Geometry and Dynamics from Entanglement Entropy

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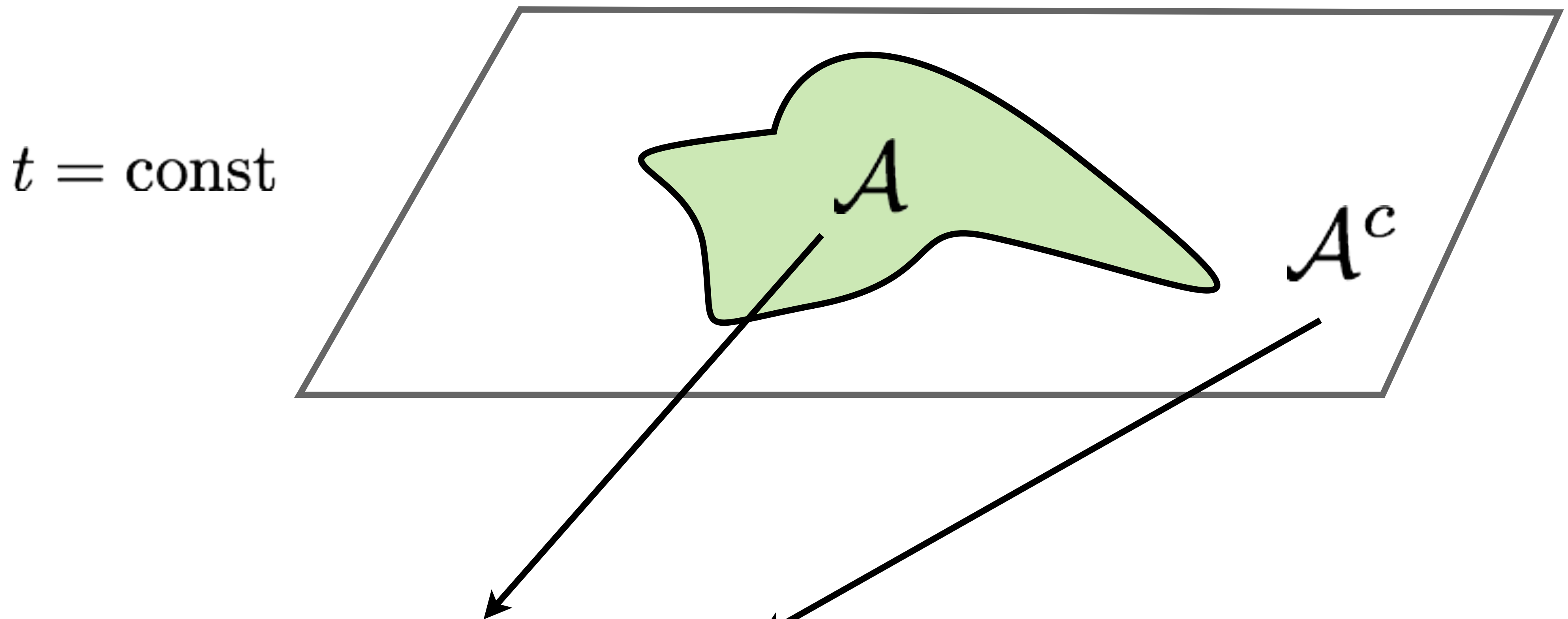
based on arXiv:1303.7221

and arXiv:1312.7856

with: M. Guica, T. Hartman, R. Myers, M. Van Raamsdonk

Entanglement Entropy

Characterize entanglement in QFT:

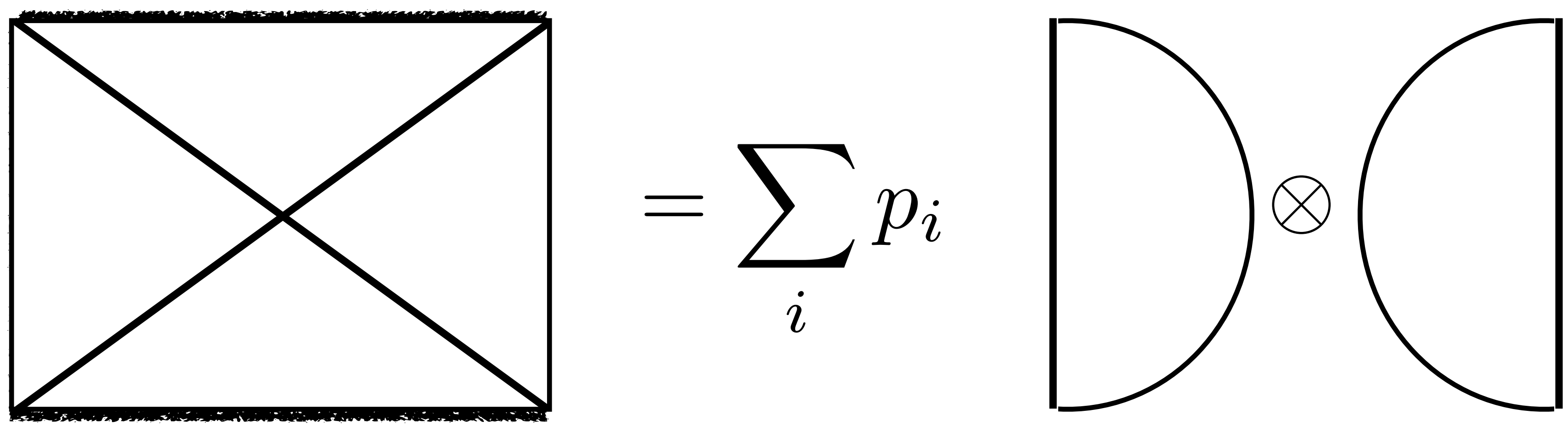


$$\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{A}^c} \qquad \rho_{\mathcal{A}} = \text{tr}_{\mathcal{A}^c} \left|0\right\rangle \left\langle 0\right|$$

$$S_{EE} = -\text{tr}\rho_{\mathcal{A}} \ln \rho_{\mathcal{A}}$$

Emergent Geometry & Entanglement

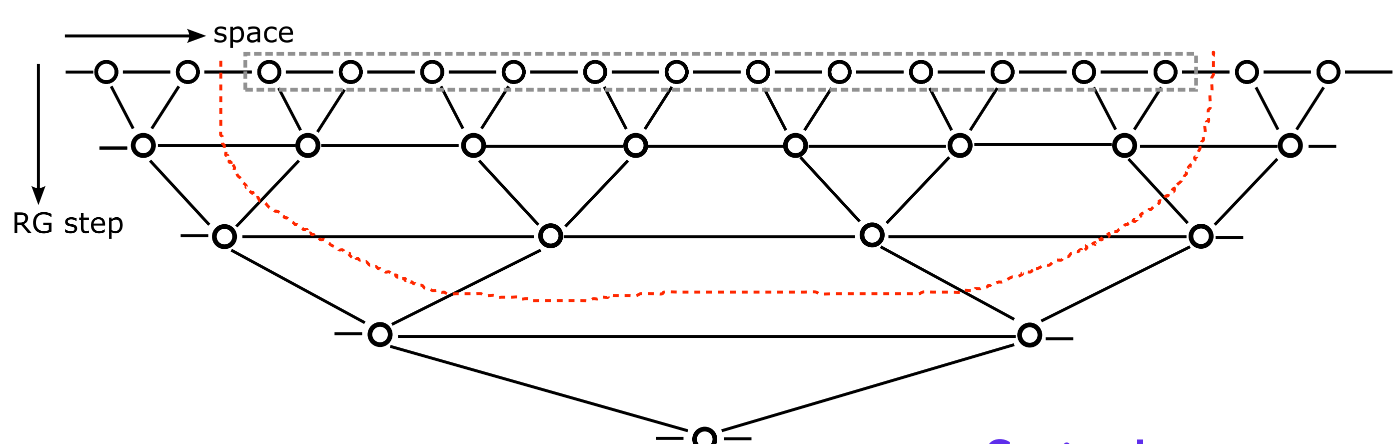
Some relationship between entanglement and emergent geometry:



$$= \sum_i p_i$$

Maldacena; Van Raamsdonk; Maldacena-Susskind (EPR = ER) ...

