Huot
- interesting questions for mathematicians — Gangl, Goncharov; Brown; Broadhurst Kreimer,...
- strong coupling: integrable Y-system — Alday, Maldacena, Sever, Vieira (2010)

Example: choice of integral basis

three-loop N=4 SYM form factor

$$
\begin{aligned}
F_S^{(3)} = R_\epsilon^3 \Bigg[&+ \frac{(3D-14)^2}{(D-4)(5D-22)} A_{9,1} - \frac{2(3D-14)}{5D-22} A_{9,2} - \frac{4(2D-9)(3D-14)}{(D-4)(5D-22)} A_{8,1} \\
&- \frac{20(3D-13)(D-3)}{(D-4)(2D-9)} A_{7,1} - \frac{40(D-3)}{D-4} A_{7,2} + \frac{8(D-4)}{(2D-9)(5D-22)} A_{7,3} \\
&- \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,4} - \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,5} \\
&- \frac{128(2D-7)(D-3)^2}{3(D-4)(3D-14)(5D-22)} A_{6,1} \\
&- \frac{16(2D-7)(5D-18)\left(52D^2-485D+1128\right)}{9(D-4)^2(2D-9)(5D-22)} A_{6,2} \\
&- \frac{16(2D-7)(3D-14)(3D-10)(D-3)}{(D-4)^3(5D-22)} A_{6,3} \\
&- \frac{128(2D-7)(3D-8)\left(91D^2-821D+1851\right)(D-3)^2}{3(D-4)^4(2D-9)(5D-22)} A_{5,1} \\
&- \frac{128(2D-7)\left(1497D^3-20423D^2+92824D-140556\right)(D-3)^3}{9(D-4)^4(2D-9)(3D-14)(5D-22)} A_{5,2} \\
&+ \frac{4(D-3)}{D-4} B_{8,1} + \frac{64(D-3)^3}{(D-4)^3} B_{6,1} + \frac{48(3D-10)(D-3)^2}{(D-4)^3} B_{6,2} \\
&- \frac{16(3D-10)(3D-8)\left(144D^2-1285D+2866\right)(D-3)^2}{(D-4)^4(2D-9)(5D-22)} B_{5,1} \\
&+ \frac{128(2D-7)\left(177D^2-1584D+3542\right)(D-3)^3}{3(D-4)^4(2D-9)(5D-22)} B_{5,2} \\
&+ \frac{64(2D-5)(3D-8)(D-3)}{9(D-4)^5(2D-9)(3D-14)(5D-22)} \\
&\times \left(2502D^5-51273D^4+419539D^3-1713688D^2+3495112D-2848104\right) B_{4,1} \\
&+ \frac{4(D-3)}{D-4} C_{8,1} + \frac{48(3D-10)(D-3)^2}{(D-4)^3} C_{6,1} \Bigg] . \qquad \text{(B.1)}
\end{aligned}
$$

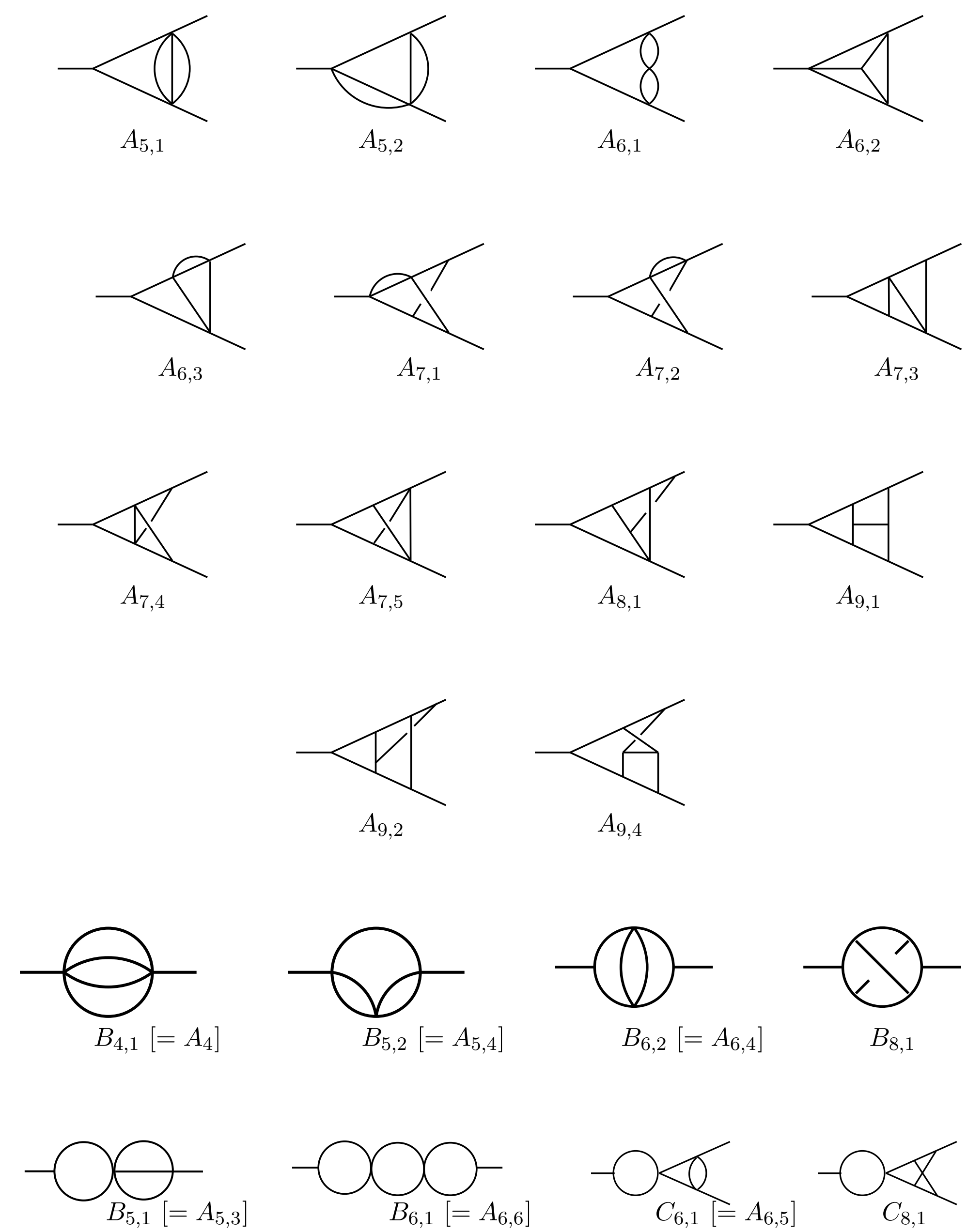


Figure 1: Master integrals for the three-loop form factors. Labels in brackets indicate the naming convention of Ref. [25].

