Causality Sum Rules in Quantum Field Theory

Tom Hartman
Cornell University

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Causality imposes constraints on effective field theories.

Example:

$$L = (\partial\phi)^2 + \lambda(\partial\phi)^4 + \cdots$$

for $\lambda < 0$ cannot be embedded in any consistent UV theory.

This plays a key role in the proof of the *a* theorem.

Similar constraints on higher curvature gravity are even stronger:

$$L = \sqrt{g}(R + \alpha_{GB}R^2 + \cdots) \quad \Rightarrow \alpha_{GB} = 0!$$

[Adams, Arkani-Hamed, Dubovsky, Nicolas, Rattazzi]
[Komargodski, Schwimmer]
[Camanho, Edelstein, Maldacena, Zhiboedov]

Two arguments for this type of constraint:

- Causality in non-trivial backgrounds

- Optical theorem / dispersion relations

Both are indirect: to any order perturbatively around vacuum,

$$[\phi(x), \phi(y)] = 0 \quad \text{(spacelike)}$$

This must fail at some point (UV); but why and where, exactly?

Also, both arguments are inherently *Lorentzian* signature.

But good Lorentzian theories are in one-to-one correspondence with good Euclidean theories:

Minkowski space:

Lorentz invariant

Unitary

Causal

Schwinger,
Wightman,
Osterwalder,
Schrader

Euclidean:

SO(d) invariant

Reflection positive

So where is the problem in the Euclidean theory?

This talk: Sum rules for ***position space*** correlators

- $d > 2$
- strong or weak coupling (CFT, EFT)

The sum rules + reflection positivity imply infrared constraints.

In various limits, these constraints reproduce many known facts and some new ones:

In EFT:

- "Almost"-Euclidean derivation of $\lambda(\partial\phi)^4$ and *a*-theorem
- Maldacena-Shenker-Stanford Chaos bound
- Constraints on fine-tuning of non-renormalizable operators

In CFT:

- If *N* large: holographic dual of $\lambda(\partial\phi)^4$
- Sign constraints on OPE coefficients / anomalous dimensions of high-spin operators
- Bootstrap derivation of Hofman-Maldacena bounds on *a/c*
- Maldacena-Zhiboedov: no higher spin currents in CFT

CFT:

1509.00014 and 1601.07904

with Sachin Jain and Sandipan Kundu

Perturbative QFT:

work in progress

with Nima Afkhami-Jeddi, Venkatesa Chandrasekaran, and Amir Tajdini

In most of the talk I will use CFT language, but almost everything has a corresponding statement in (non-conformal) perturbative EFT.

Holographic Motivation

Holographic 4d CFTs have *a=c*.

Anomaly coefficients in <TTT>

This is dictated by locality in the bulk:

$$S = \int \sqrt{g}(R + \text{small})$$

Can we derive this from CFT?

Two ingredients are probably needed:

- Bootstrap (some successes in d=2)
- Causality (important in bulk argument)
  [Camanho, Edelstein, Maldacena, Zhiboedov]

Causality review

Causality:

$$\langle\Psi|[O(x),\ O(y)]|\Psi\rangle = 0 \qquad\qquad (x-y)^2 < 0\ .$$

This is a Lorentzian statement.

But bootstrap is usually formulated in terms of Euclidean correlation functions.

So first:

How is causality encoded in Euclidean correlators?

This was understood long ago [*eg, Streater and Wightman].*

Euclidean correlators

$$G(x_1, x_2, \dots) \equiv \langle O(x_1) O(x_2) \dots \rangle$$

are:

- Permutation invariant $G(x_1, x_2, \dots) = G(x_2, x_1, \dots)$
- With singularities only at coincident points
- and no branch cuts (ie, single-valued).

