Recent progress in N=2 4d field theory

DG, G. Moore, A. Neitzke: arXiv:0907.3987

DG: arXiv:0904.2715

L.F.Alday, DG, Y.Tachikawa: arXiv:0906.3219

L.F.Alday, DG, S.Gukov, Y.Tachikawa, H.Verlinde:
arXiv:0909.0945

Non-rational 2d CFT
compute protected quantities
in 4d N=2 gauge theory

A strange relation

Pick a 2d CFT correlation function

- Pick a conformal Toda theory
  - ADE classification, Liouville theory is A_1
- Pick a Riemann surface
- Place vertex operators at punctures
- Add twist lines for outer automorhisms

Each choice of correlation function selects a specific N=2 4d field theory

What does it compute?

4d N=2 gauge theory

- S^4 partition function
- Instanton partition function(s)
- Surface operators
- ‘t Hooft-Wilson loops

2d Toda CFT

- Correlation function
- Conformal blocks
- Degenerate field insertions
- Loop operators

Liouville theory examples

Sphere four point function

- SU(2) gauge theory with $N_f=4$

Sphere n point function

- Linear quiver of SU(2) gauge groups

Torus one point function

- SU(2) gauge theory coupled to one adjoint

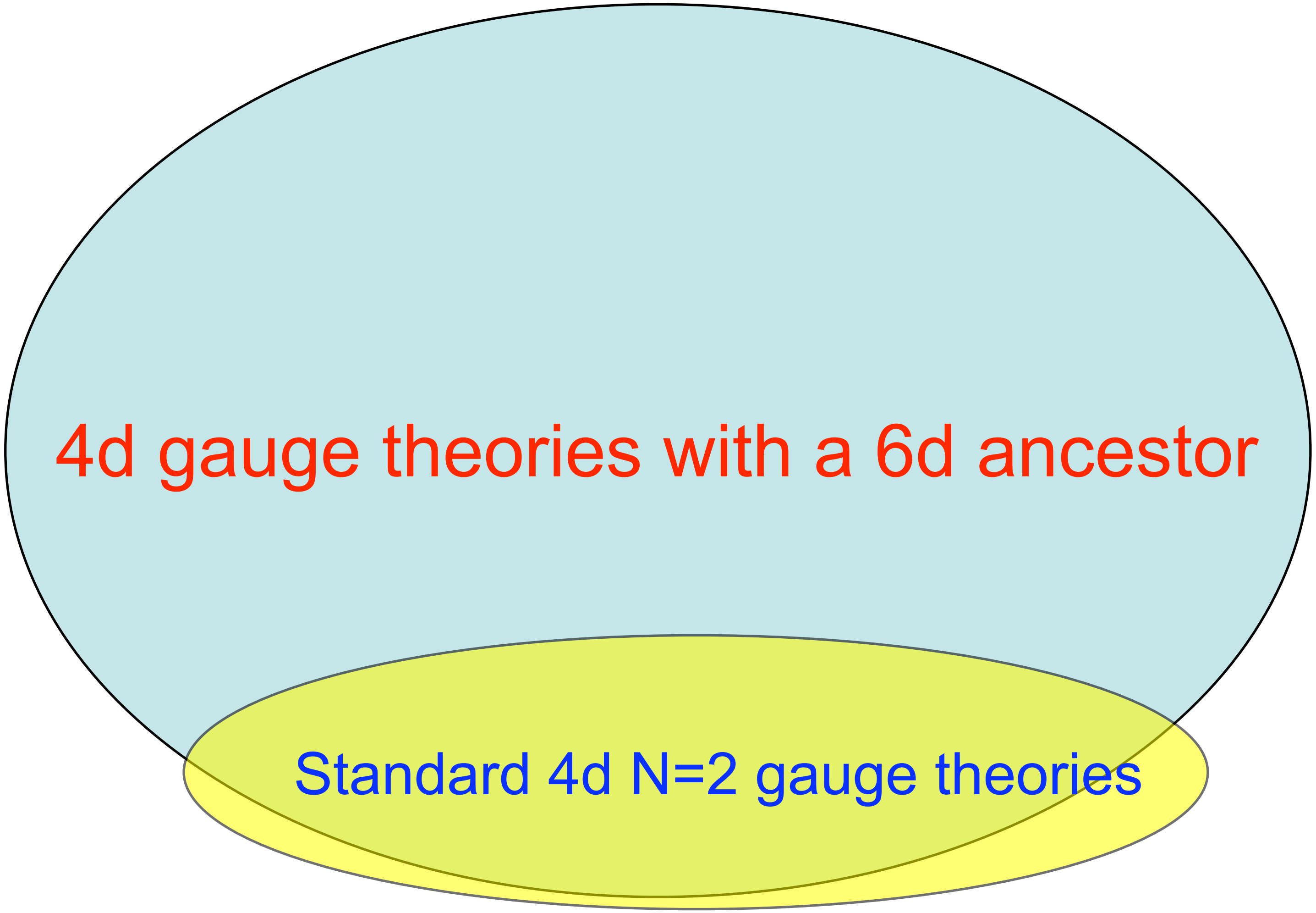
Which N=2 gauge theories?

4d gauge theories with a ``6d ancestor”

- Twisted compactification of 6d (2,0) ADE SCFT
  - Pick a Riemann surface C
  - Place defects at points in C
  - Add twist lines if appropriate