Tensor Networks:



Swingle, ...

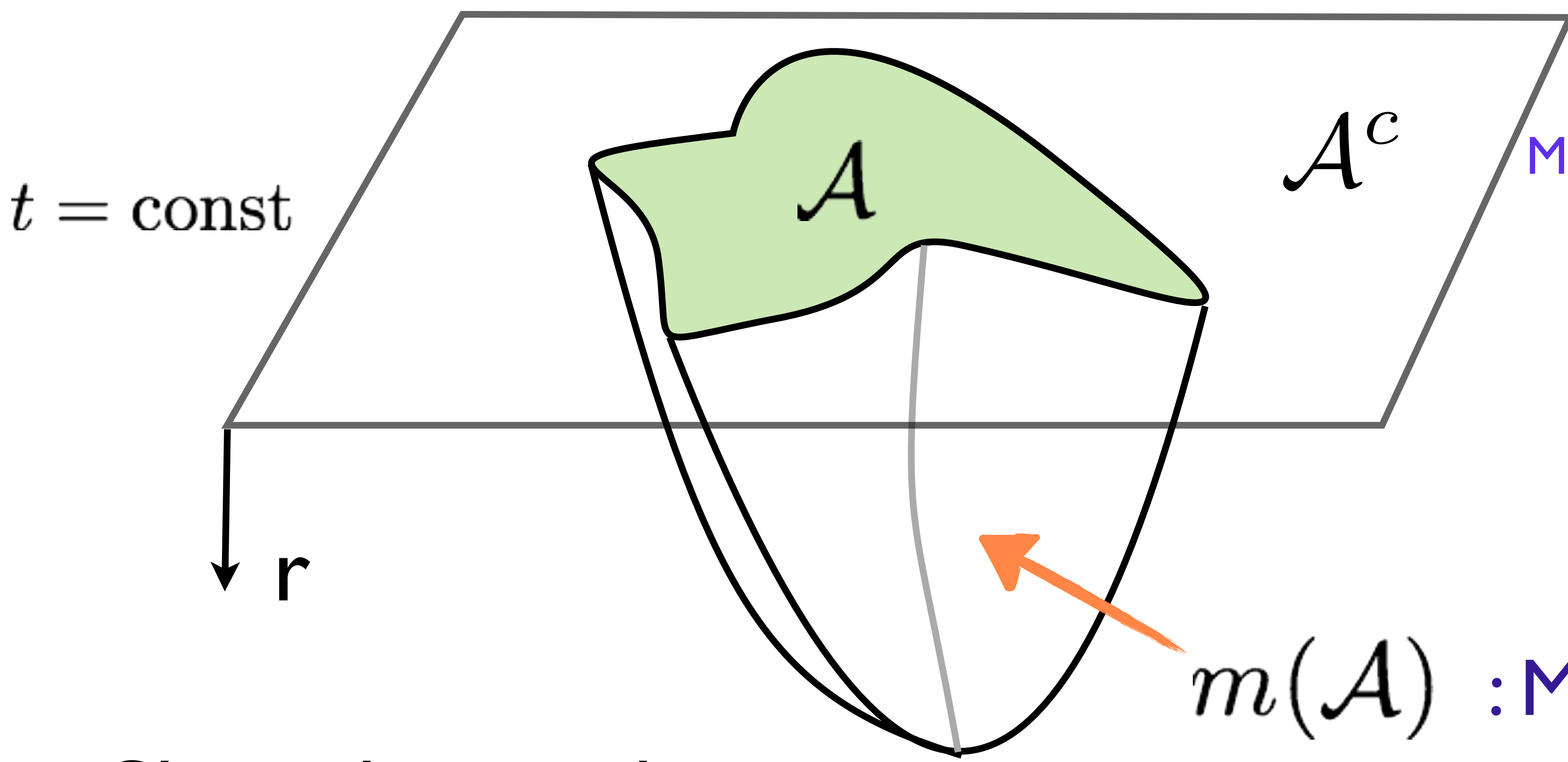
Ryu-Takayanagi formula

AdS/CFT: geometry emerges out of CFT

Ryu, Takayanagi `06

proven:

Maldacena, Lewkowycz `13



$m(\mathcal{A})$: Minimal surface

Classical gravity limit:

$$S_{EE} = \frac{\text{area}\,(m(\mathcal{A}))}{4G_N}$$

Beautiful generalization of Bekenstein-Hawking area law for Black Holes

Entanglement = probe of emergent geometry!

Plan:

Two interesting consequences/lessons of this formula:

- Large-N phase transitions in Entanglement Entropy
- Spacetime dynamics (Einstein's Equations) from the Entanglement First Law