Gehrmann, J.M.H., Huber (2011)

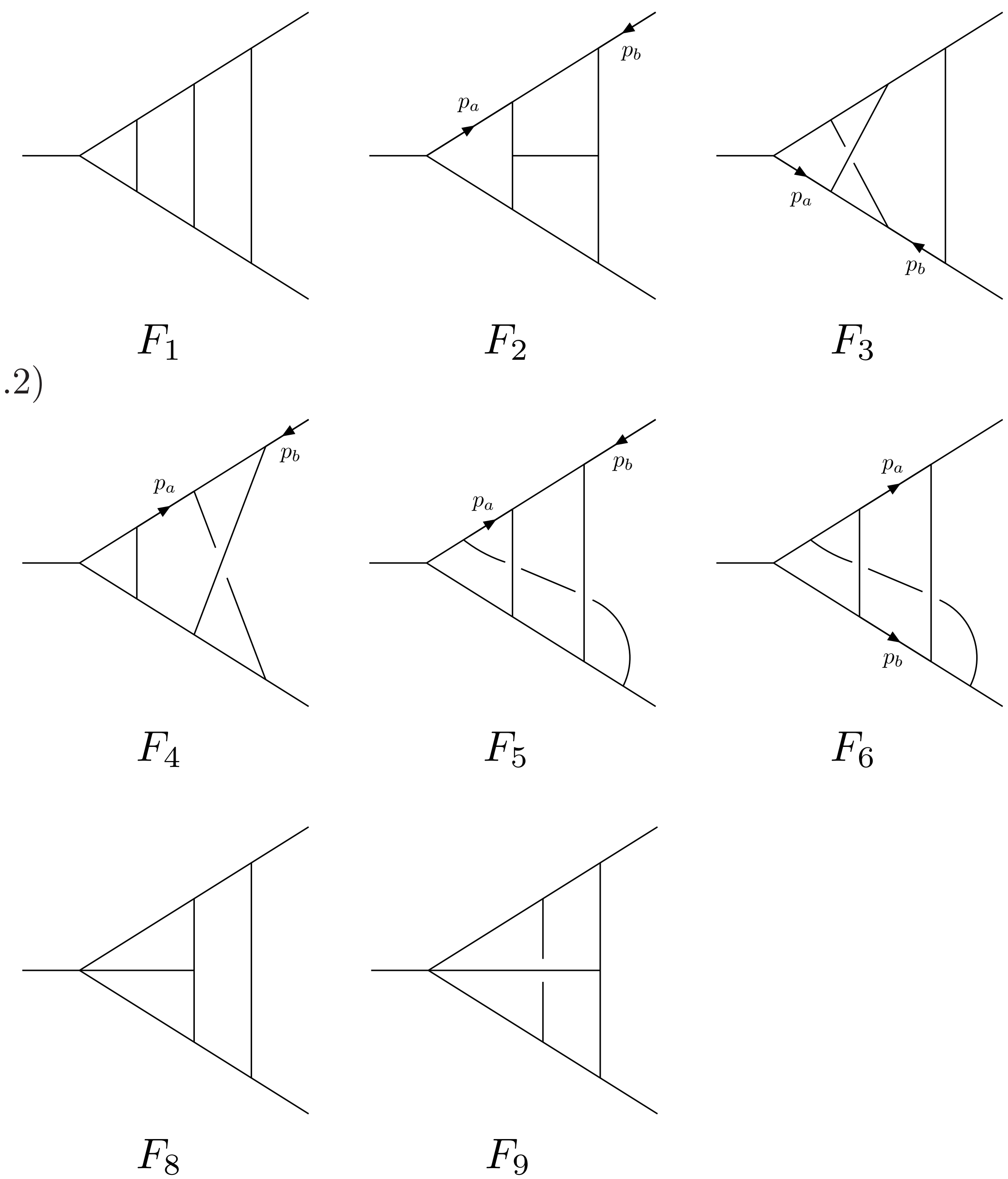
Gehrmann, Glover, Huber, Ikizlerli, Studerus;
Lee, Smirnov & Smirnov

Example: choice of integral basis

three-loop N=4 SYM form factor

$$F_S^{(3)} = R_\epsilon^3 \cdot [8\,F_1^{\rm exp} - 2\,F_2^{\rm exp} + 4\,F_3^{\rm exp} + 4\,F_4^{\rm exp} - 4\,F_5^{\rm exp} - 4\,F_6^{\rm exp} - 4\,F_8^{\rm exp} + 2\,F_9^{\rm exp}]$$

$$\begin{aligned} F_S^{(3)} &= R_\epsilon^3 \cdot [8\,F_1^{\rm exp} - 2\,F_2^{\rm exp} + 4\,F_3^{\rm exp} + 4\,F_4^{\rm exp} - 4\,F_5^{\rm exp} - 4\,F_6^{\rm exp} - 4\,F_8^{\rm exp} + 2\,F_9^{\rm exp}] \\ &= -\frac{1}{6\epsilon^6} + \frac{11\zeta_3}{12\epsilon^3} + \frac{247\pi^4}{25920\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{85\pi^2\zeta_3}{432} - \frac{439\zeta_5}{60}\right) \\ &\quad -\frac{883\zeta_3^2}{36} - \frac{22523\pi^6}{466560} + \epsilon\left(-\frac{47803\pi^4\zeta_3}{51840} + \frac{2449\pi^2\zeta_5}{432} - \frac{385579\zeta_7}{1008}\right) \\ &\quad +\epsilon^2\left(\frac{1549}{45}\zeta_{5,3} - \frac{22499\zeta_3\zeta_5}{30} + \frac{496\pi^2\zeta_3^2}{27} - \frac{1183759981\pi^8}{7838208000}\right) + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{5.2}$$



- each integral has uniform (and maximal) ``transcendentality''
  T[Zeta[n]] = n
  T[eps^-n] = n
  T[A B] = T[A] + T[B]

- for theories with less susy, other integrals also needed

Gehrmann, J.M.H., Huber (2011)