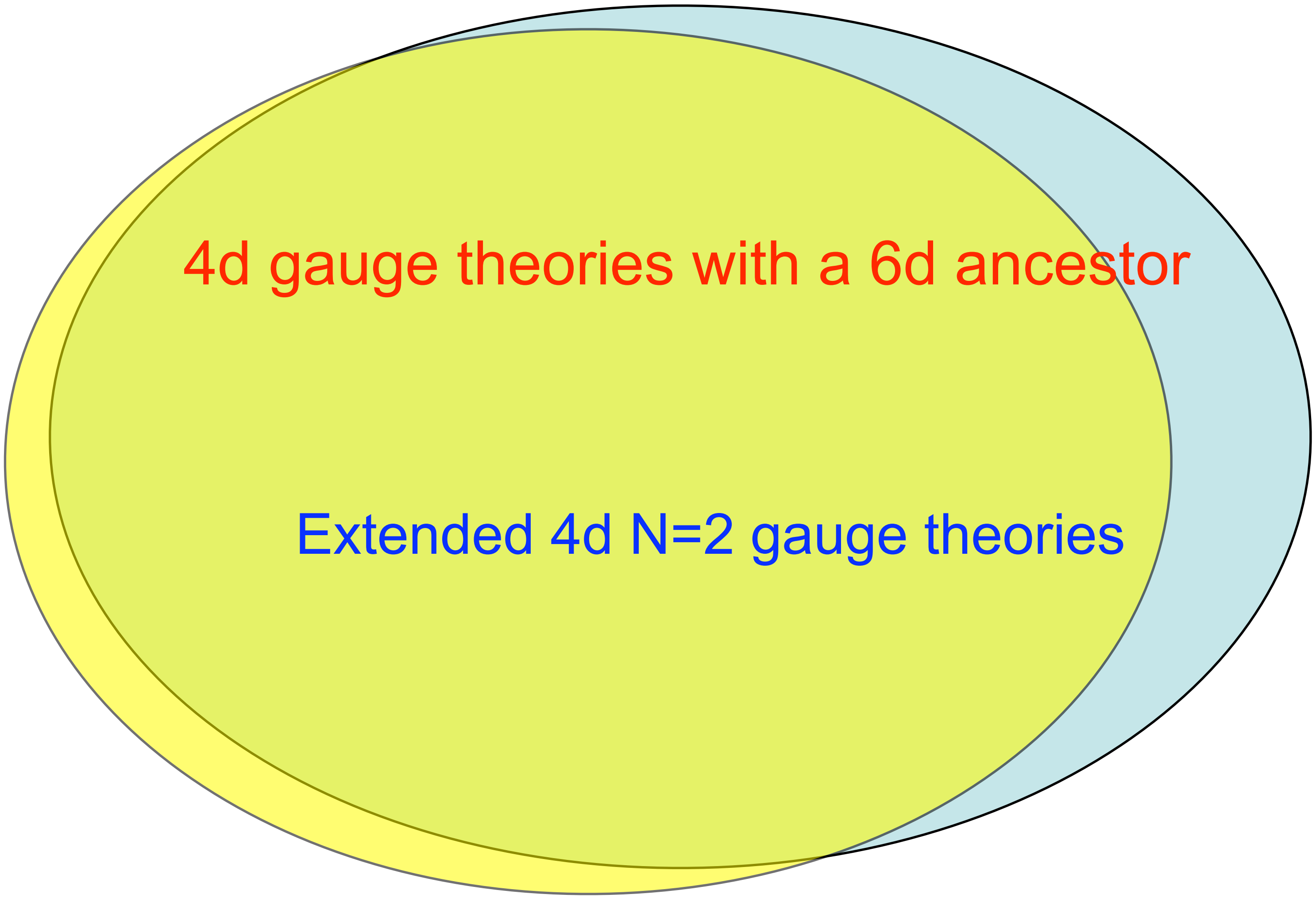
- Flows in IR to a 4d theory with N=2 SUSY

Which N=2 gauge theories?



Do all N=2 theories have a 6d ancestor?

Which N=2 gauge theories?



A step back

Seiberg and Witten solution

- N=2 SU(2) gauge theory
  - Exact low energy massless Lagrangian
  - Lagrangian described through SW curve and SW differential
  - Exact spectrum of massive dyonic BPS particles

- SW curve/differential for many more theories is available
  - No systematic field theory approach
    - M-theory, Type IIB engineering!
  - BPS spectrum is poorly understood

A step back

KS Wall Crossing Formula

- Recent mathematical progress
  - About "motivic Donaldson-Thomas invariants"
- Predicts behavior of BPS spectrum in N=2 field theories

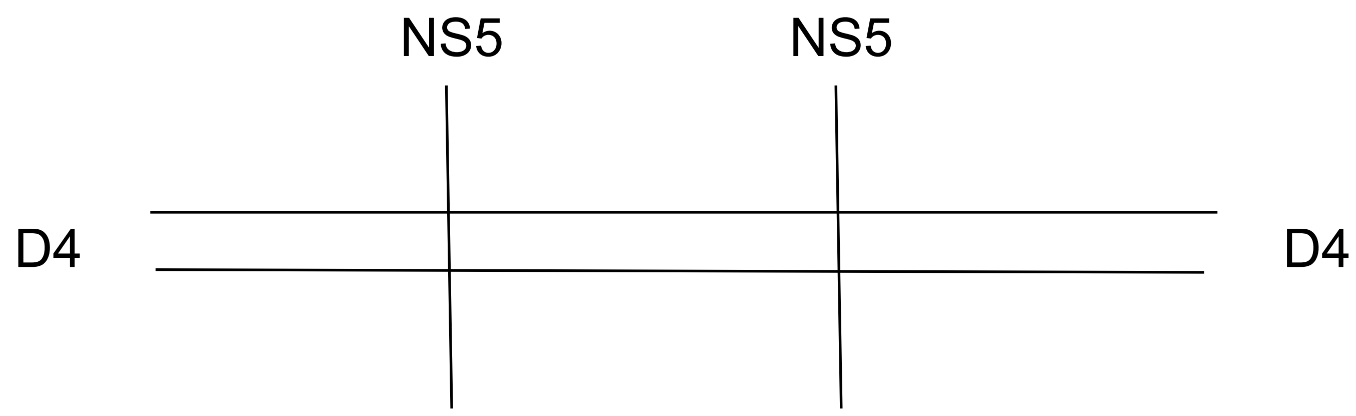
Physical explanation? GMN

- Compactify the N=2 4d theory on a circle.
- BPS particles loops correct the metric on moduli space
- BPS spectrum jumps across walls of marginal stability
- Metric is continuous! Wall crossing formula follows

A step back

How to test this idea?