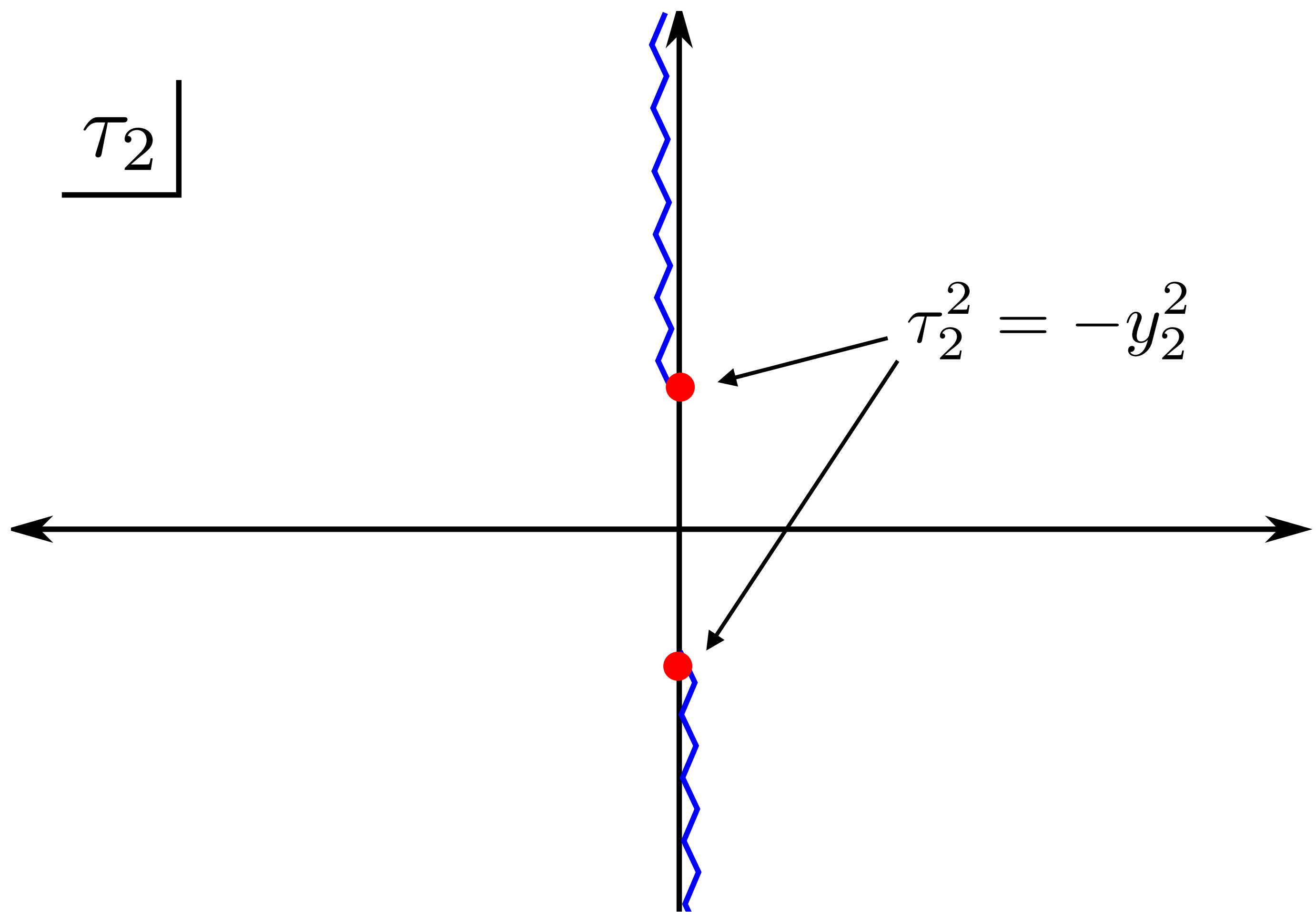
Ex: conformal scalar

$$\langle O(0,0) O(\tau_2, y_2) \rangle = (\tau_2^2 + y_2^2)^{-2\Delta}$$

But if we analytically continue to complex time: $\tau_i \in \mathbf{C}$

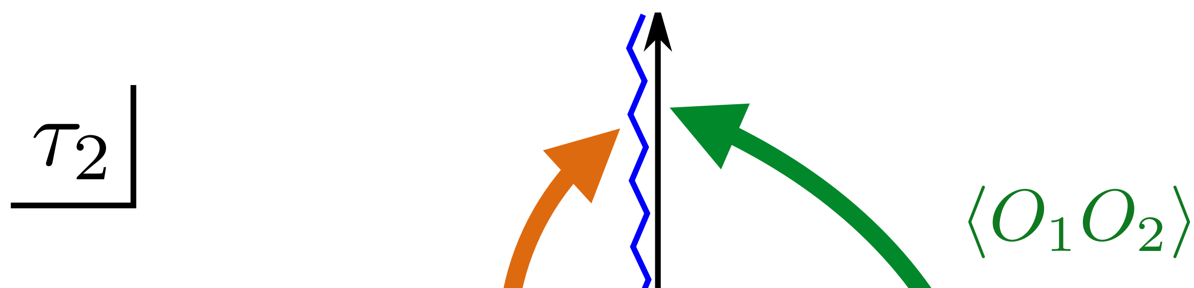
then there is an intricate structure of singularities and branch cuts.

Ex: conformal scalar 2pt function $G = (\tau_2^2 + y_2^2)^{-2\Delta}$



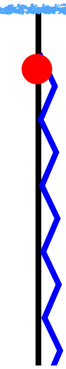
Therefore the analytic continuation to Lorentzian signature is ambiguous.

This ambiguity is why operators do not commute in Lorentzian QFT.



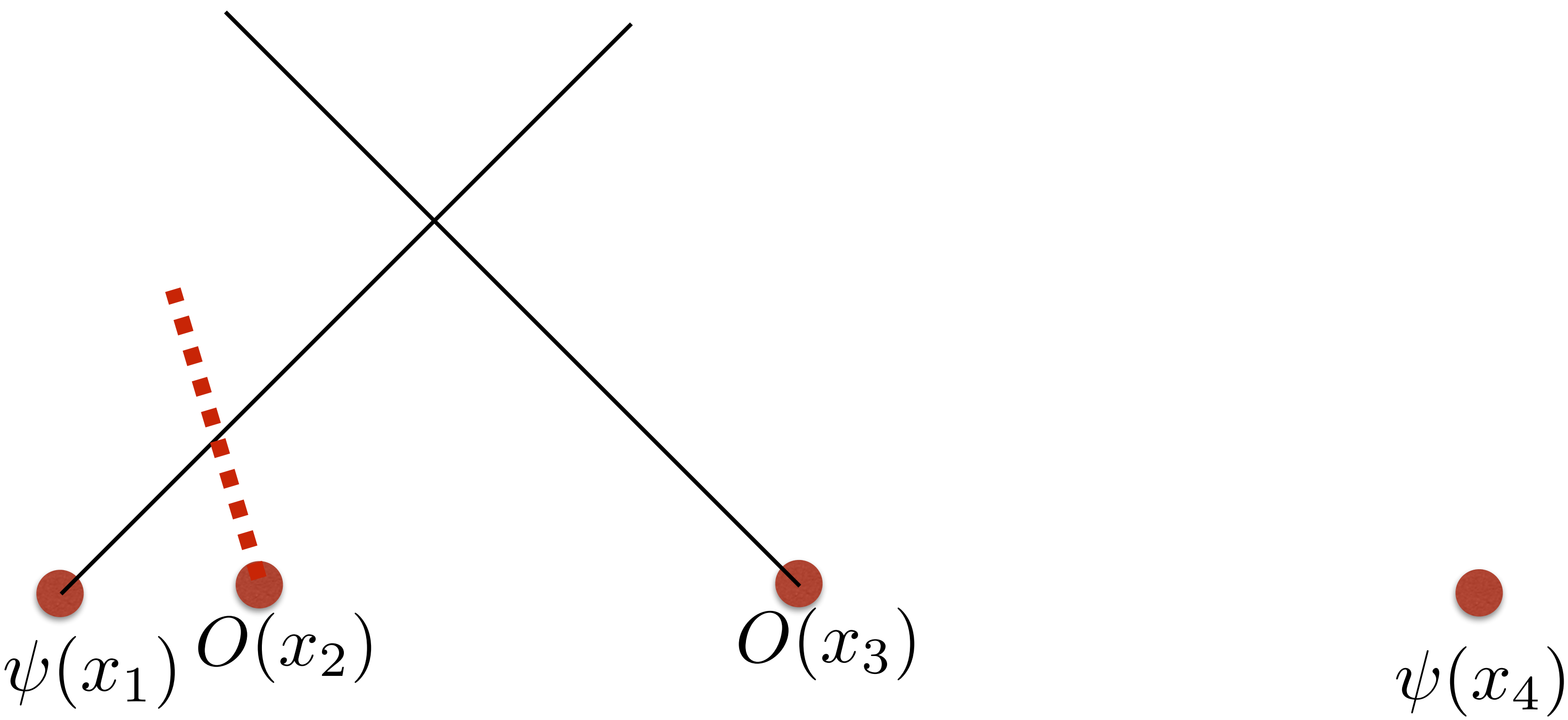
So: Commutator $\langle [O_1, O_2] \rangle$ = discontinuity across the cut.