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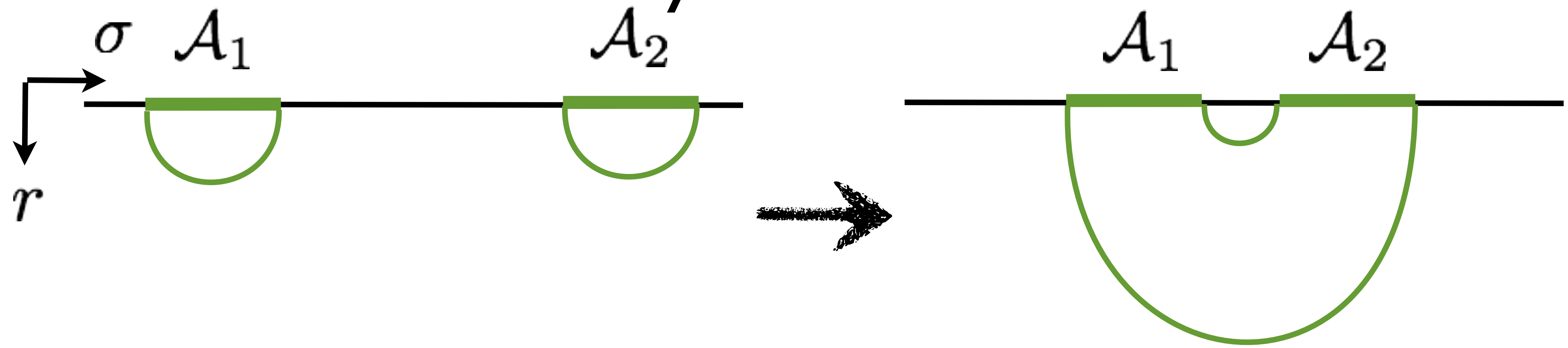
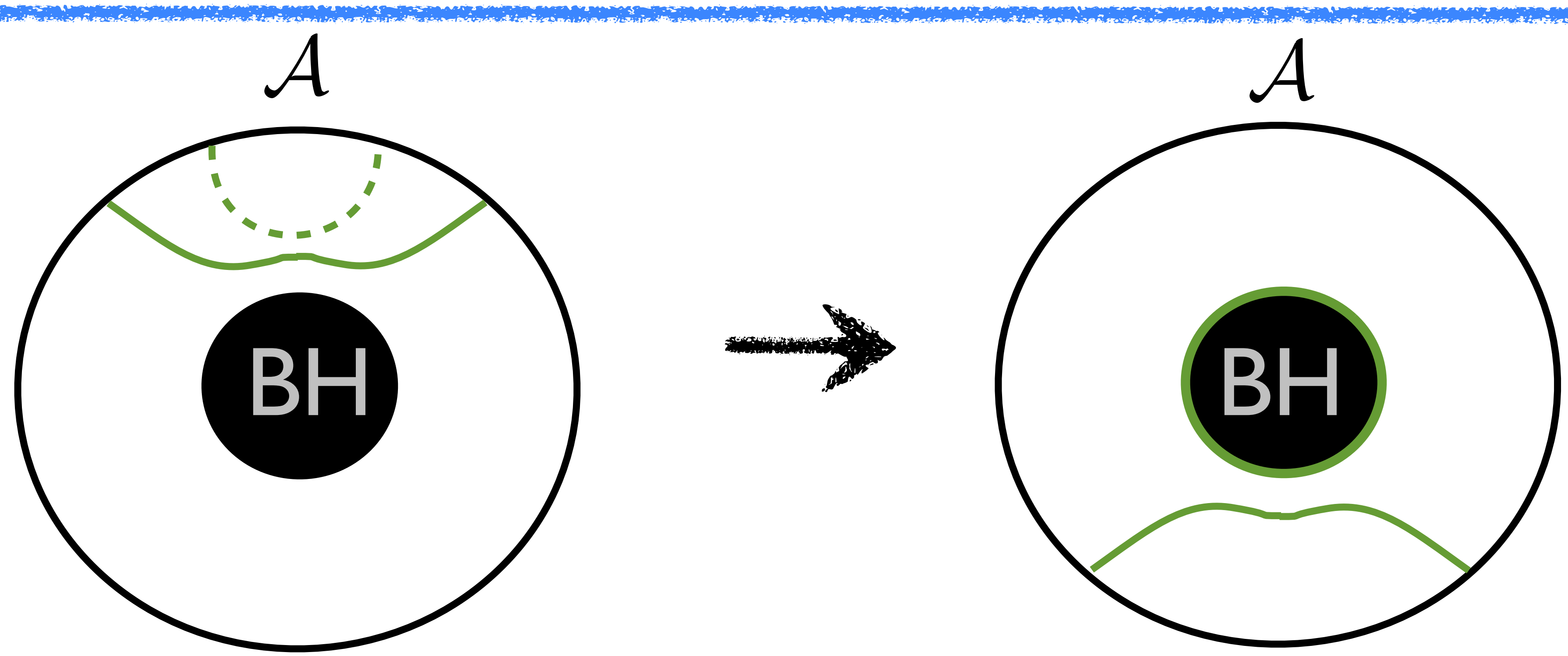
Minimization procedure gives rise to geometric phase transitions:

Two locally minimal surfaces:





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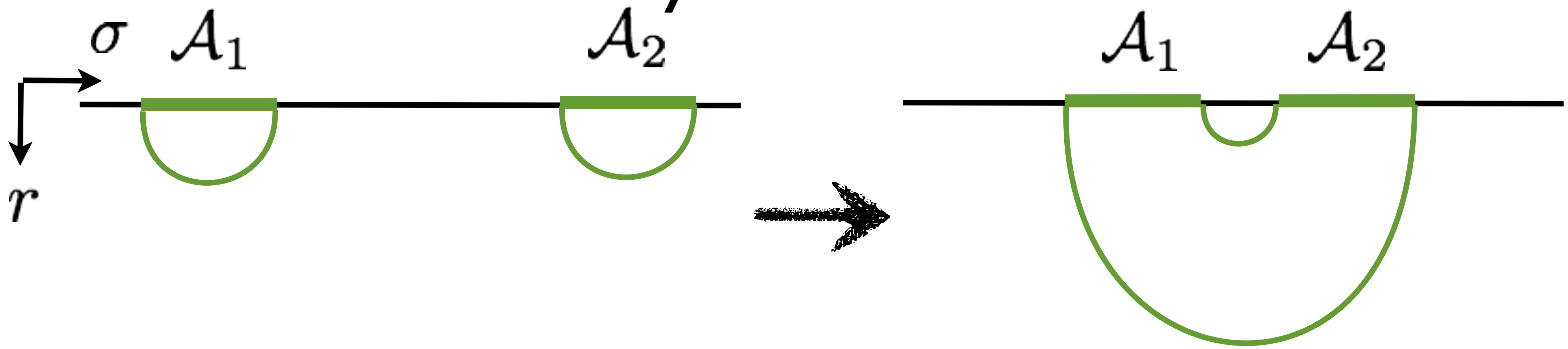

Order parameter: Mutual Information

$$I(\mathcal{A}_1, \mathcal{A}_2) = S(\mathcal{A}_1) + S(\mathcal{A}_2) - S(\mathcal{A}_1 \cup \mathcal{A}_2)$$

$$I = 0 \longrightarrow I \neq 0$$

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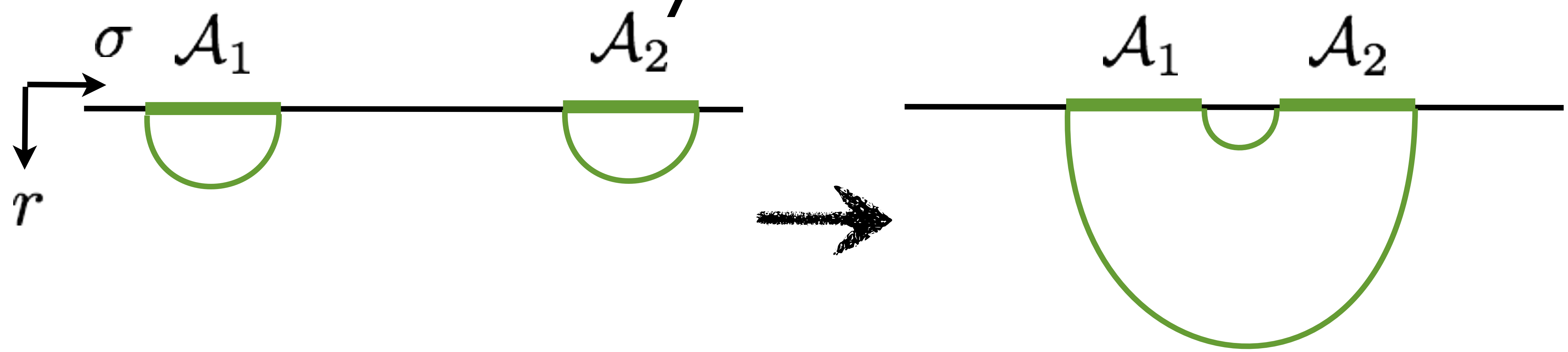
Two locally minimal surfaces:



Order parameter: Mutual Information

$$I(\mathcal{A}_1, \mathcal{A}_2) = S(\mathcal{A}_1) + S(\mathcal{A}_2) - S(\mathcal{A}_1 \cup \mathcal{A}_2)$$

$$I = \mathcal{O}(G_N^0) \longrightarrow I = \mathcal{O}(G_N^{-1})$$

Bulk Quantum Corrections to RT

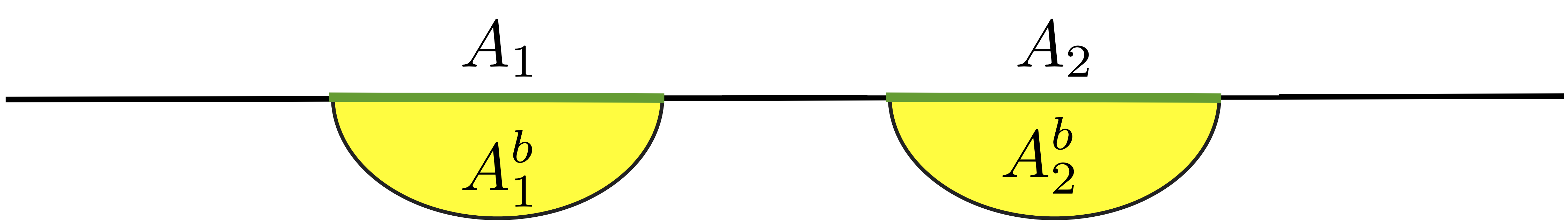
In a local QFT: Mutual Information can never be zero

$$I \geq \left(\left\langle \hat{O}_{\mathcal{A}_1} \hat{O}_{\mathcal{A}_2} \right\rangle_c \right)^2$$

In fact we argued leading correction comes from mutual information of bulk fields:

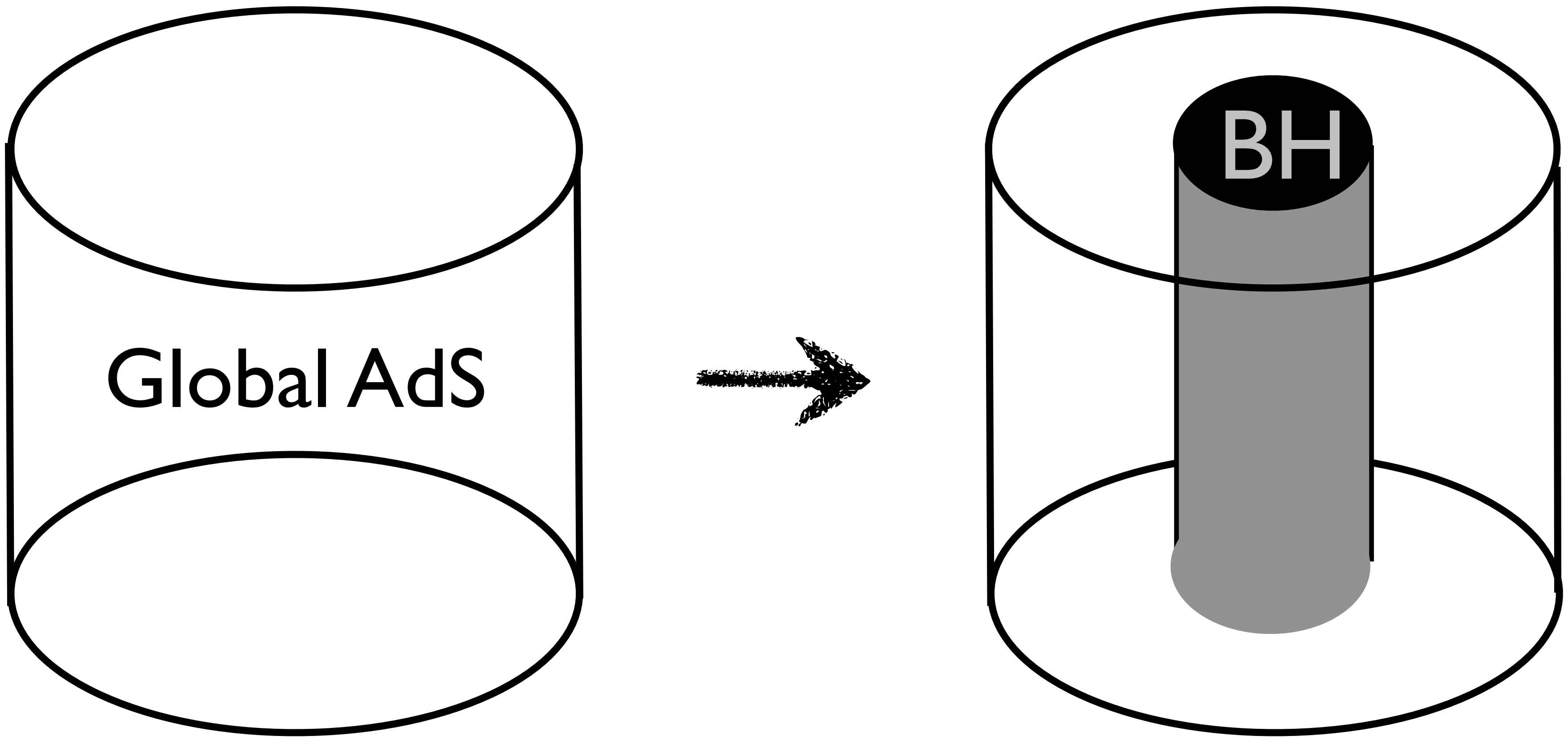
$$S_{EE}^{1-\text{loop}} = S_{EE}(A_b) + S_{\text{loc}}$$