Example: choice of integral basis

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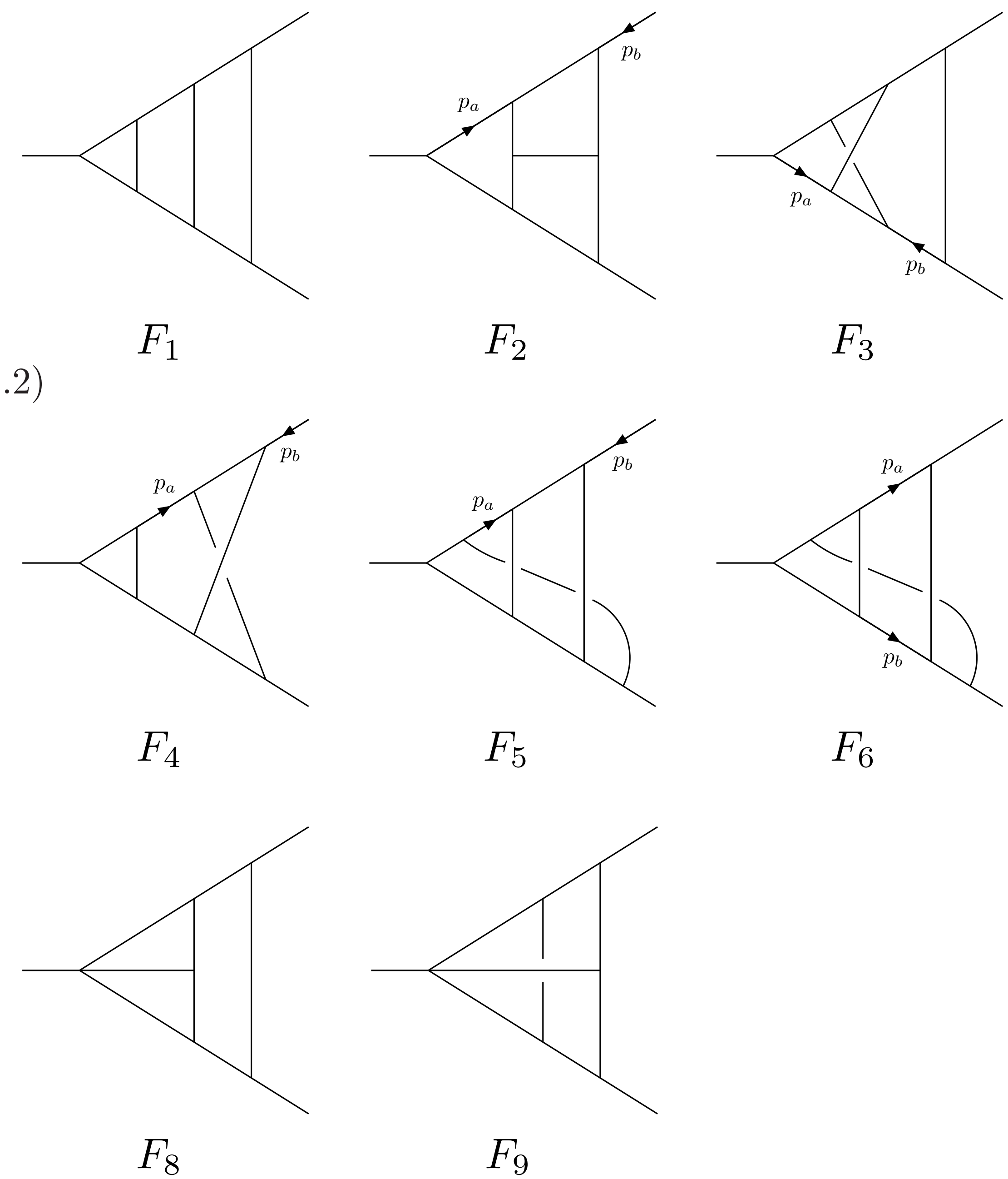

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Gehrmann, J.M.H., Huber (2011)

Why scattering amplitudes? (II)

all-order results?

- results based on new symmetries
- iterative structures:

- for loop integrals — Drummond, J.M.H. , Trnka (2010); Dixon, Drummond, J.M.H. (2011)

- for N=4 SYM Wilson loops — Caron-Huot (2011); Bullimore, Skinner (2011)

- can we resum the perturbative series?
- test the AdS/CFT correspondence

Amplitudes - where are we?

- tree-level and one-loop essentially completely understood

  methods: analytic properties (BCFW recursion, unitarity)

  unexpected simplicity: dual conformal symmetry

- four- and five-point amplitudes known to all orders:

  dual conformal symmetry & AdS/CFT

- n>5 points:

  partial results at two and three loops,

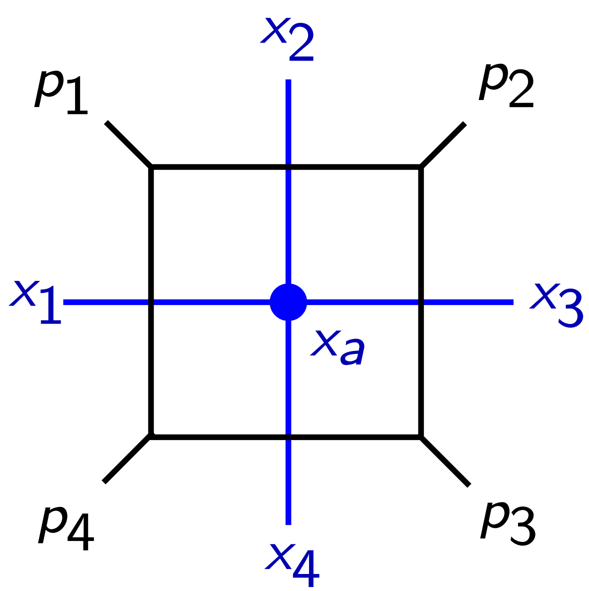
  especially for n=6

- strong coupling: integrable Y-system

Dual conformal symmetry

- first seen in planar loop integrals

Broadhurst (1993); Drummond, J.M.H., Smirnov, Sokatchev (2006)



$$x_i^\mu - x_{i+1}^\mu = p_i^\mu$$

- natural at strong coupling: isometry of T-dual AdS space

Alday Maldacena (2007)

- symmetry unbroken in "Higgsed" N=4 SYM

Alday, J.M.H., Plefka, Schuster (2009)

The remainder function

- 4 of 5-point amplitudes fixed by dual conformal Ward identity

Drummond, J.M.H., Korchemsky, Sokatchev (2007)

- remainder function:

$$A_{MHV} = A^{\text{tree}}_{MHV} M_{MHV} \,,$$

$$\log(M_{MHV}) = A_{IR} + A_{BDS} + R(u,v,w) + \mathcal{O}(\epsilon)$$

variables: $x_i^\mu - x_{i+1}^\mu = p_i^\mu$

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \,, \qquad v = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} \,, \qquad w = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{25}^2} \,.$$

- modification of BDS ansatz was expected

Bern, Dixon, Smirnov (2005)

Alday, Maldacena (2007); Drummond, J.M.H., Korchemsky, Sokatchev (2007), Bartels, Lipatov, Sabio-Vera (2008), Bern et al (2008)

The ratio function

- define "ratio" $\mathcal{A}_{\rm NMHV} = \mathcal{A}_{\rm MHV}\,\mathcal{P}_{\rm NMHV}$

(similar for generic non-MHV)

Drummond, J.M.H., Korchemsky, Sokatchev (2008)

$\mathcal{A}_{\rm MHV}$ removes IR divergences from $\mathcal{A}_{\rm NMHV}$

$\mathcal{P}_{\rm NMHV}$ is IR finite, expected to be dual conformal

Drummond, J.M.H., Korchemsky, Sokatchev (2008)

Elvang, Freedman, Kiermaier (2009) Branduber, Heslop, Travaglini (2009)

Roiban, Kosower, Vergu (2010)

- at six points

$$\mathcal{P}_{\rm NMHV} = \frac{1}{2}(1)\left[V(u,v,w) + \tilde{V}(y_u,y_v,y_w)\right] + {\rm cyclic}$$

helicity factors: $(1) = R_{1;3,5}$ $\quad R_{r;ab} = \dfrac{\langle a,\, a-1\rangle\langle b,\, b-1\rangle\ \delta^4\big(\langle r|x_{ra}x_{ab}|\theta_{br}\rangle + \langle r|x_{rb}x_{ba}|\theta_{ar}\rangle\big)}{x_{ab}^2\,\langle r|x_{ra}x_{ab}|b\rangle\,\langle r|x_{ra}x_{ab}|b-1\rangle\,\langle r|x_{rb}x_{ba}|a\rangle\,\langle r|x_{rb}x_{ba}|a-1\rangle}$

variables:

$$y_u = \frac{u-z_+}{u-z_-}\,, \qquad z_\pm = \frac{1}{2}\left[-1+u+v+w\pm\sqrt{\Delta}\right]\,, \qquad \Delta = (1-u-v-w)^2 - 4uvw\,.$$