- We need both the BPS spectra and the 3d metric
  - Certain N=2 theories can be engineered in IIA



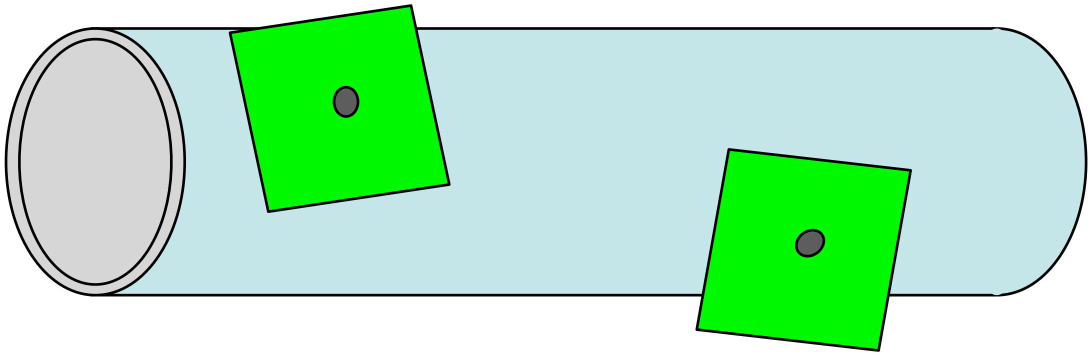
N_f=4 SU(2) gauge theory

- SW curve/differential from lift to M-theory (Witten)
- BPS spectra and the 3d metric can also be derived!

A step back

IIA to M-theory

- M-theory lift: M5 branes wrapping a cylinder or torus C



- 4d gauge theory determined by choice of defects
- SW curves, differential, are constructed systematically

A broad generalization

Restrictive lifts

- Only special combinations of defects from IIA lift
- Only genus g=0,1 surfaces

Why not any surface, any defects combination?

- The M-theory setup is sensible
- We can apply the same analysis
  - Reasonable SW curve, BPS spectrum, etc.
  - Are these actual 4d field theories? Which ones?

A broad generalization

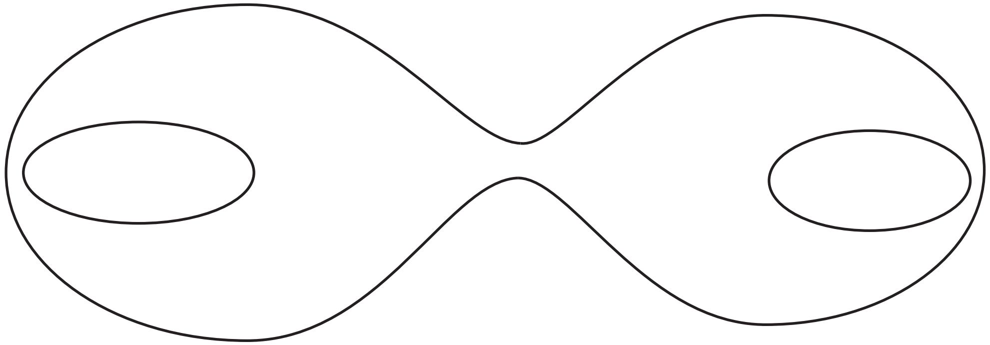
Divide and conquer

- Identify all UV gauge couplings
- Make them infinitely weak
- Read off weakly coupled gauge groups
- Read off matter theories

A broad generalization

Divide and conquer

- Gauge couplings control complex structure of C
- At weak coupling, C degenerates



``Pair of pants decomposition of C''

- Matter theories associate to fragments of C
- Matter theories are often strongly interacting SCFTs!

Basic SCFTs in A_1 class

A simple example: $T_{g,n}[A_1]$

- Only one type of defect, carries SU(2) flavor symmetry
- No twist lines
- Admit Lagrangian descriptions: matter theories are free.

$T_{0,3}$ the three punctured sphere

- Four free hypermultiplets
- $SU(2)_a$ x $SU(2)_b$ x $SU(2)_c$ flavor symmetry (3 punctures!)
- In components, (Q_{abc} , Q^*_{abc})
- A ``pair of pants"

Sewing pants together

Connecting the pants

- To join pant leg a and pant leg b....
- Gauge diagonal SU(2) in $SU(2)_a \times SU(2)_b$
- Instanton factor $q=\exp 2\pi i\tau$ is sewing parameter
- Weak coupling when tube is long

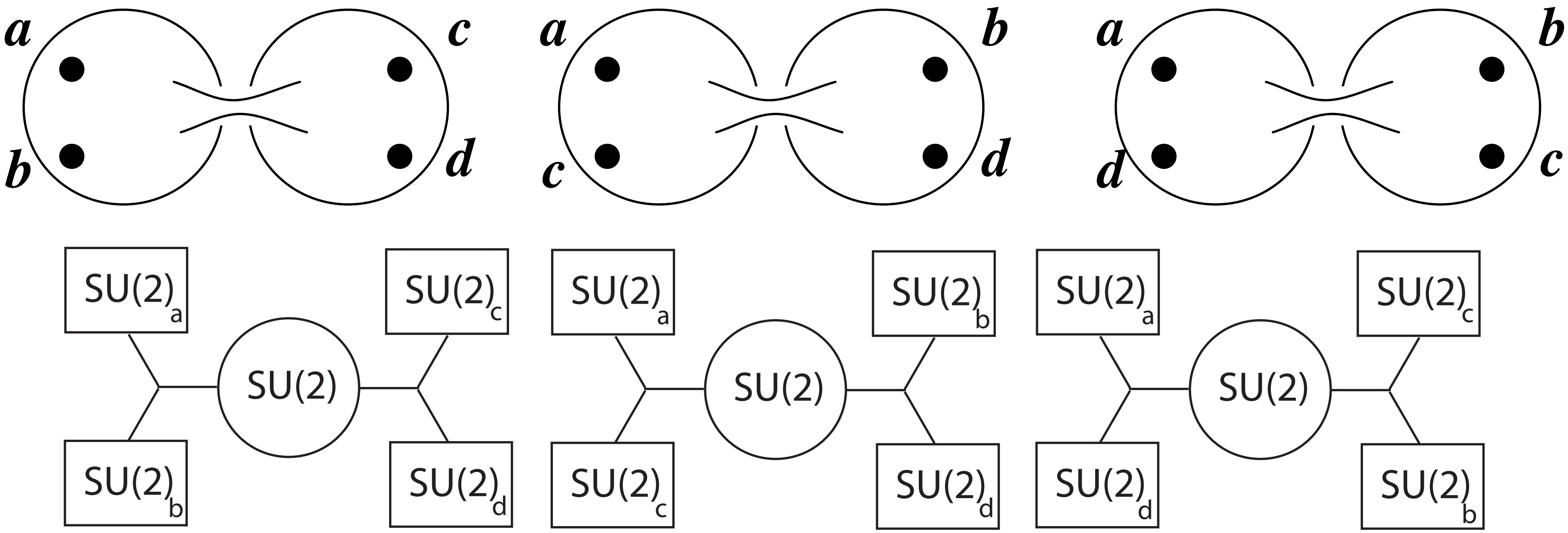
Lagrangians and sewing

- Any Riemann surface can be sewn from pair of pants
- For every sewing graph there is a distinct Lagrangian!
- Every theory $T_{g,n}$ has multiple Lagrangian descriptions!