TF, Lewcowycz, Maldacena



$$I(A_1; A_2) = I_{\text{bulk}}(A_1^b; A_2^b)$$

Many similarities with Hawking-Page



As a function of TR

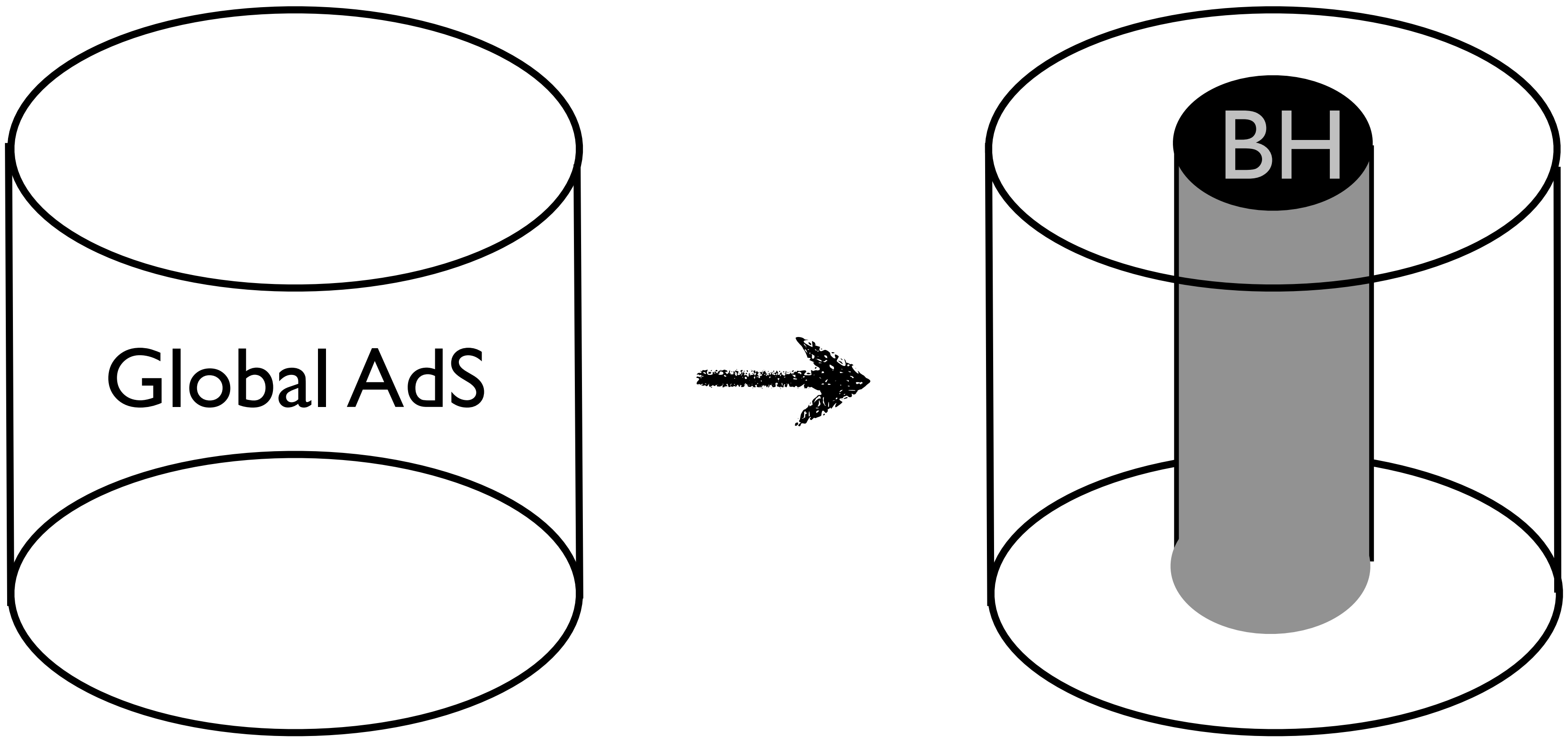
$S = \mathcal{O}(G_N^0)$ $\qquad\qquad$ $S = \mathcal{O}(G_N^{-1})$

Deep connections with deconfinement transition of large-N gauge theories

Witten; Sundborg; Aharony, Marsano, Minwalla, Papadodimas, van Raamsdonk

In 2d EE connection is very strong!

Many similarities with Hawking-Page



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The replica trick

Main computational tool for EE in QFT

$$S_{EE}(\mathcal{A}) = -\mathrm{Tr}\rho_{\mathcal{A}} \ln \rho_{\mathcal{A}}$$

hard to deal with

Introduce Entanglement Renyi Entropy:

$$S_n(\mathcal{A}) = -\frac{1}{n-1} \ln \mathrm{Tr}\rho_{\mathcal{A}}^n$$

Compute for integer $n \geq 2$ ***attempt*** to continue to non-integer .... take the limit

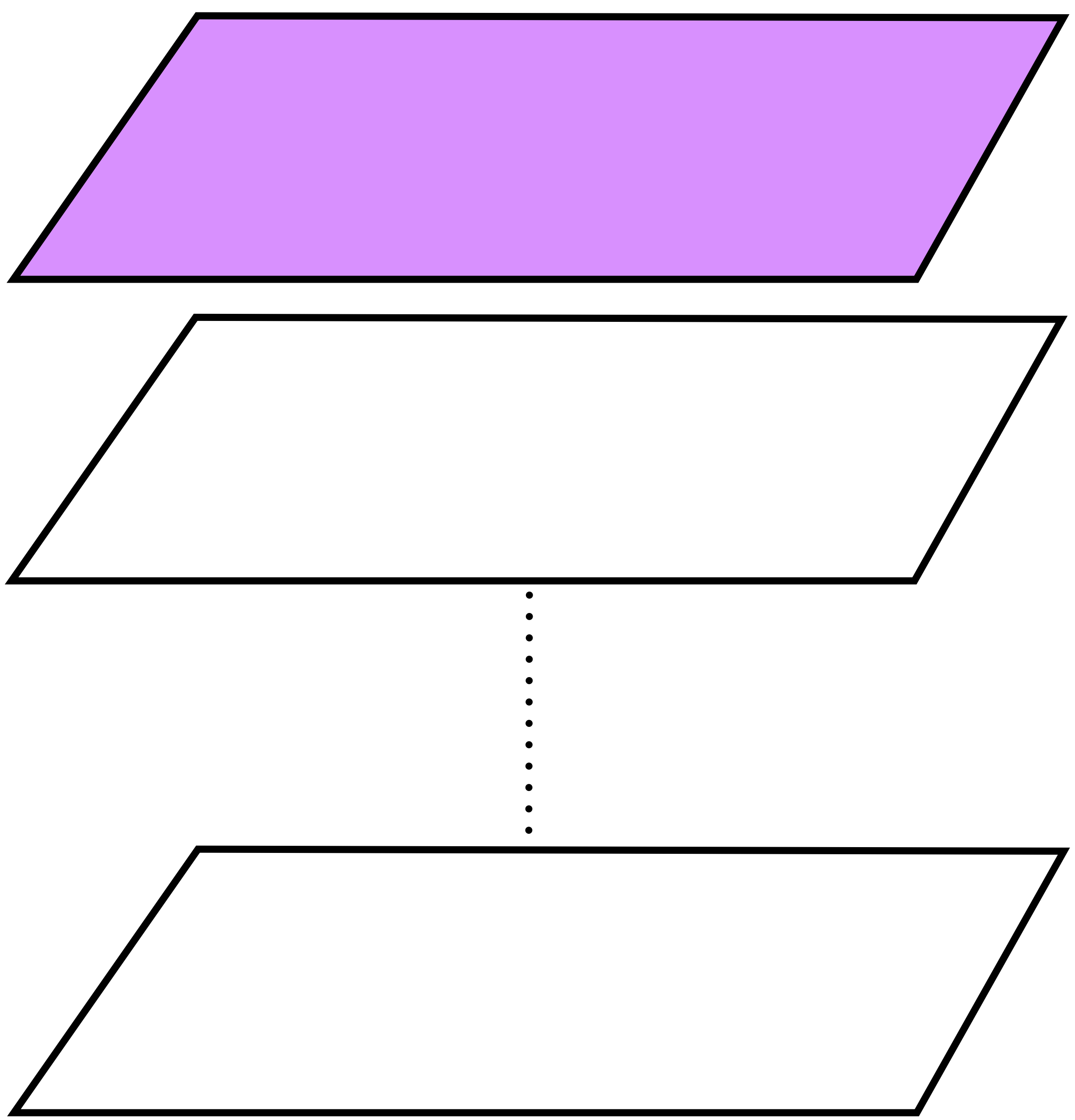
$$\lim_{n \to 1} S_n(\mathcal{A}) = S_{EE}(\mathcal{A})$$