- symmetry $\quad V(w,v,u) = V(u,v,w)\,, \qquad \tilde{V}(y_w,y_v,y_u) = -\tilde{V}(y_u,y_v,y_w)\,.$

Constraints on NMHV ratio function

- vanishing in collinear limit

$$\left[V(u,v,w)+V(w,u,v)+\tilde V(y_u,y_v,y_w)-\tilde V(y_w,y_u,y_v)\right]_{w\to 0,\ v\to 1-u}=0\,.$$

- absence of spurious poles

Korchemsky, Sokatchev (2009)

Dixon, Drummond, J.M.H. (2011)

$$\left[V(u,v,w)-V(w,u,v)+\tilde V(y_u,y_v,y_w)-\tilde V(y_w,y_u,y_v)\right]_{(*)}=0\,.$$

$$(*):\qquad w\to 1\,,\qquad y_u\to(1-w)\frac{u(1-v)}{(u-v)^2}\,,\qquad y_v\to\frac{1}{(1-w)}\frac{(u-v)^2}{v(1-u)}\,,\qquad y_w\to\frac{1-u}{1-v}\,.$$

Example:

$$V^{(1)}=\frac{1}{2}\Big[-\log u\log w+\log(uw)\log v+\mathrm{Li}_2(1-u)+\mathrm{Li}_2(1-v)+\mathrm{Li}_2(1-w)-2\zeta_2\Big]\,,\qquad \tilde V^{(1)}=0\,.$$

- consistency with OPE expansion of (super) Wilson loops: L-th discontinuity annihilated by differential operator

Alday, Gaiotto, Maldacena, Sever, Vieira (2010) Sever, Vieira, Wang (2011)

- expect at L loops: 2L-fold iterated integrals

What are symbols?

- in loop calculations, complicated iterated integrals appear, that are generalizations of the polylogarithm:

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t}\,\mathrm{Li}_{n-1}(t)\,, \qquad \mathrm{Li}_1(x) = -\log(1-x)$$

- the symbol S(f) captures important properties of a function f
  e.g. derivatives, locations of branch cuts
  while forgetting precise integration contours, numerical values
  (can be reconstructed later)

- symbol reduces complicated identities between functions,
  e.g. polylogarithm identities to simple algebra

Pure functions and symbols

- pure function: derivatives can be written as:

$$d\,f^{(k)} = \sum_r f_r^{(k-1)}\, d\log\phi_r \,.$$

with some algebraic functions ϕ_r

- define symbol recursively in degree k: Goncharov (2009); also: F. Brown

$$\mathcal{S}(f^{(k)}) = \sum_{\vec{\alpha}} \phi_{\alpha_1} \otimes \ldots \otimes \phi_{\alpha_k} \,,$$

- Examples:

- by definition: $\mathcal{S}(\log x) = x\,,\qquad \mathcal{S}(\log(1-x)) = 1-x$

- if derivative is known, symbol is known:

$$\frac{d}{dx}\mathrm{Li}_2(x) = -\frac{\log(1-x)}{x} \quad\longrightarrow\quad \mathcal{S}(\mathrm{Li}_2(x)) = -[(1-x)\otimes x]$$

- symbols of products from factors:

$$\mathcal{S}(\log x \log y) = x\otimes y + y \otimes x$$

Useful symbol properties

- factorization (inherited from logarithm)

$$\ldots \otimes x\, y \otimes \ldots = \ldots \otimes x \otimes \ldots + \ldots \otimes y \otimes \ldots$$

- integrability   not every (multi-variables) symbol is a function

$$\mathcal{S}(\log x \log y) = x \otimes y + y \otimes x$$

but no function has symbol $x \otimes y - y \otimes x$

- integrability test   Goncharov   Gaiotto, Maldacena, Sever, Vieira (2011)

$$\phi_1 \otimes \ldots \otimes \phi_i \otimes \phi_{i+1} \otimes \ldots \otimes \phi_k$$

$$\to (d \log \phi_i \wedge d \log \phi_{i+1})[\phi_1 \otimes \ldots \otimes \ldots \otimes \phi_k] = 0$$

for symbols of functions

- first entry controls branch cut location   Gaiotto, Maldacena, Sever, Vieira (2011)

can only be u, v, w

Reason: cuts start at $x_{ij}^2 = 0$ for massless particles

Two-loop remainder function

Drummond, J.M.H., Korchemsky, Sokatchev (2007/8) Bern et al (2008) Del Duca, Duhr, Smirnov (2009) Drummond, J.M.H. (2010)

- can be expressed in terms of classical polylogarithms

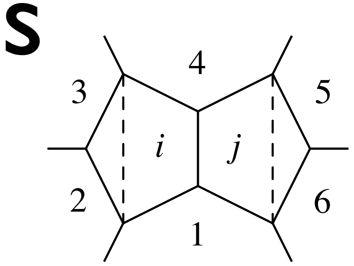
Goncharov, Spradlin, Vergu, Volovich (2010)

- symbol is remarkably simple:

$$\mathcal{S}(R_6^{(2)}) = -\frac{1}{8}\Big\{\Big[u\otimes(1-u)\otimes\frac{u}{(1-u)^2}+2\,(u\otimes v+v\otimes u)\otimes\frac{w}{1-v}+2\,v\otimes\frac{w}{1-v}\otimes u\Big]\otimes\frac{u}{1-u} + \Big[u\otimes(1-u)\otimes y_u y_v y_w - 2\,u\otimes v\otimes y_w\Big]\otimes y_u y_v y_w\Big\} + \text{permutations}\,, \qquad (21)$$

$$y_u = \frac{u-z_+}{u-z_-}\,, \qquad z_\pm = \frac{1}{2}\Big[-1+u+v+w\pm\sqrt{\Delta}\Big]\,, \qquad \Delta = (1-u-v-w)^2-4uvw\,.$$

- relation to finite loop integrals

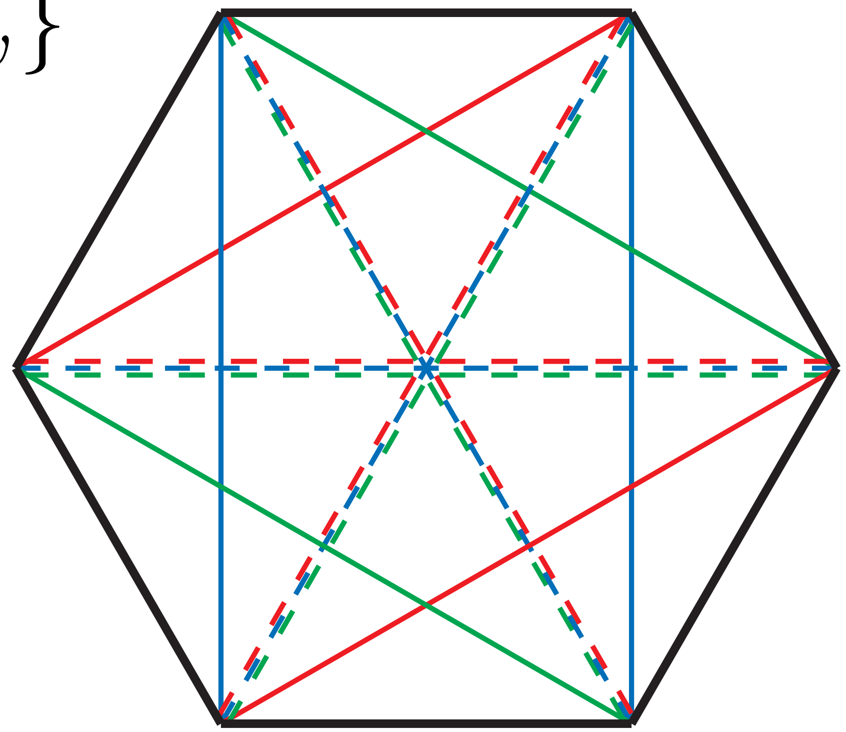


Arkani-Hamed et al (2010)
Dixon, Drummond, J.M.H. (2011)