The branch point is exactly at the Minkowski lightcone, so the 2pt function is trivially causal.



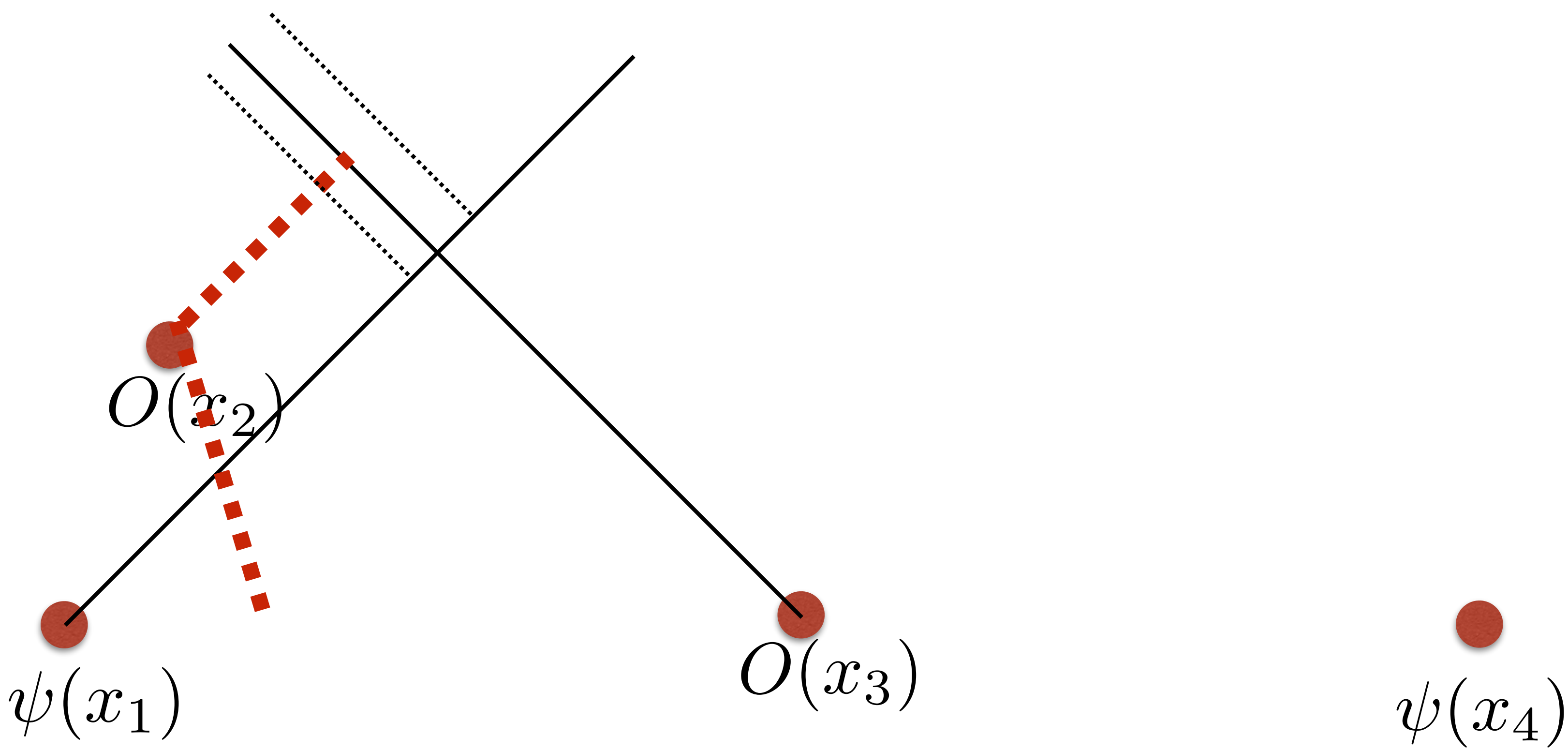
4-point functions

More generally, there is a branch cut whenever an operator passes the lightcone of another operator:



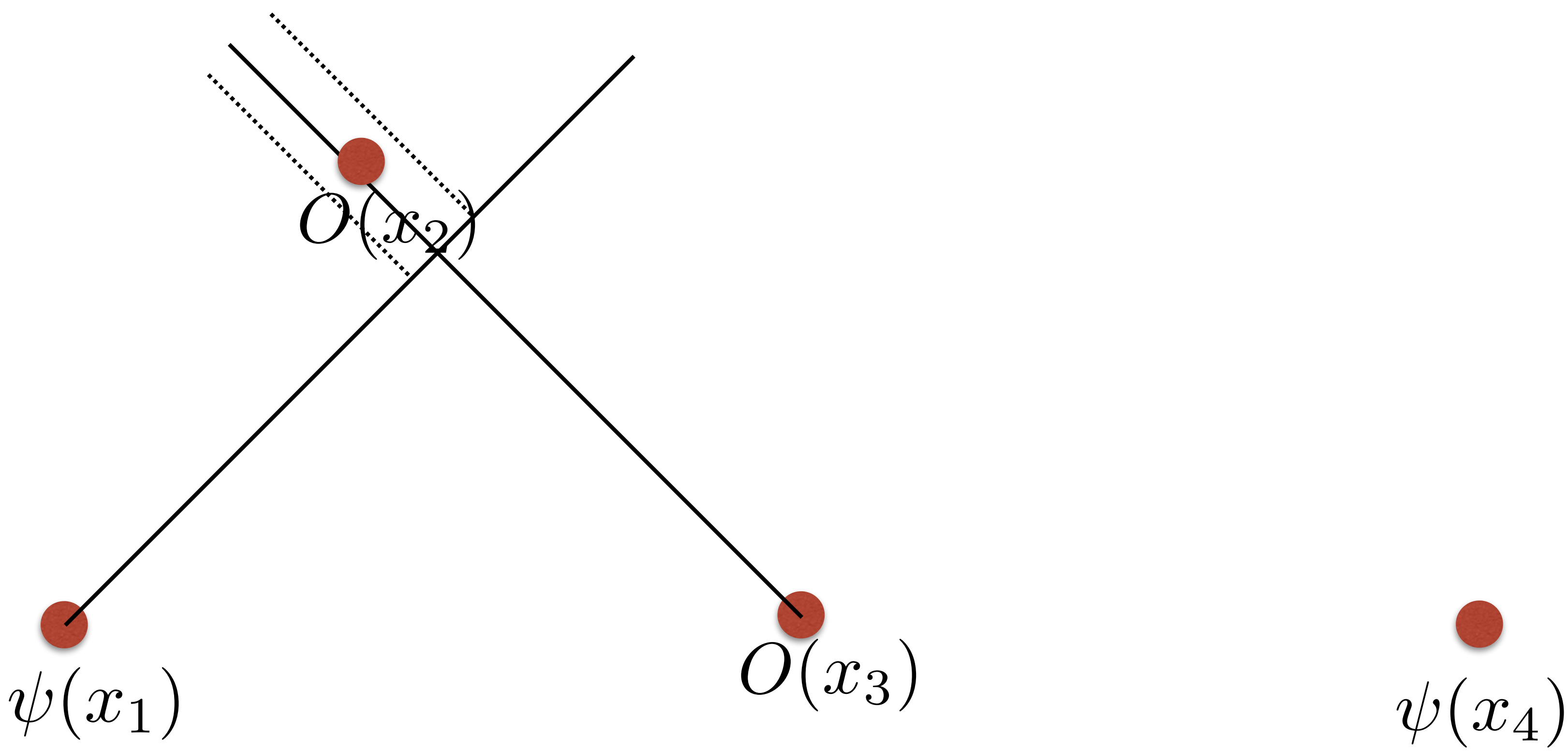
SO(d) invariance of the Euclidean correlator automatically implies that the first branch cuts you encounter are as expected.

But once you pass the first branch cut, symmetries do not tell you the location of other branch cuts.