Why? One can formulate $\mathrm{Tr}\rho_{\mathcal{A}}^n$ as a *euclidean* path-integral

The replica trick

$\mathrm{tr}\rho_{\mathcal{A}}^{n}$

- Take n-copies of CFT on the Euclidean plane:



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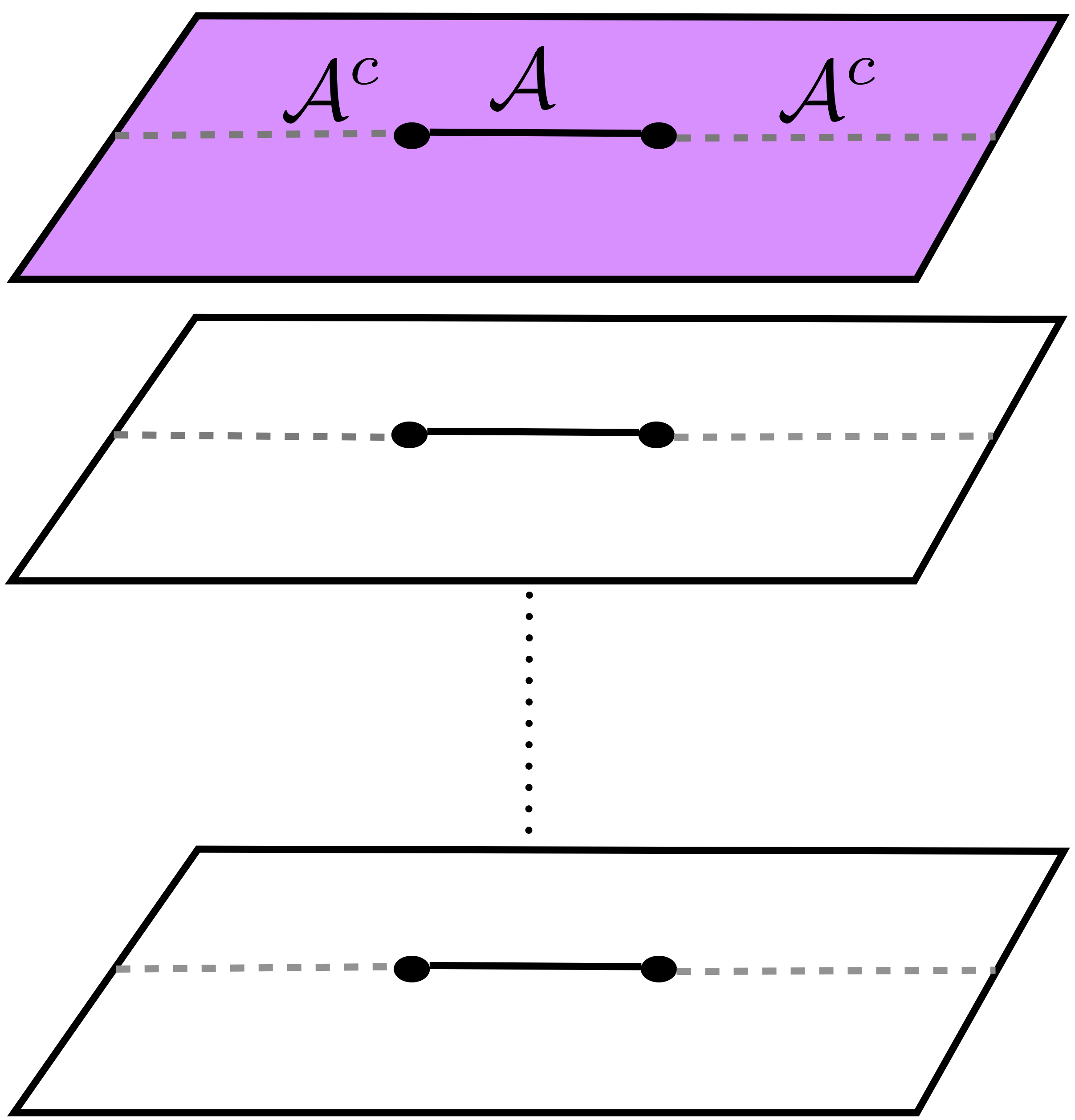
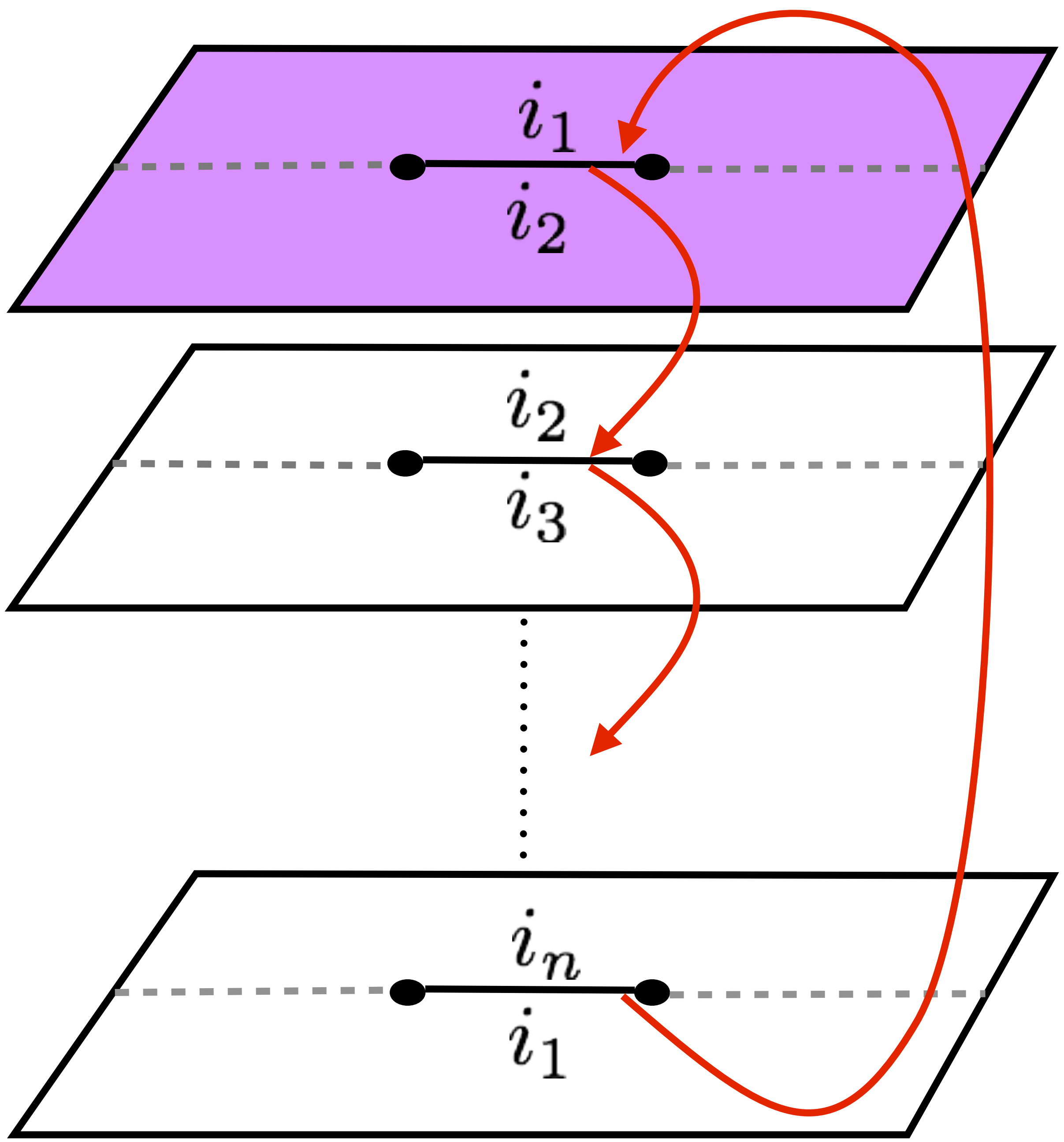