Ansatz for the symbol

- assume symbol is built from the following nine letters

$$\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$



$$u = \frac{y_u(1-y_v)(1-y_w)}{(1-y_w y_u)(1-y_u y_v)}, \quad 1-u = \frac{(1-y_u)(1-y_u y_v y_w)}{(1-y_w y_u)(1-y_u y_v)},$$
$$v = \frac{y_v(1-y_w)(1-y_u)}{(1-y_u y_v)(1-y_v y_w)}, \quad 1-v = \frac{(1-y_v)(1-y_u y_v y_w)}{(1-y_u y_v)(1-y_v y_w)},$$
$$w = \frac{y_w(1-y_u)(1-y_v)}{(1-y_v y_w)(1-y_w y_u)}, \quad 1-w = \frac{(1-y_w)(1-y_u y_v y_w)}{(1-y_v y_w)(1-y_w y_u)}.$$

Motivation: two-loop remainder function — Goncharov, Spradlin, Vergu, Volovich (2010)

explicitly known six-point loop integrals — Dixon, Drummond, J.M.H. (2011); Del Duca, Duhr, Smirnov (2011)

Imposing constraints on the ansatz

Summary of constraints on symbol:

- integrability
- first entry (absence of unphysical branch cuts)
- symmetry
- collinear behavior
- absence of spurious divergences
- consistency with OPE

} imply non-zero odd part at two loops!

Result:
$$\mathcal{S}(V) \;=\; \alpha_X\,\mathcal{S}(V_X) + \sum_{i=1}^{9} \alpha_i\,\mathcal{S}(f_i), \qquad \mathcal{S}(\tilde{V}) \;=\; \alpha_X\,\mathcal{S}(\tilde{V}_X) + \alpha_8\,\mathcal{S}(\tilde{f})\,,$$

Corollary:
$$\Delta_v \Delta_v \mathcal{P}^{(2356)}_{\text{NMHV}} \;\propto\; \frac{1}{(2356)}\Big[\log^2 u + \log^2 w + 4\,\log u\,\log w + 2\,\log^2(1-v) - 4\log(uw)\,\log(1-v) - 2\Big(\text{Li}_2(1-u) + \text{Li}_2(1-w) - 2\,\zeta_2\Big)\Big]\,.$$

From symbols to functions

Procedure:

- write down candidate function that has the correct symbol
- parametrize the beyond the symbol ambiguities
- apply constraints for the functions

Example:

$$X := u \otimes (1-u) \otimes w \otimes (1-w) + u \otimes w \otimes (1-u) \otimes (1-w) + u \otimes w \otimes (1-w) \otimes (1-u)$$
$$+ w \otimes u \otimes (1-u) \otimes (1-w) + w \otimes u \otimes (1-w) \otimes (1-u) + w \otimes (1-w) \otimes u \otimes (1-u)$$

$$f = \mathrm{Li}_2(1-u)\mathrm{Li}_2(1-w)\,, \qquad \mathcal{S}(f) = X$$

$$\hat{f} = -\frac{\zeta_2}{2}\left[\mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w)\right] + \frac{5}{4}\zeta_4$$

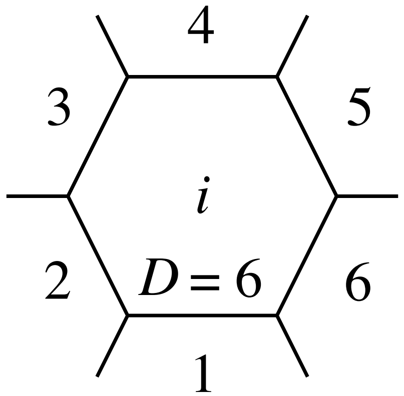
$f + \hat{f}$ satisfies collinear and spurious constraints

New functions appearing in the ansatz

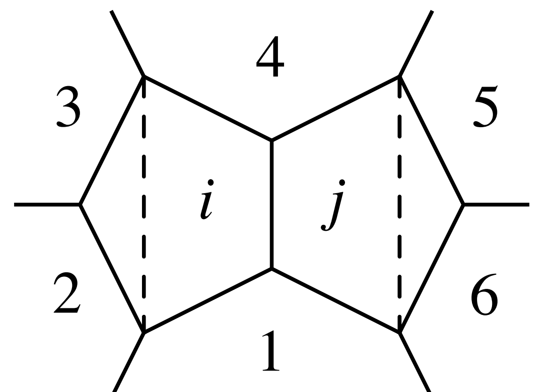
- after imposing constraints at function level:
  10 coefficients left to be determined
- most functions are expressible using polylogarithms with rational arguments, e.g.

$$\mathrm{Li}_2(1-u)\mathrm{Li}_2(1-w) - \frac{\zeta_2}{2}\left[\mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w)\right] + \frac{5}{4}\zeta_4$$

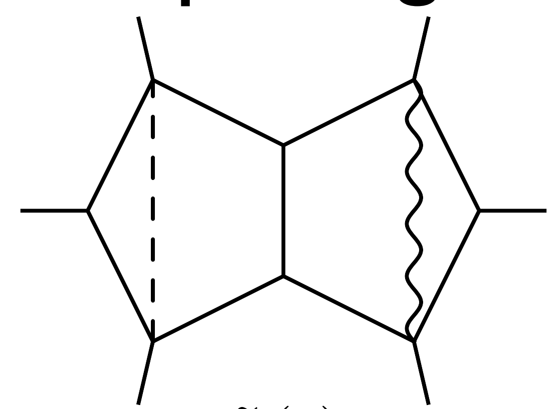
- other functions: can be identified with finite loop integrals!