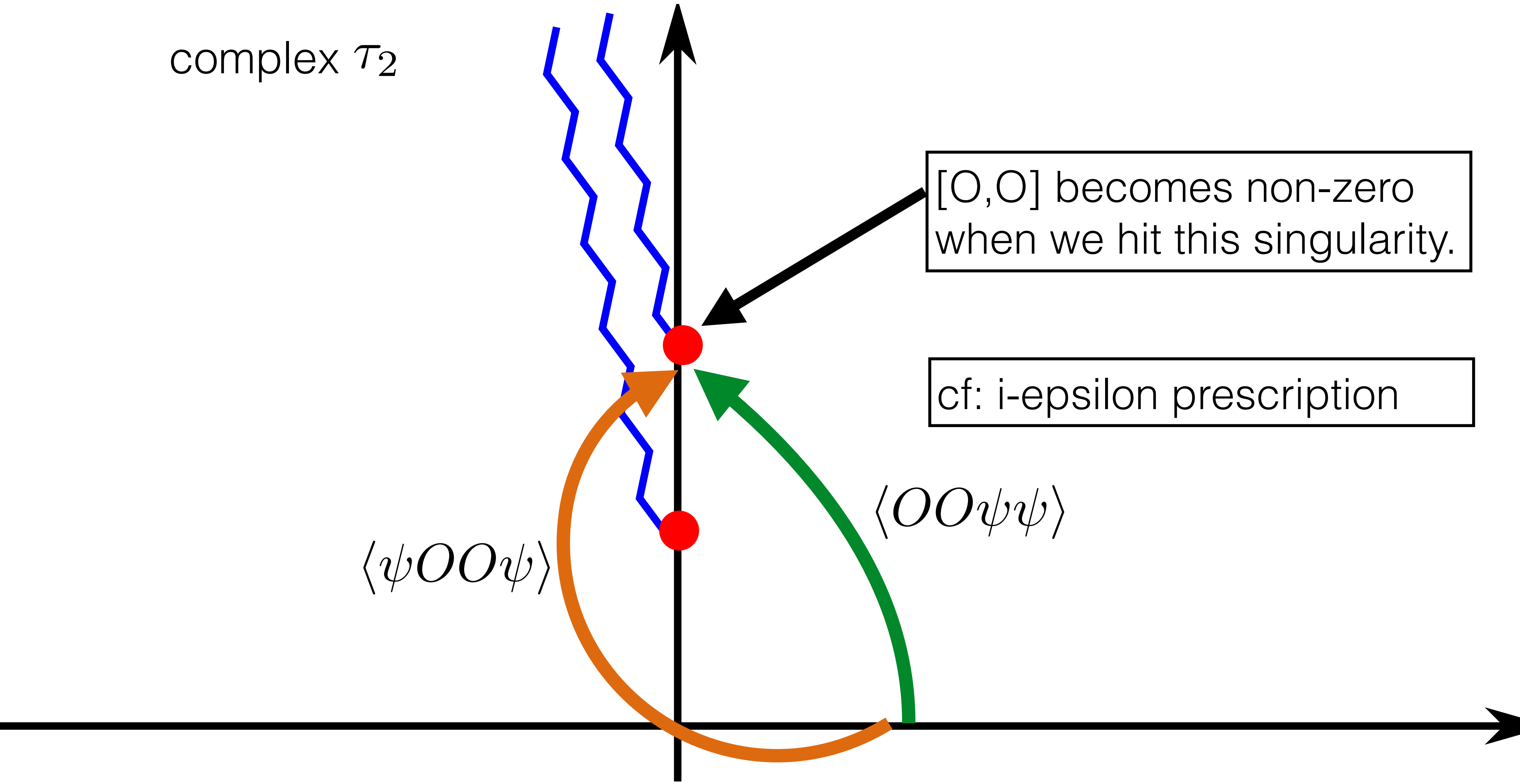


Causality is the statement that the lightcone singularity in this situation cannot appear "too soon."

This is a statement that the correlator is analytic on some region of complexified spacetime.



Same picture on the complex time plane:



CFT

[Luscher, Mack '74]

This was for a general QFT.

In CFT, can phrase in terms of the cross ratios:

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \ , \quad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

and causality is statement about where $G(z, \bar{z})$ is analytic, on a multi-sheeted ***C x C***.

(Example later.)

The key ingredient in Euclidean QFT that prevents singularities from being in the "wrong place" is reflection positivity:

Reflection-positive Euclidean theories

$$\Updownarrow$$

Unitary, causal Lorentzian theories

[Schwinger, Wightman, Osterwalder, Schrader, etc.]

We will first “rediscover” this result in CFT in a way amenable to bootstrap, then extend it to derive *low energy* constraints.

Building on lightcone bootstrap,

[Komargodski, Zhiboedov;
Fitzpatrick et al; Alday et al; etc]

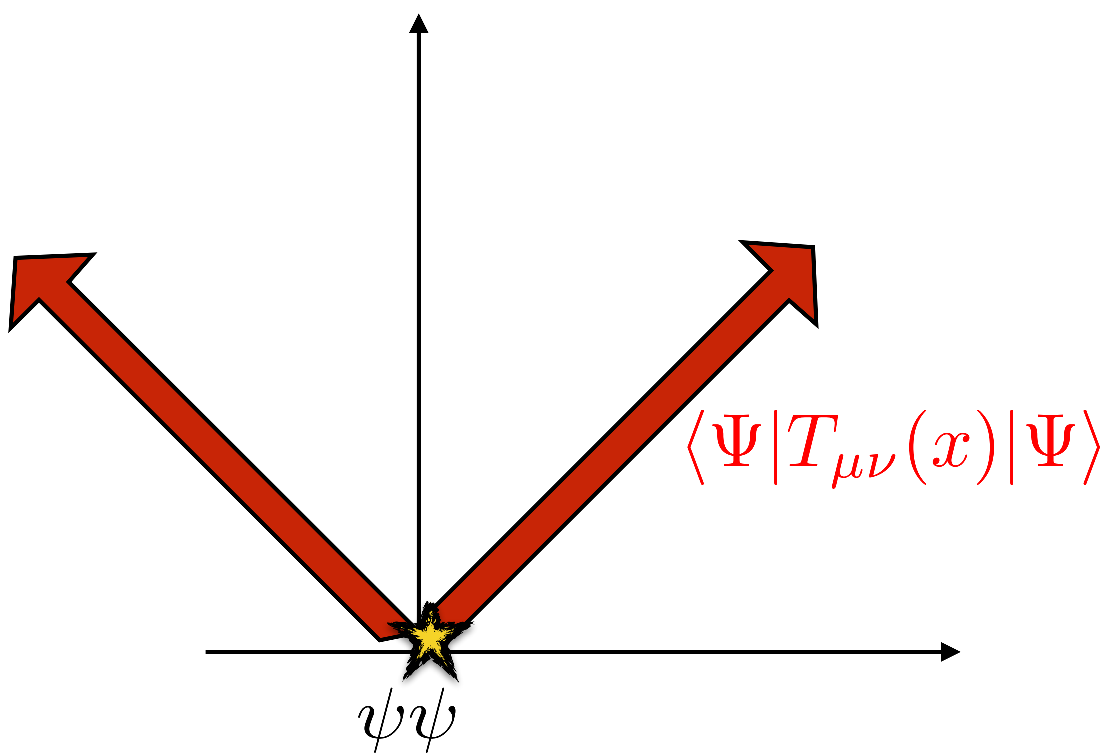
but allowing for timelike-separated operators.