Examples

$T_{0,4}$: sphere with four defects.

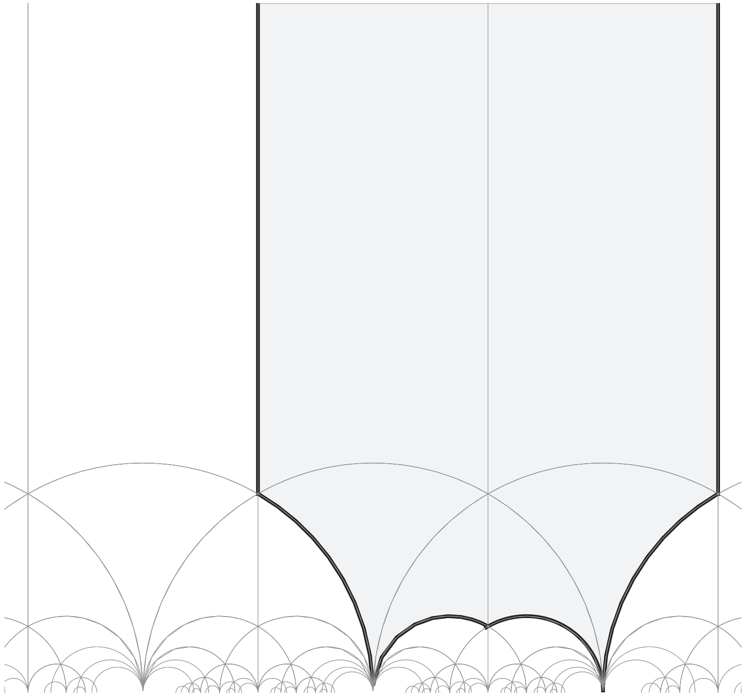
- Two pairs of pants
- One SU(2) gauge group, 8 hypers: SU(2) N_f=4



S-duality in SU(2) $N_f=4$

SO(8) flavor symmetry

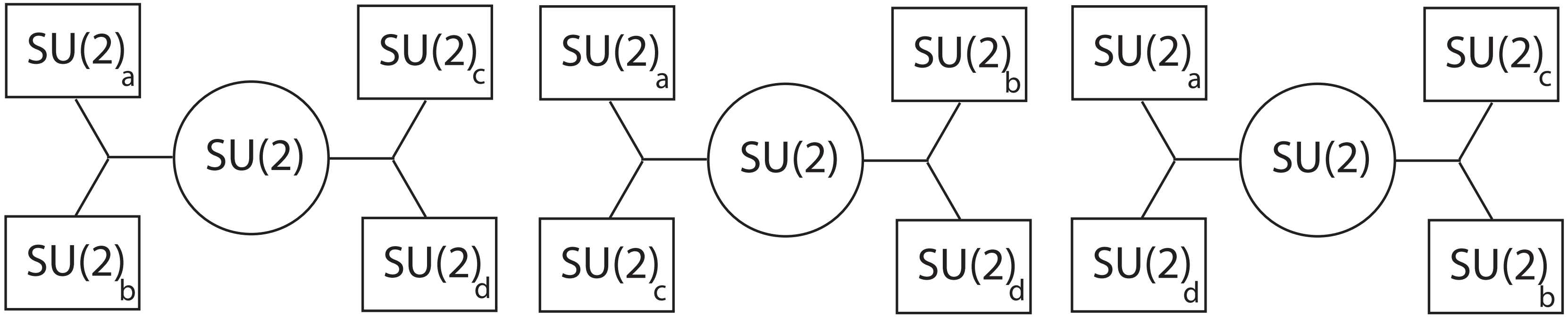
- SL(2,Z) acts through triality on SO(8)
- Exchanges electrons in 8_v, monopoles in 8_s,dyons in 8_c



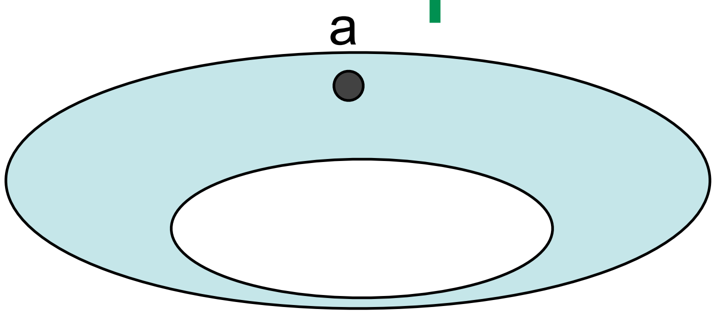
S-duality in SU(2) N_f=4

Reformulating triality

- Consider subgroup SO(4)xSO(4) in SO(8)
- Rewrite it as $[SU(2)_a \times SU(2)_b] \times [SU(2)_c \times SU(2)_d]$
- SL(2,Z) permutes (a,b,c,d)
  - $8_v = (2_a \times 2_b) + (2_c \times 2_d)$
  - $8_s = (2_a \times 2_c) + (2_b \times 2_d)$
  - $8_c = (2_a \times 2_d) + (2_c \times 2_b)$