$\tilde{\Phi}$



$\Omega^{(2)}$


$\tilde{\Omega}^{(2)}$

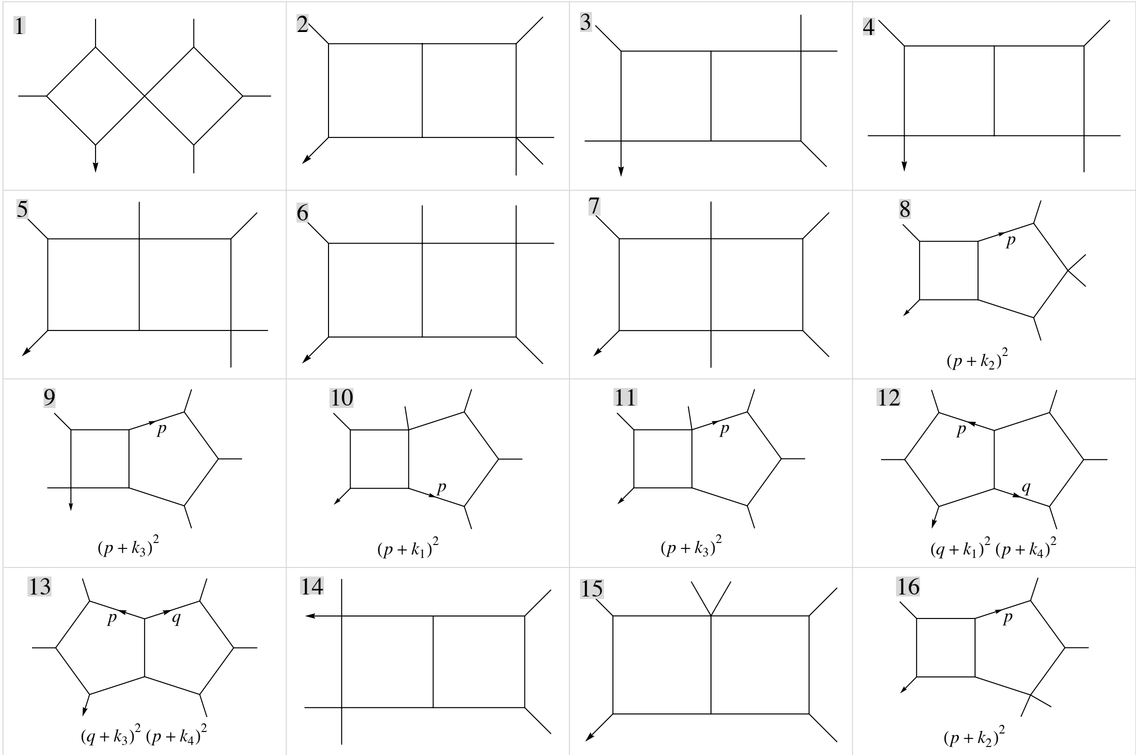
- new form of MHV remainder function

$$\mathcal{R}_6^{(2)}(u,v,w) = \frac{1}{4}\Big[\Omega^{(2)}(u,v,w) + \Omega^{(2)}(v,w,u) + \Omega^{(2)}(w,u,v)\Big] + \mathcal{R}_{6,\mathrm{rat}}^{(2)}$$

How to fix the remaining coefficients?

Analytic computation via loop integrals

- loop integral expression for even part Roiban, Kosower, Vergu (2010)



- NMHV and MHV similar

  eliminate cumbersome integrals 12 and 13
  using known MHV answer (for equal cross ratios)

- remaining integrals easy to evaluate in massive regularization

  Drummond, J.M.H. (2010)
  J.M.H., Naculich Schnitzer, Spradlin (2010)

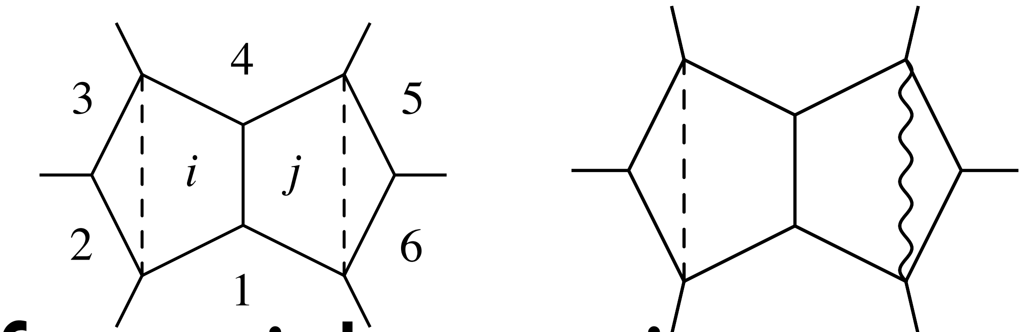
- analytic result in limits, e.g.

$$\lim_{u\to 0} S_*^{(2)}|_{\log^0 m^2} = \frac{5}{32}\log^4 u + \log^3 u\left[\frac{3}{4}u + \frac{7}{8}u^2 + \frac{7}{4}u^3 + \frac{71}{16}u^4 + \frac{253}{20}u^5 + \mathcal{O}(u^6)\right]$$
$$+ \log^2 u\left[-\frac{\pi^2}{12} + \frac{7}{4}u^2 + \frac{19}{4}u^3 + \frac{653}{48}u^4 + \frac{995}{24}u^5 + \mathcal{O}(u^6)\right] + \mathcal{O}(\log u)\,,$$

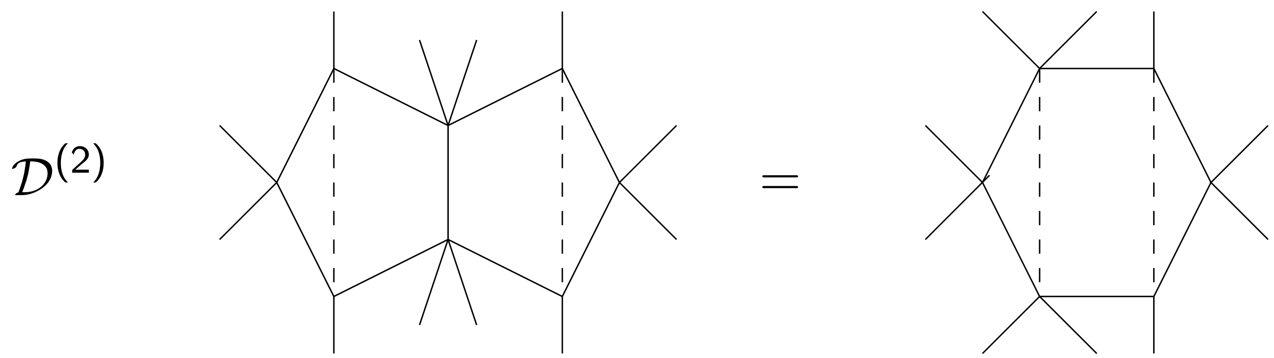
- fixes all coefficients in our ansatz; nontrivial check!
- numerical checks for generic u

New functions $\Omega^{(2)}, \tilde{\Omega}^{(2)}$

- "most complicated" part of amplitude
- finite, dual conformal integrals



- hints of simplicity: 2nd-order differential equations



Drummond, J.M.H, Trnka (2010)

- solution $\Omega^{(2)}(u,v,w) = -6\,\zeta_4 + \int_1^u \frac{du_t}{u_t(u_t-1)}\, Q_\phi(u_t,v_t,w_t)\,.$

Drummond, J.M.H. Dixon (2010)

$$
\begin{aligned}
Q_\phi(u,v,w) \;=\;& 2\left[\mathrm{Li}_3(1-w) + \mathrm{Li}_3\left(1-\frac{1}{w}\right)\right] \qquad (4.6)\\
&+ \log w \left[-\mathrm{Li}_2(1-w) + \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \log u\, \log v - 2\,\zeta_2\right]\\
&- \frac{1}{3}\log^3 w - 2\,\mathrm{Li}_3(1-u) - \mathrm{Li}_3\left(1-\frac{1}{u}\right) - 2\,\mathrm{Li}_3(1-v) - \mathrm{Li}_3\left(1-\frac{1}{v}\right)\\
&+ \log\left(\frac{u}{v}\right)\left[\mathrm{Li}_2(1-u) - \mathrm{Li}_2(1-v)\right] + \frac{1}{6}\log^3 u + \frac{1}{6}\log^3 v\\
&- \frac{1}{2}\log u\, \log v\, \log(uv)\,.
\end{aligned}
$$

$$
\begin{aligned}
v_t &= \frac{(1-u)\,v\,u_t}{u\,(1-v) + (v-u)\,u_t}\,,\\
w_t &= 1 - \frac{(1-w)\,u_t\,(1-u_t)}{u\,(1-v) + (v-u)\,u_t}\,.
\end{aligned}
$$