- Split t=0 surface into regions A and A^c

The replica trick

$$\mathrm{tr}\rho_{\mathcal{A}}^{n} = \sum_{i_1 \in \mathcal{A}} \langle i_1 | \rho_{\mathcal{A}} \sum_{i_2 \in \mathcal{A}} |i_2\rangle \langle i_2 | \rho_{\mathcal{A}} \ldots \sum_{i_n \in \mathcal{A}} |i_n\rangle \langle i_n | \rho_{\mathcal{A}} |i_1\rangle$$

- Take n-copies of CFT on the Euclidean plane:



- Split t=0 surface into regions A and A^c
- Cut and join according to products and traces

The replica trick

$$\mathrm{tr}\rho_{\mathcal{A}}^n = \sum_{i_1 \in \mathcal{A}} \langle i_1 | \rho_{\mathcal{A}} \sum_{i_2 \in \mathcal{A}} |i_2\rangle \langle i_2| \rho_{\mathcal{A}} \dots \sum_{i_n \in \mathcal{A}} |i_n\rangle \langle i_n| \rho_{\mathcal{A}} |i_1\rangle$$

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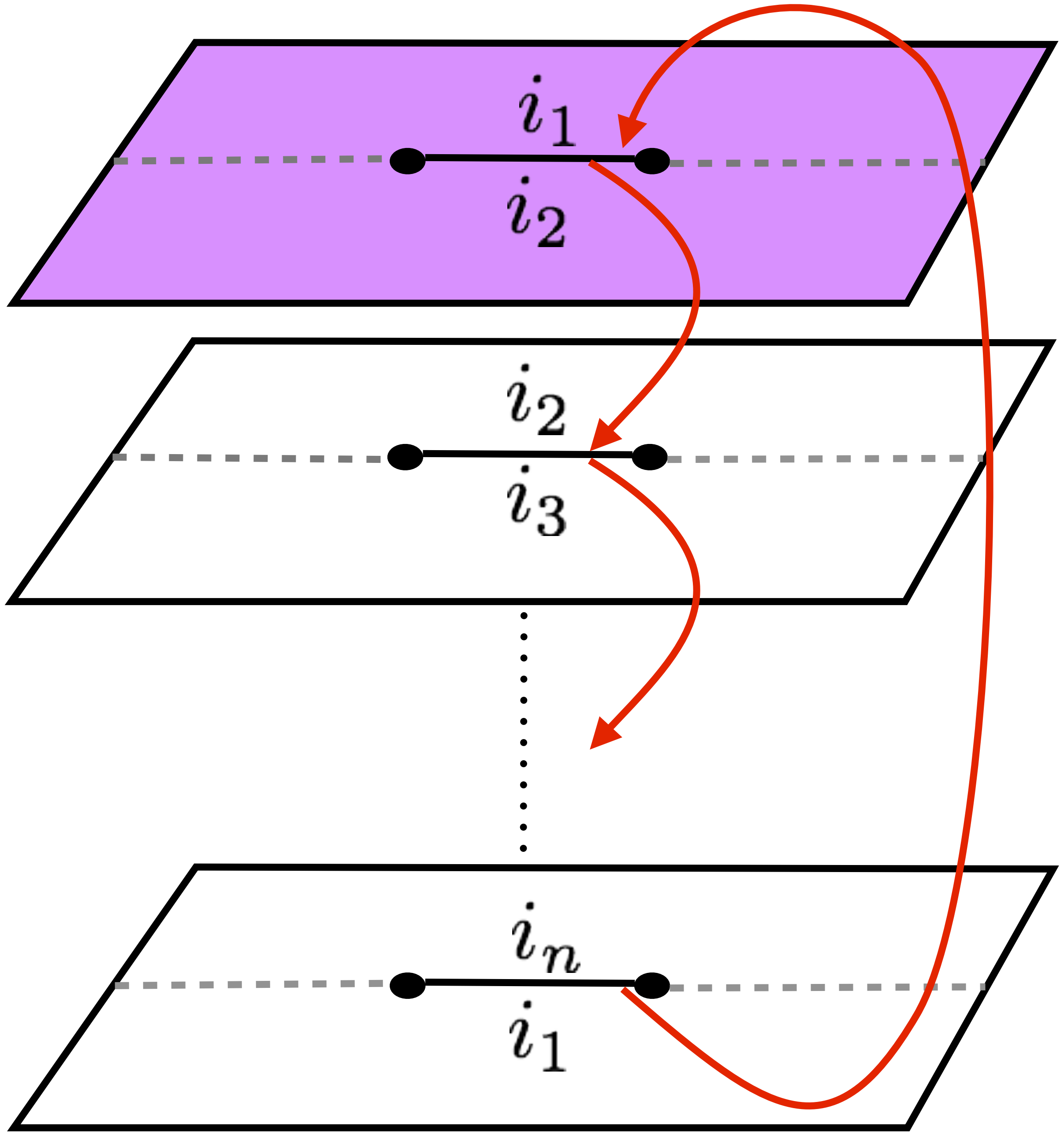


- Split t=0 surface into regions A and A^c
- Cut and join according to products and traces
- Partition function on this manifold:

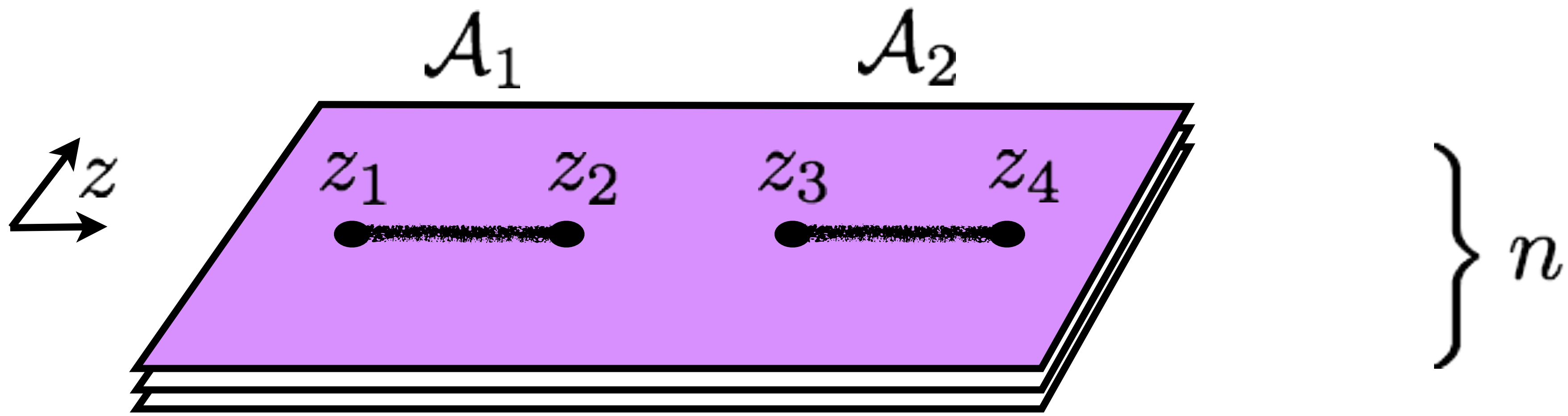
$$\mathrm{tr}\rho_{\mathcal{A}}^n = \frac{Z_{\mathcal{M}_n}}{Z_1^n}$$