Examples



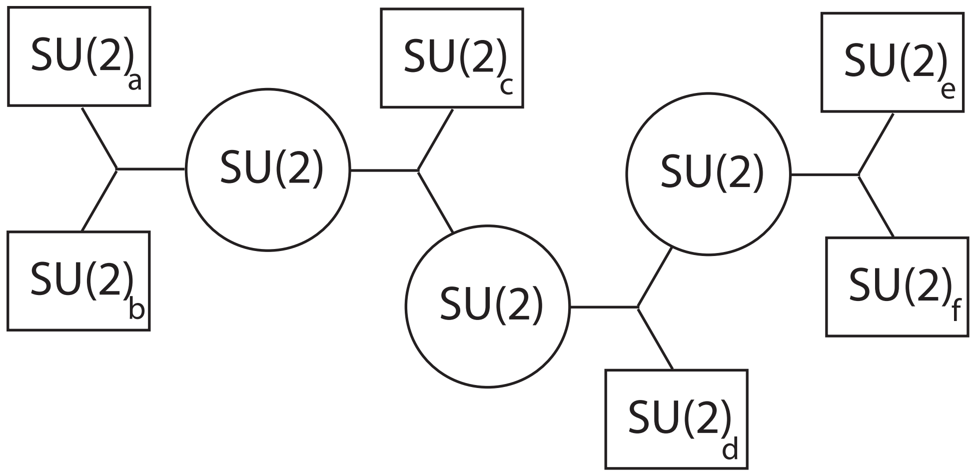
$T_{1,1}$: one-punctured torus

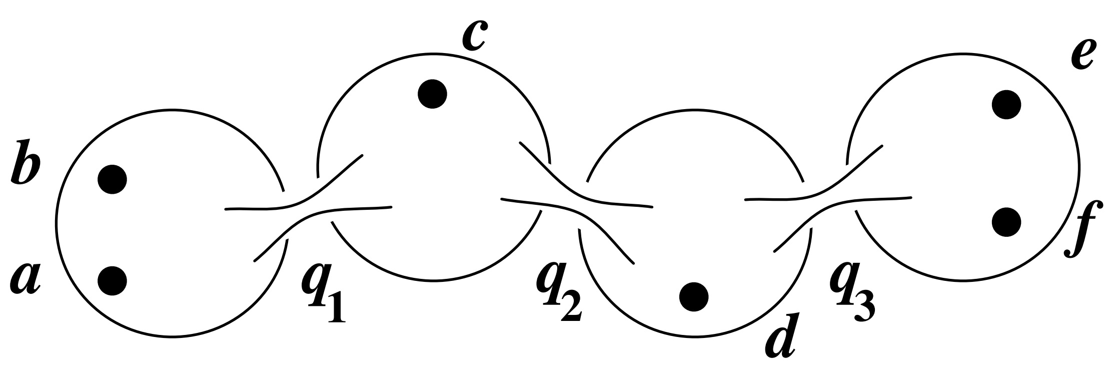
- One pair of pants, glue two legs
  - (2_a 2_b 2_c) goes to (2_a, 3) + (2_a,1)
  - One adjoint of SU(2) gauge group and one free hyper

N=2* theory: mass deformed N=4 SYM

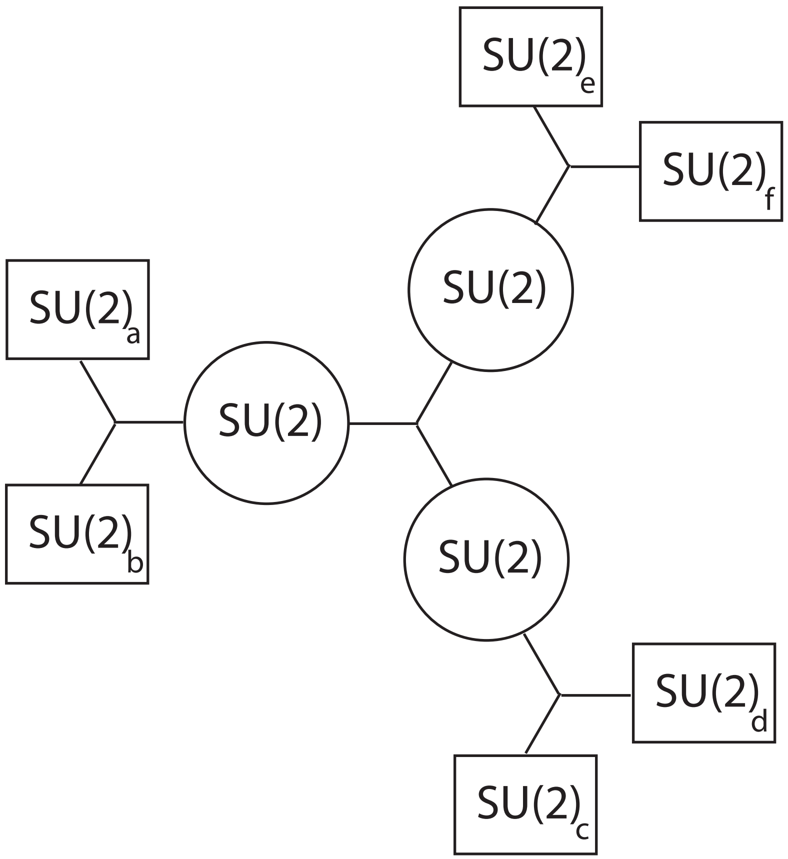
- Standard SL(2,Z) S-duality group acting on torus