6-point 2-loop NMHV function

$$\mathcal{A}_{\rm NMHV} = \mathcal{A}_{\rm MHV}\, \mathcal{P}_{\rm NMHV}$$

$$\mathcal{P}_{\rm NMHV} = \frac{1}{2}(1)\left[V(u,v,w) + \tilde{V}(y_u, y_v, y_w)\right] + \text{cyclic}$$

analytic formula

$$V + \tilde{V} = -\frac{1}{2}\left[\Omega^{(2)}(w,u,v) + \tilde{\Omega}^{(2)}(1/y_w, 1/y_u, 1/y_v)\right] + T(u,v,w)$$

T given by polylogarithms with rational arguments

$$\begin{aligned}
\left[V + \mathcal{R}_6^{(2)}\right]\left(\tfrac{16}{5}, \tfrac{112}{85}, \tfrac{28}{17}\right) &= 14.4289552936316184 92\,, \\
\left[V + \mathcal{R}_6^{(2)}\right]\left(\tfrac{112}{85}, \tfrac{28}{17}, \tfrac{16}{5}\right) &= 12.613874875030471932\,, \\
\left[V + \mathcal{R}_6^{(2)}\right]\left(\tfrac{28}{17}, \tfrac{16}{5}, \tfrac{112}{85}\right) &= 11.705797993389994692\,.
\end{aligned}$$

perfect agreement with numerical values from Roiban, Kosower, Vergu (2010)

new predictions for odd part

6-point two-loop NMHV ratio

$$V + \tilde{V} \;=\; -\frac{1}{2}\left[\Omega^{(2)}(w,u,v) + \tilde{\Omega}^{(2)}(1/y_w, 1/y_u, 1/y_v)\right] + T(u,v,w)$$

$$T(u,v,w) = T^A(u,v,w) + T^A(w,v,u) + T^B(u,v,w)$$

$$\begin{aligned}
T^A(u,v,w) \;=\;& -\frac{1}{2}\,\mathrm{Li}_4\left(1-\frac{1}{u}\right) - \frac{3}{2}\,\mathrm{Li}_4(1-u) + \frac{1}{2}\,\mathrm{Li}_4(u) + \frac{1}{12}\,\ln^3 u\,\ln(1-u) + \ln\left(\frac{uv}{w}\right)\mathrm{Li}_3(1-u)\\
&+ \frac{1}{2}\,\ln\left(\frac{v}{w}\right)\mathrm{Li}_3\left(1-\frac{1}{u}\right) + \frac{3}{8}\,[\mathrm{Li}_2(1-u)]^2 + \frac{1}{8}\left[4\,\mathrm{Li}_2(1-u) + \ln^2 u\right]\mathrm{Li}_2(1-v)\\
&+ \frac{1}{8}\left[6\,\ln v\,\ln w - 2\,\ln u\,\ln\left(\frac{v}{w}\right) - \ln^2 v - \ln^2 w - 12\,\zeta_2\right]\mathrm{Li}_2(1-u)
\end{aligned}$$

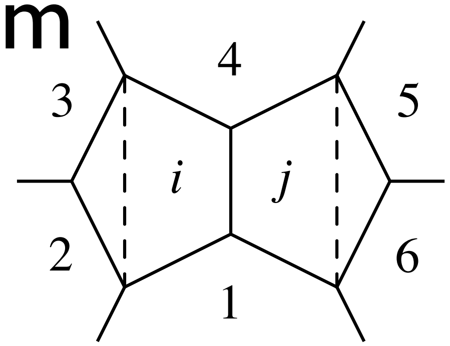
$$\begin{aligned}
T^B(u,v,w) \;=\;& \mathrm{Li}_4\left(1-\frac{1}{v}\right) + \frac{1}{2}\,\mathrm{Li}_4(1-v) + \frac{1}{2}\,\mathrm{Li}_4(v) + \frac{1}{12}\,\ln^3 v\,\ln(1-v) + \frac{1}{2}\,\ln v\,\mathrm{Li}_3\left(1-\frac{1}{v}\right)\\
&+ \frac{1}{8}\,[\mathrm{Li}_2(1-v)]^2 + \frac{1}{4}\left[\ln(uw)\,\ln v - \ln u\,\ln w - 2\,\zeta_2\right]\left[\mathrm{Li}_2(1-v) - 6\,\zeta_2\right] - 3\,\zeta_4\\
&+ \frac{1}{2}\,\mathrm{Li}_2(1-u)\,\mathrm{Li}_2(1-w) - \frac{1}{48}\,\ln^4\left(\frac{u}{w}\right) + \frac{1}{16}\,\ln^2 u\,\ln^2 w - \frac{1}{12}\,(\ln^3 u + \ln^3 w)\,\ln v\\
&+ \frac{1}{16}\,(\ln^2 u + \ln^2 w + 4\,\ln u\,\ln w)\,\ln^2 v - \frac{1}{24}\,\ln^4 v - \frac{\zeta_2}{4}\,(\ln^2 u + \ln^2 w - \ln^2 v) - \frac{\zeta_3}{2}\,\ln(uvw)
\end{aligned}$$

$$\begin{aligned}
\left[V + \mathcal{R}_6^{(2)}\right]\left(\tfrac{16}{5}, \tfrac{112}{85}, \tfrac{28}{17}\right) \;&=\; 14.428955293631618492\,,\\
\left[V + \mathcal{R}_6^{(2)}\right]\left(\tfrac{112}{85}, \tfrac{28}{17}, \tfrac{16}{5}\right) \;&=\; 12.613874875030471932\,,\\
\left[V + \mathcal{R}_6^{(2)}\right]\left(\tfrac{28}{17}, \tfrac{16}{5}, \tfrac{112}{85}\right) \;&=\; 11.705797993389994692\,.
\end{aligned}$$

Conclusions and outlook

Analytic formula for 2-loop NMHV ratio function

- analytic result for six-point two-loop NMHV ratio function
- finite loop integrals play key role; classical polylogarithms not sufficient; simple integral representations derived from differential equations
- new representation of two-loop remainder function