The "Shockwave State"

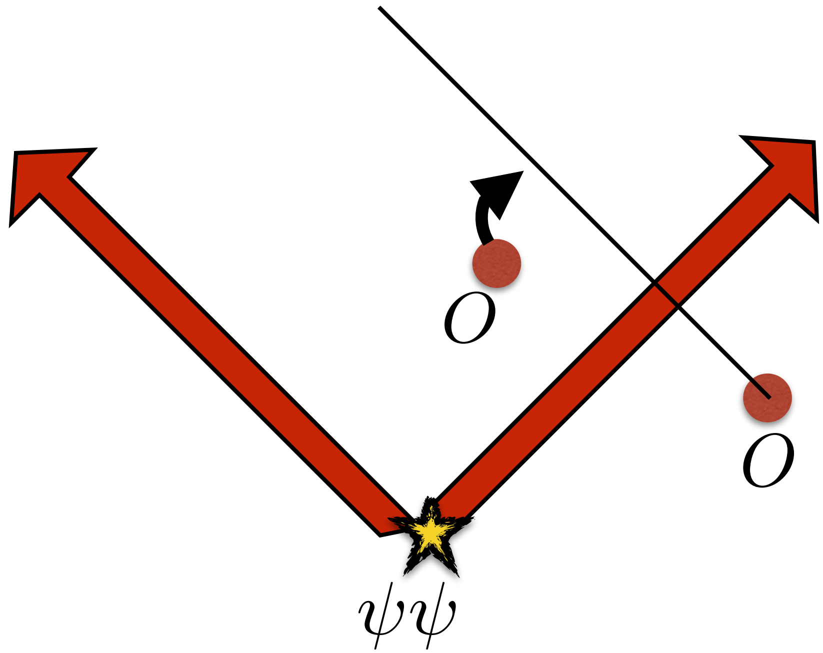
Define the “shockwave state”:

$$|\Psi\rangle \equiv \psi(t = i\delta, \vec{x} = 0)|0\rangle$$

For small δ this creates a stress tensor with support on an expanding null shell:



Probe the shockwave with an operator *O*:



Causality is a statement about the commutator

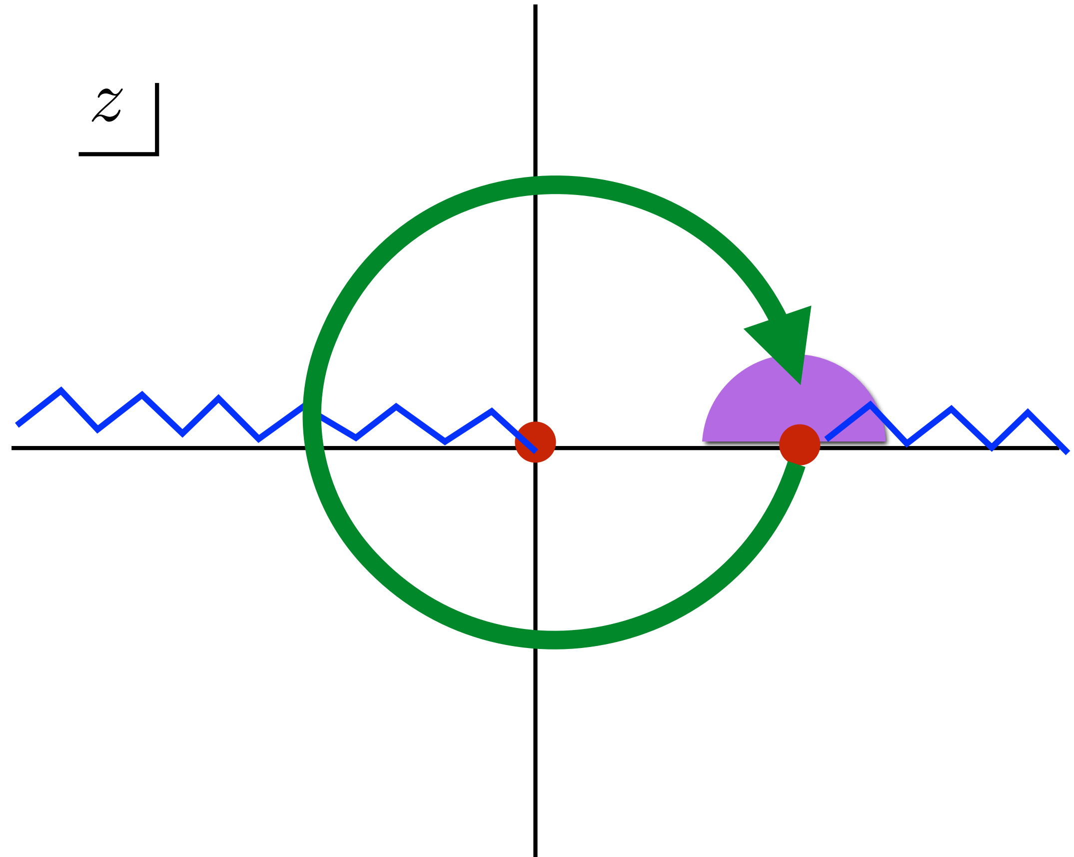
$$\langle\Psi|[O(x_2),O(x_3)]|\Psi\rangle$$

$$= \text{disc. } \langle\psi(-i\delta)O(x_2)O(x_3)\psi(i\delta)\rangle$$

==> This 4pt function must be analytic before the lightcone

The Causality Requirement:

After taking z around zero (holding $\bar{z}$ fixed),



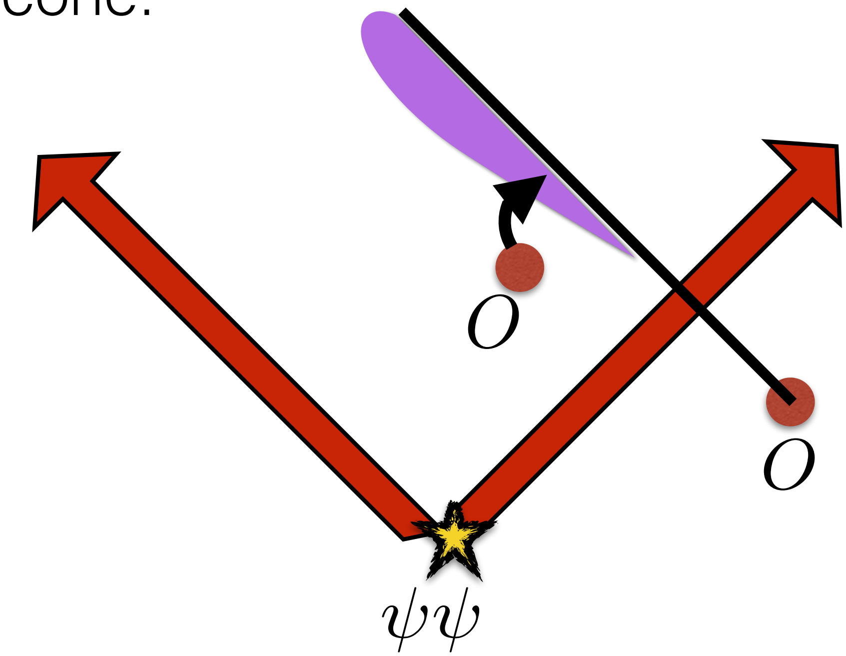
The lightcone singularity as O —> O must not appear in the purple region.

ie, it appears exactly at the red dot (=the Minkowski lightcone) or below it (=time delay), but not above it (=time advance)

So far, we've just translated causality into a statement about a particular CFT correlator.

Next: analyze this correlator using the conformal block expansion.

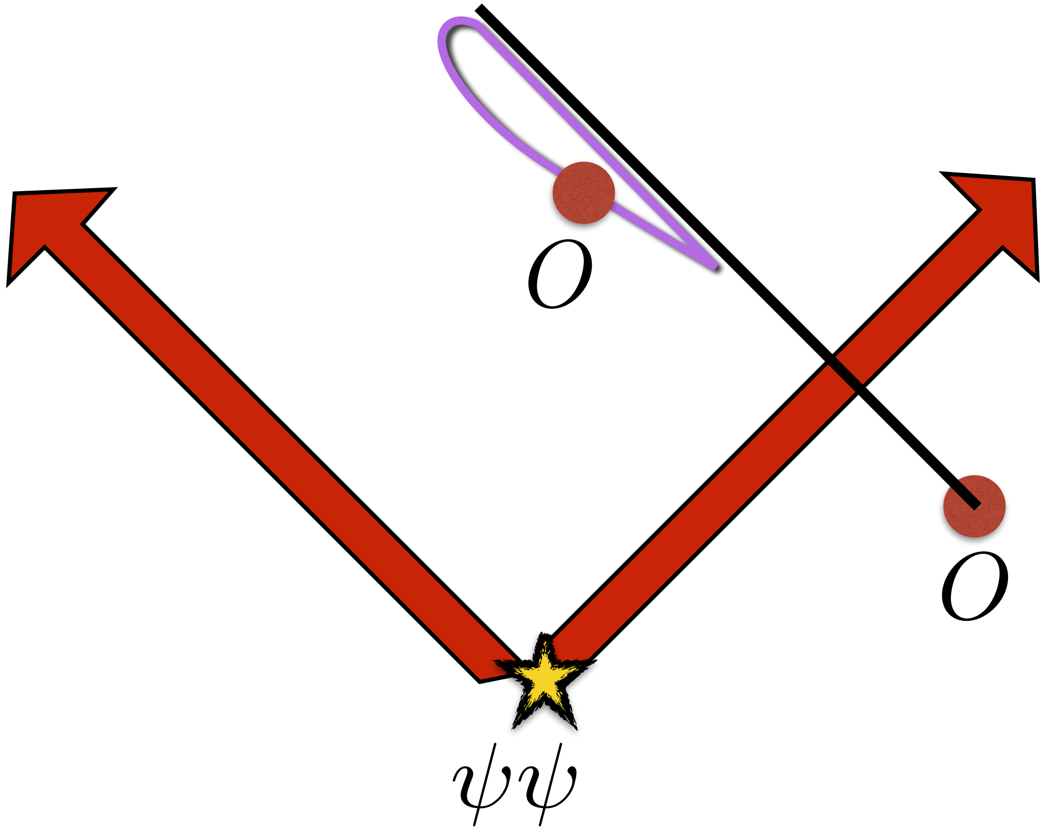
The purple region is a complexified region of spacetime "just before" the lightcone:



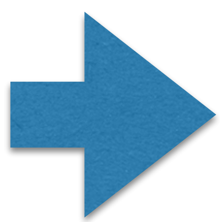
- In a reflection-positive CFT, the s-channel OPE $O \to \psi$ converges in this Lorentzian configuration.
- Therefore $G(z, \bar{z})$ is analytic on the purple region.
- Therefore this correlator is causal.

[TH, Jain, Kundu].

To derive IR constraints, integrate this analytic function on a closed contour:



And require $\oint G = 0$



Position space
"dispersion relation"

When the dust settles, the sum rule + crossing symmetry implies

$$\lambda_{IR} = \int_{UV} dx |\text{something}|^2 \boxed{x^{\ell-2}}$$

$$\geq 0$$

where λ_{IR} is an OPE coefficient.