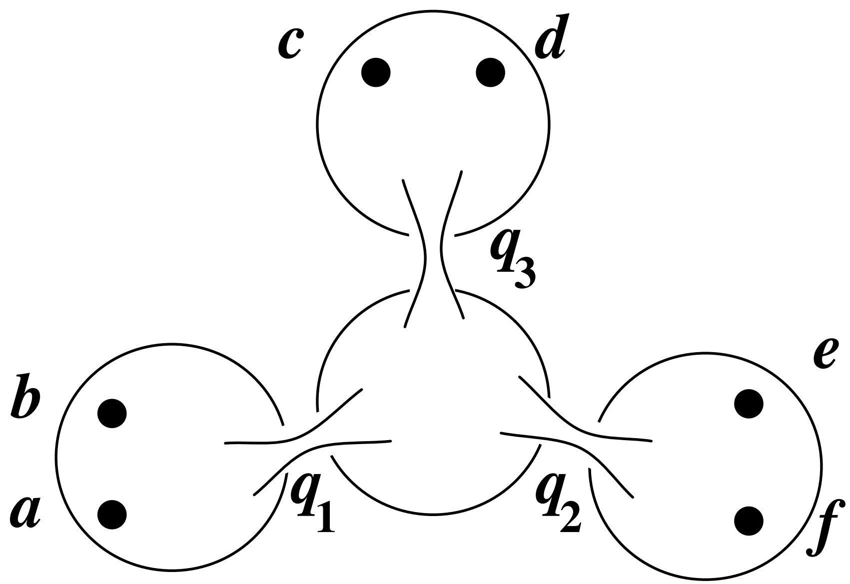
Sphere with six punctures



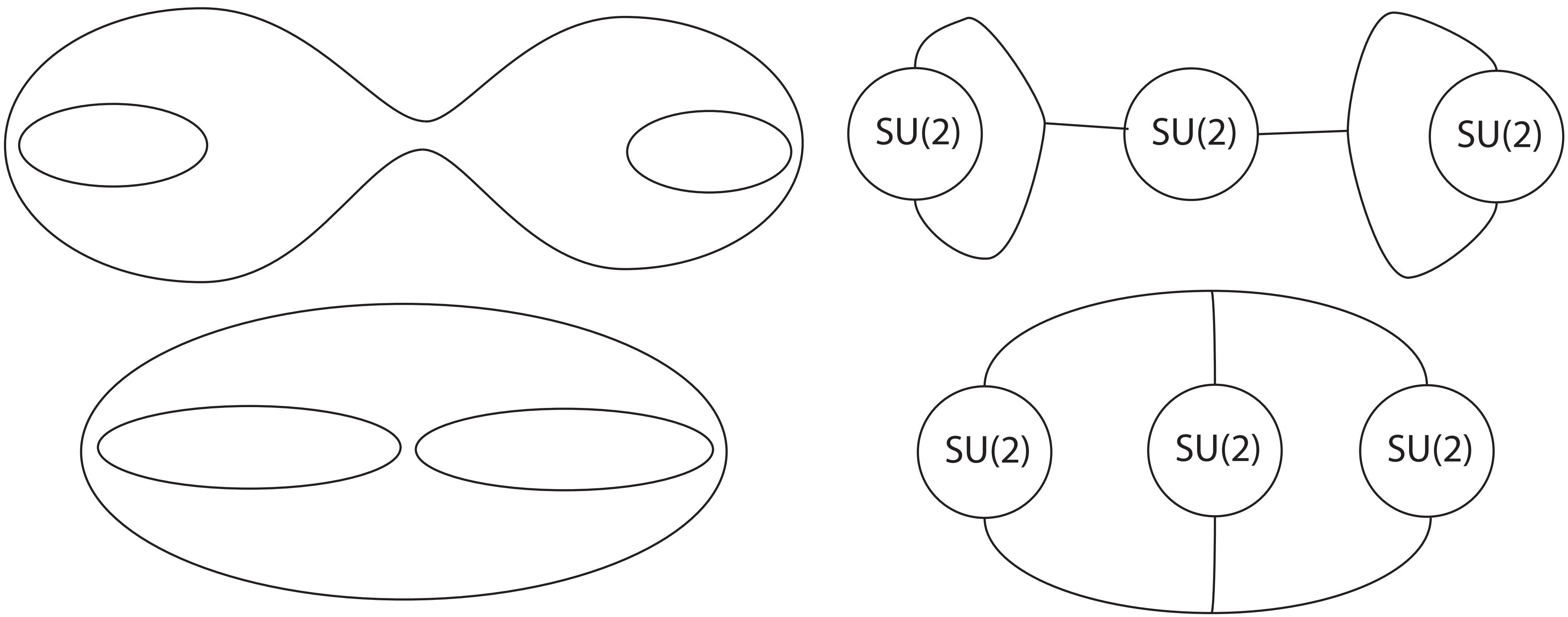


Sphere with six punctures





Genus two surface



S-dualities

A large S-duality group(oid) relates different sewings, and Lagrangians

- Same as ``Moore-Seiberg groupoid'' from 2d CFT lore
  - Generated by few basic operations
    - A-move: s-t channel duality in 4 point function (N_f=4 S-duality)
    - B-move: permute two legs
    - S-move: duality of one punctured torus ($N=2^*$ S-duality)
  - Crucial relations:
    - pentagon, hexagons at genus 0
    - genus 1 relations

On conformal blocks

Building blocks of 2d CFT correlation functions

- Correlation function computed by sewing
- Glue pair of pants with sum over complete set of states

$$\sum_v \sum_{i,j} |v,i\rangle B_{ij} \langle v,j| q^{L_0} \bar{q}^{\bar{L}_0}$$

- Fix primary fields v, only sum over descendants i,j

$$\mathcal{Z} = \sum_{v_a} \mathcal{F}[v_a](q_a) \bar{\mathcal{F}}[v_a](\bar{q}_a) \prod_n \langle v_{a^n} v_{b^n} v_{c^n} \rangle$$

On conformal blocks

Conformal blocks: a basis of objects which satisfy Virasoro Ward identities

- Labeled by choice of primaries v_a in the channels
- For RCFT, finite set of primaries
  - finite dimensional space of conformal blocks
  - solutions of hypergeometric-like differential eqns.

- For non-rational theories, continuous labels
  - Hilbert space of conformal blocks
  - Labeled by conformal dimensions Δ
  - Better: Liouville momenta $\Delta = \alpha(Q-\alpha)$ $c = 1+6Q^2$

Examples

Liouville three point function (DOZZ)

$$\frac{\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_1-\alpha_2+\alpha_3)\Upsilon(-\alpha_1+\alpha_2+\alpha_3)}$$

Four point conformal blocks

$$\mathcal{F}[V_1,V_2,V_3,V_4](q)=\frac{\langle V_1V_2V_a\rangle\langle V_aV_3V_4\rangle}{\langle V_a|V_a\rangle}+\frac{\langle V_1V_2(L_{-1}V_a)\rangle\langle(L_{-1}V_a)V_3V_4\rangle}{\langle(L_{-1}V_a)|(L_{-1}V_a)\rangle}q+\cdots$$

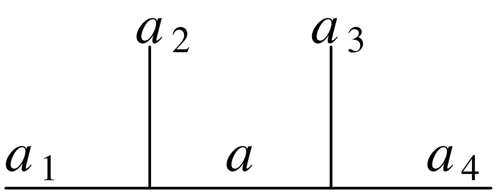
On conformal blocks

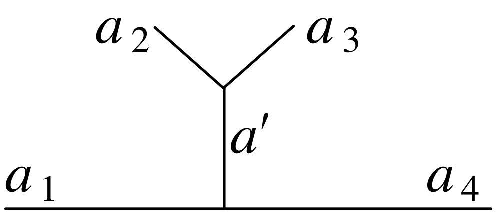
Are conformal blocks in different channels related?