Similar analysis for 3-loop MHV remainder function

- symbol fixed up to 2 parameters

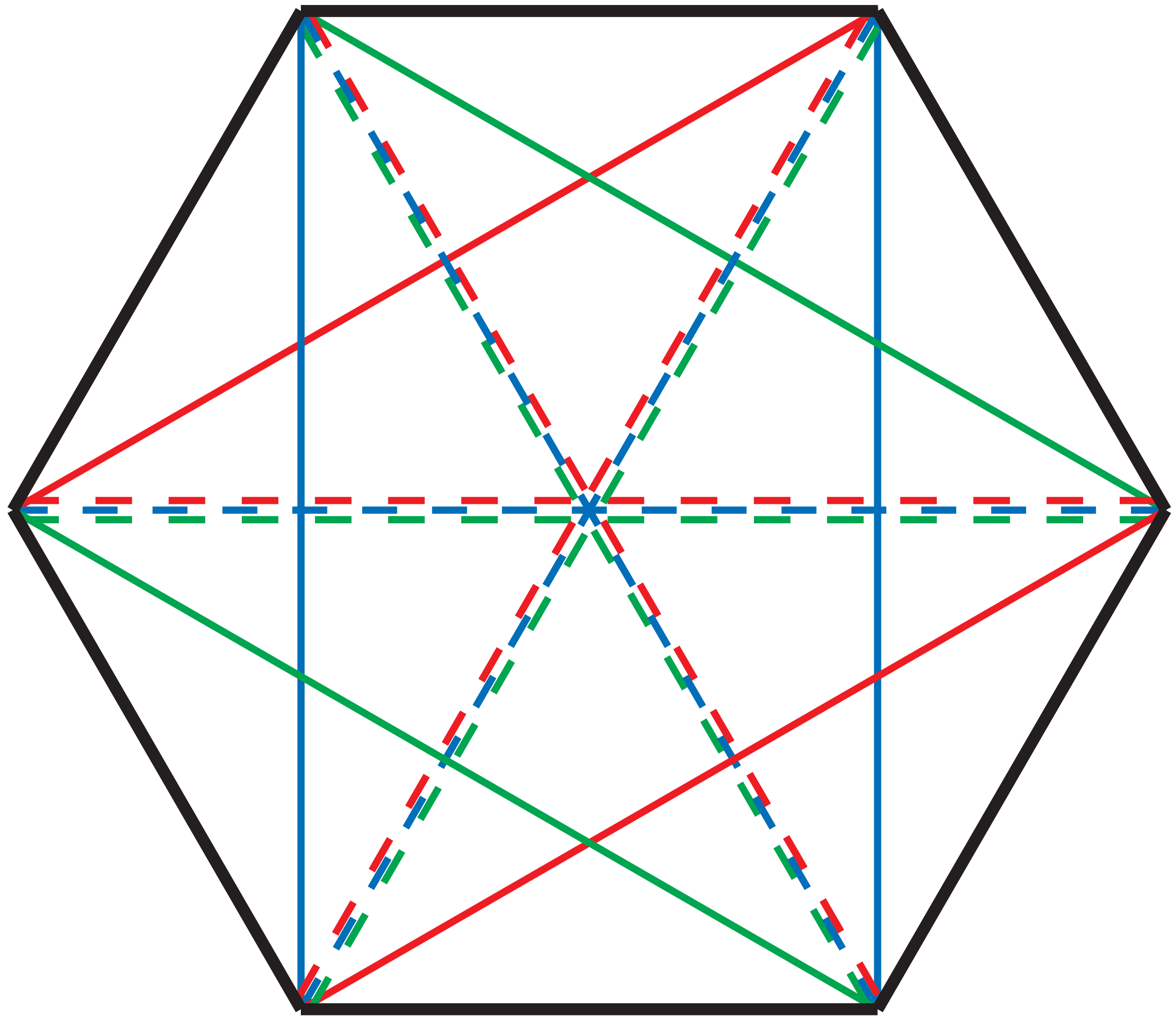
Dixon, Drummond, J.M.H. (2011)

independently confirmed: Caron-Huot (2011)

- analytic prediction for NLO and NNLO Regge limit

NLO prediction recently confirmed by Fadin, Lipatov (2011)
see also: Bartels, Lipatov, Prygarin

Extra slides



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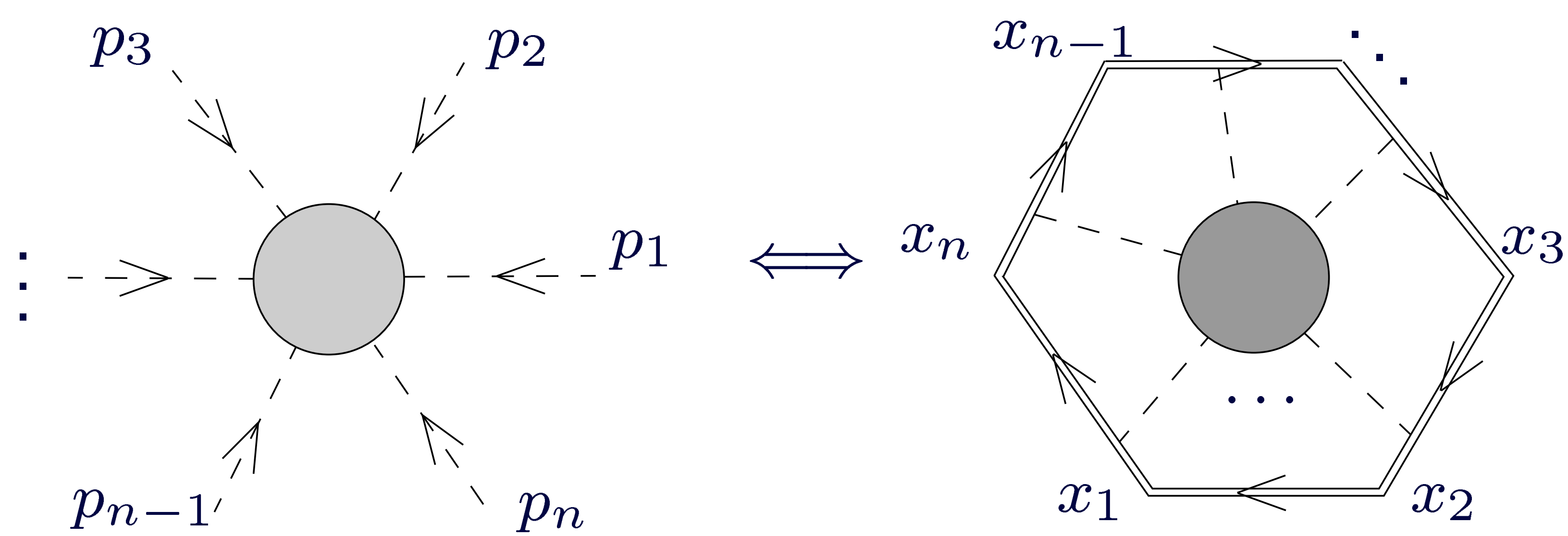
Duality between Wilson loops and scattering amplitudes



$$x^{\mu}_{i+1} - x^{\mu}_{i} = p^{\mu}_{i}$$

- motivated by AdS/CFT
  Alday, Maldacena (2007)
- also present at weak coupling
  one loop: Drummond, Korchemsky, Sokatchev; Brandhuber, Heslop Travaglini (2007),
  two loops: Drummond, J.M.H. Korchemsky, Soktachev (2007, 2008); Bern et al. (2008)
- extension to non-MHV amplitudes
  Caron-Huot; Mason, Skinner; subtleties with regularization: Belitsky, Korchemsky, Sokatchev (2010)
- (dual) conformal Ward identities Drummond, J.M.H, Korchemsky, Sokatchev (2007)
- operator product expansion (OPE) for Wilson loops
  Alday, Gaiotto, Maldacena, Sever, Vieira (2010)