For example, stress tensor exchange:

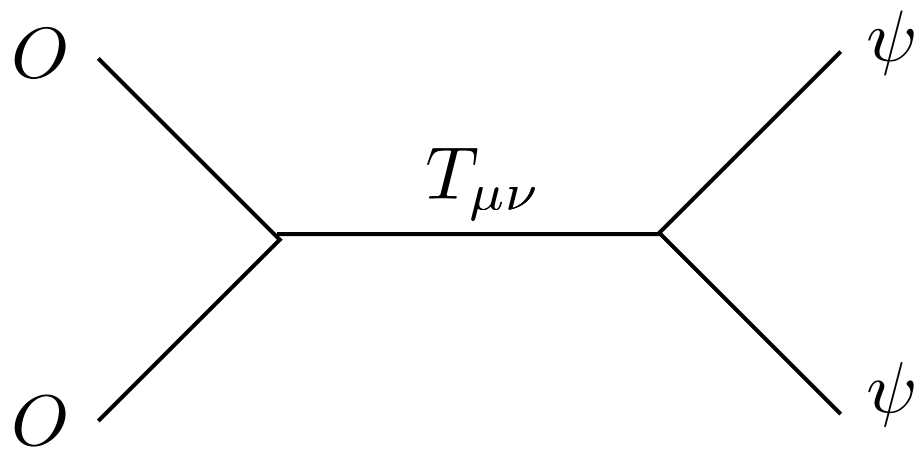
$$\lambda_{IR} \sim \langle O(0) O(1) T_{--}(\infty) \rangle$$

More generally: constraints on coupling to the lowest-dimension operator at each spin (s=2 or higher)

[TH, Jain, Kundu].

Application #1: A trivial case

For scalar probes,



the coupling is fixed by conformal Ward identity:

$$\lambda = \frac{\Delta_O \Delta_\psi}{c} > 0$$

The causality constraint gives this obvious inequality.

Application #2: Maldacena-Zhiboedov Theorem

Comparing magnitude of spin-2 to spin-L:

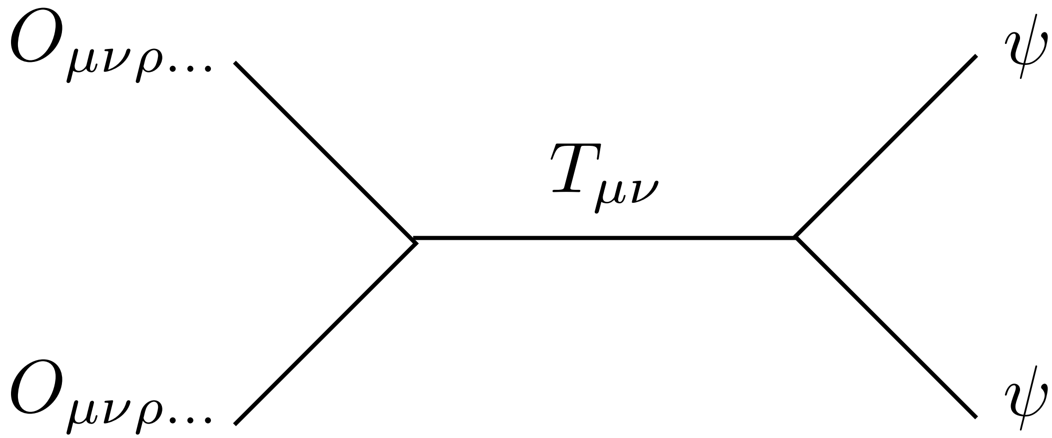
Conserved currents with spin > 2 *are not allowed.*

(Unless there is an infinite number of them, as in free theory.)

Application #3: Probes with spin

[TH, Jain, Kundu]

For probes with spin, stress-tensor exchange is nontrivial (not fixed by Ward identity):



Same logic gives a new "null energy"-like constraint

$$\langle \varepsilon \cdot O(0) \varepsilon^* \cdot O(y^+, y^-) T_{--}(\infty) \rangle > 0 \qquad \text{for} \quad y^+ \to 0$$

An interesting example is $\langle T_{\mu\nu} T_{\sigma\rho} T_{\gamma\delta} \rangle$

which includes the anomaly coefficients *a,c*

causality ==>

$$\frac{1}{3} < \frac{a}{c} < \frac{31}{18}$$

[TH, Jain, Kundu]

[Hofman, Li, Meltzer, Poland, Rejon-Barrera

This is a bootstrap derivation of the Hofman-Maldacena energy calorimeter constraints.

Also: new constraints for other external operators.

different approach: Komargodski, Kulaxizi, Parnachev, Zhiboedov

The more recent constraints

$$a \approx c$$

[Camanho, Edelstein, Maldacena, Zhiboedov]

are much stronger versions of Hofman-Maldacena.

To derive this from CFT is difficult, but plausibly within reach with existing techniques and enough effort:

1. Compute anomalous dimensions by lightcone bootstrap

2. Apply causality constraints

Application #4: Holographic Dual of $(\partial\phi)^2 + \lambda(\partial\phi)^4$

Consider a scalar theory in AdS with this contact interaction.

(Gravity is decoupled.)

If $\lambda < 0$, the dual CFT violates the sum rule.

Therefore we reproduce the Adams *et al.* result directly from the CFT bootstrap:

$$\lambda > 0$$

Application #5: Another look at $\lambda(\partial\phi)^4$

Most of what I said does not require CFT.

Can also apply sum rule *directly* to flat-space, perturbative QFT. For example:

$$L = (\partial\phi)^2 + \lambda(\partial\phi)^4 + \cdots$$

Sum rule says that wrong-sign leads to causality-violating commutators.

This is an "almost-Euclidean" derivation of the *a*-theorem.

(Different from previous slide, because we're not using holography here!)

Application #6: Fine tuning

Consider

$$L = (\partial\phi)^2 + \frac{\lambda_4}{f^4}(\partial\phi)^4 + \frac{\lambda_8}{f^8}(\partial\phi)^8 + \cdots$$

Compare magnitude of spin-2 and spin-4 sum rules:

$\lambda_4 \ll \lambda_8$ is disallowed

"Non-renormalizable EFT cannot be arbitrarily fine-tuned"

Summary:

In a causal/unitary QFT, position-space correlation functions have analogues of

- “optical theorem” positivity conditions
- “dispersion relations” relating UV <—> IR

This imposes constraints on the IR couplings of both conformal and non-conformal QFTs.

Thank you