- Action of Moore-Seiberg groupoid
- A,B,S moves implemented by ``braiding,fusion matrices''
  - For non-rational CFT, really intricate integration kernels
  - Act on primary labels
  - Do NOT depend on moduli of surface!

On conformal blocks

Example: four points on sphere





$$\mathcal{F}_s[a_1, a_2, a_3, a_4](q, a) = \int da' F[a_1, a_2, a_3, a_4](a, a')\mathcal{F}_t[a_3, a_2, a_1, a_4](1 \square\ q, a')$$

First dictionary entry

Two occurrences of Moore-Seiberg. Any relation?

- We need a gauge theory quantity which is holomorphic
  - Ideally, a power series in q
    - sewing parameter, but also instanton factor

Nekrasov instanton partition function

- A deformation of flat space partition function
- Two deformation parameters ε_1, ε_2
- Precise definition would brings too far
- Coefficient of q^k computed from k-instanton moduli space

Nekrasov partition function and conformal blocks

Gauge theory partition function depends on a_i, m_j

- a_i is Coulomb branch parameter of i-th SU(2) gauge group
- m_j is mass parameter for $SU(2)_j$

``Experimental" observation:

- Z_{inst} coincides with conformal blocks up to irrelevant factor!
- Dictionary:
  - $\varepsilon_1 = b\ \varepsilon$ $\varepsilon_2 = \varepsilon/b$ c= 1+ 6 $(b+b^{-1})$^2
  - $\alpha = b/2+b^{-1}/2 + a/\varepsilon$ Liouville momenta
  - $\Delta = \alpha(b+1/b-\alpha)$

Explicit formulae: SU(2) N_f=4

Z_{inst}

$$1-q\left(\frac{(a-\mathrm{m1}+\epsilon1+\epsilon2)(a-\mathrm{m2}+\epsilon1+\epsilon2)(a-\mathrm{m3}+\epsilon1+\epsilon2)(a-\mathrm{m4}+\epsilon1+\epsilon2)}{2a\epsilon1\epsilon2(2a+\epsilon1+\epsilon2)}+a\leftrightarrow -a\right)$$

Conformal block

$$1+\frac{(\Delta-\Delta_1+\Delta_2)(\Delta+\Delta_3-\Delta_4)}{2\Delta}q$$

Relation

$$Z_{\mathrm{inst}}=(1-q)^{\delta}\mathcal{F}$$

A neat use of Z_{inst}

Pestun computation of S^4 partition function

- Careful localization procedure

$$\int a^2 da |Z_{tree} Z_{1-loop} Z_{inst}|^2$$

Does it have a 2d CFT interpretation?

- $Z_{1\text{-loop}}$ and Vandermonde measure give DOZZs!
- Z_{tree} Z_{inst} provide overall powers of q, conformal blocks
- $Z[S_4]$ is Liouville correlation function! (b=1)

S-duality invariance from crossing symmetry

Correlation functions are independent of sewing

- Then $Z[S_4]$ is also S-duality invariant
- Very hard to prove directly!

Can we extend this result?

- Pestun also inserted Wilson loops along equator

$$\int a^2 da \cos \pi a |Z_{tree} Z_{1-loop} Z_{inst}|^2$$

S-duality and line operators

Can we do an S-duality transformation?

- Answer involving integral kernels is not very useful
- Hard to simplify by hand
- We need a direct 2d CFT interpretation of Wilson loop
- A function of the Liouville momentum in a channel

How do you ``measure'' the primary in a channel?

- Erik Verlinde strategy:
- monodromies of degenerate insertions

Degenerate fields

Primaries of momentum $2\alpha=(1-n)b+(1-m)b^{-1}$

- Null vector at level nm in Verma module
- In correlation functions, satisfy deg. nm differential eqn.

Example

- $V_{1,1}$ is identity operator, $\partial V_{1,1}=0$
- We'll look at $V_{2,1}$
  - $\partial^2 V_{2,1} + b^2{:}T\, V_{2,1}{:} =0$
  - Restricted fusion: $V_{2,1}\, V_\alpha = V_{\alpha+b/2} + V_{\alpha-b/2}$

Degenerate fields and Conformal blocks

Degenerate fields can be inserted in a channel

- Liouville momentum jumps by ±b/2 from left to right

Degenerate fields can be moved between channels

- ``Degenerate'' fusion and braiding operators
- Act as 2 x 2 matrices on ±b/2 choice
- Transport matrices for $\partial^2 V_{2,1} + b^2{:}T\, V_{2,1}{:} = 0$
- Easy to compute and use! No integral kernels

E. Verlinde move

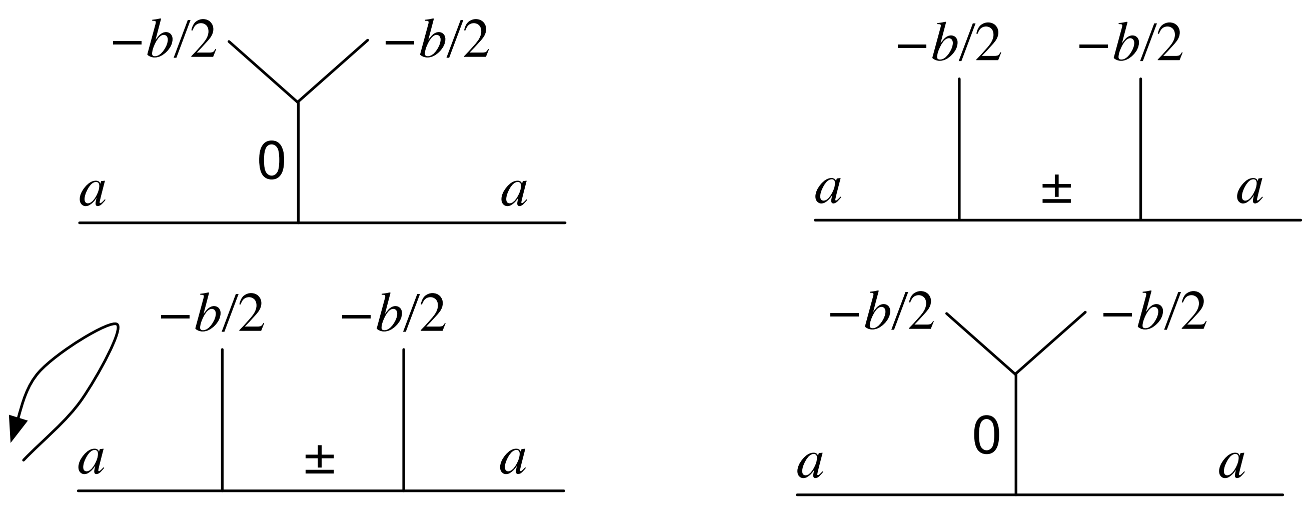
Insert identity in conformal block as fusion $V_{2,1}$ $V_{2,1}$

Transport one $V_{2,1}$ along closed path p on C

Fuse back to identity

Defines a ``loop operator'' L_p

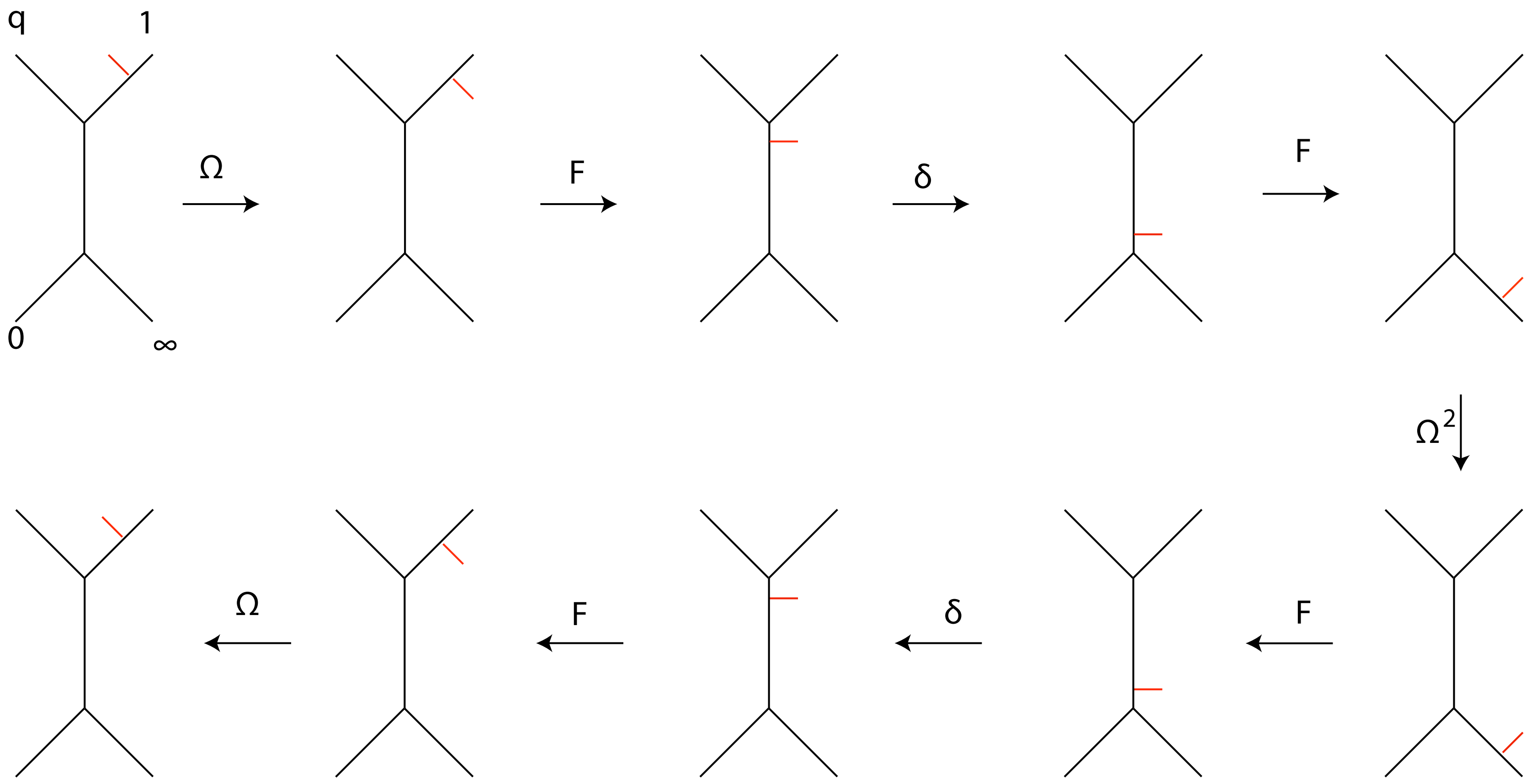
Wilson loops and L_p



Consider p around a leg

- No degenerate fusions, braiding gives a phase $\pm\pi b \alpha$
- Fusion to identity acts as a trace, gives $\cos \pi b \alpha$
- Wilson loop insertion! Z_{inst} goes to $\cos \pi b \alpha \; Z_{inst}$

Degenerate fields and Conformal blocks



Wilson loops and L_p

To compute S-dual,

- Pick a different pants decomposition
- Compute L_p on the same path
- More intricate sequence of elementary moves
  - Degenerate fusion matrices appear,
  - L_p involves shifts of α

  - $L_p Z_{inst}(a) = \sum c_n(a) Z_{inst}(a + n\, b/2)$

Time is up

Degenerate insertions are surface operators

- A whole story there.....

Which loops did we compute in S-dual frame?

- Drukker: ‘tHooft-Wilson loops labeled by paths p....

Is the result sensible?

- A direct gauge theory computation would be neat

Time is up

More open questions

- Direct proof of 2d-4d correspondence
- Why 6d? Find pure 4d gauge theory interpretation
- Classification of N=2 theories?