$$S_n(\mathcal{A}) = -\frac{1}{n-1}\left(\ln Z_{\mathcal{M}_n} - n \ln Z_1\right)$$

Two intervals leads to a complicated surface:

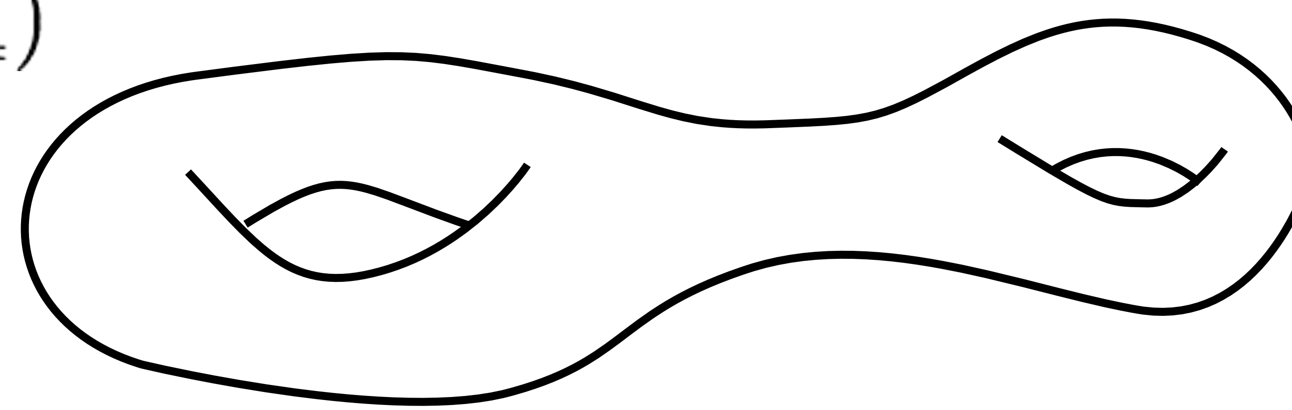
Riemann surface:

$\mathcal{M}_n =$



$$y^n = \frac{(z-z_1)(z-z_3)}{(z-z_2)(z-z_4)}$$

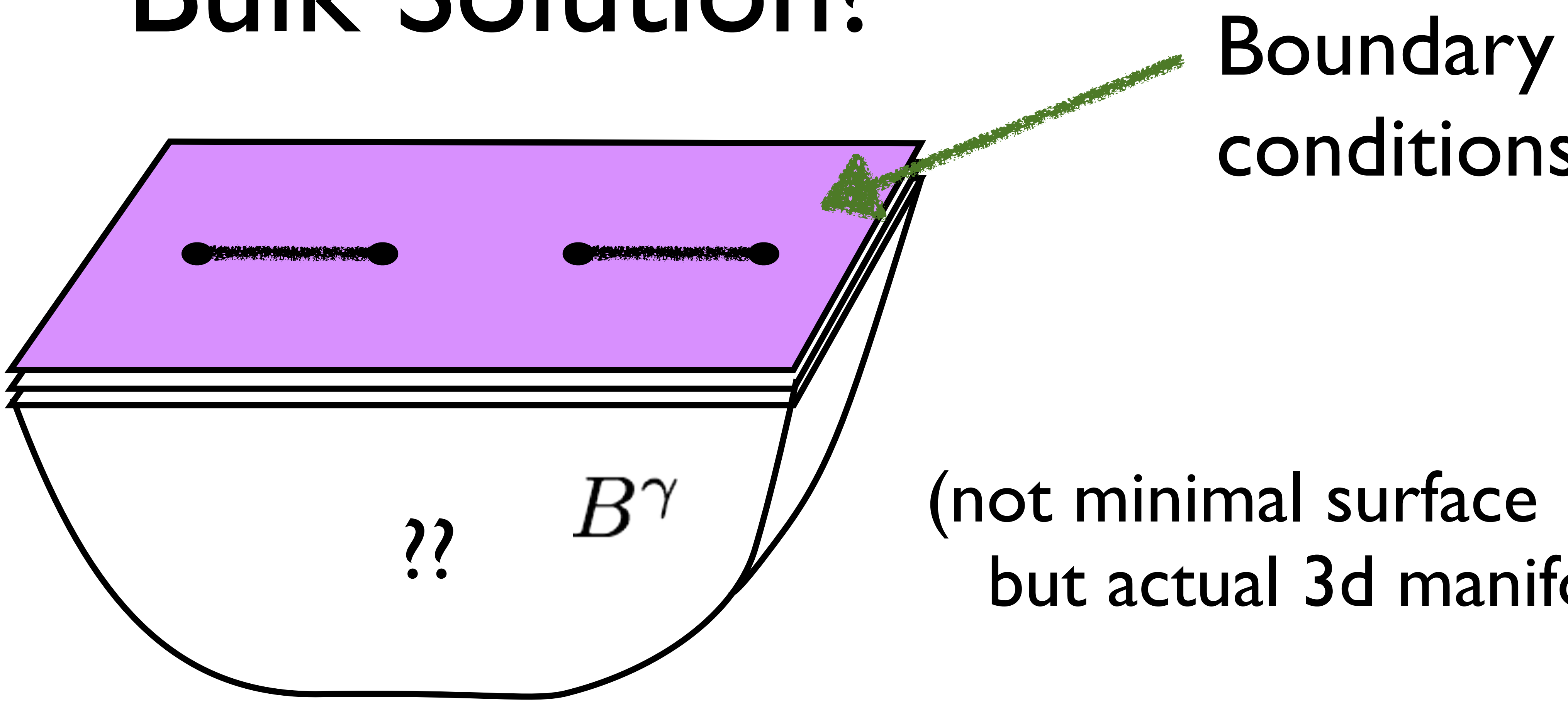
Genus (n-1)



$Z(\mathcal{M}_n) =$ complicated!

- Goal: use usual rules of AdS/CFT to compute the partition function: $Z_{\mathrm{CFT}}(\mathcal{M}_n)$

Bulk Solution?



(not minimal surface
but actual 3d manifold)

Solve Einstein's equations subject to boundary conditions and bulk regularity. $\partial B^\gamma = \mathcal{M}_n$

Many solutions! $\mathcal{O}(c)$

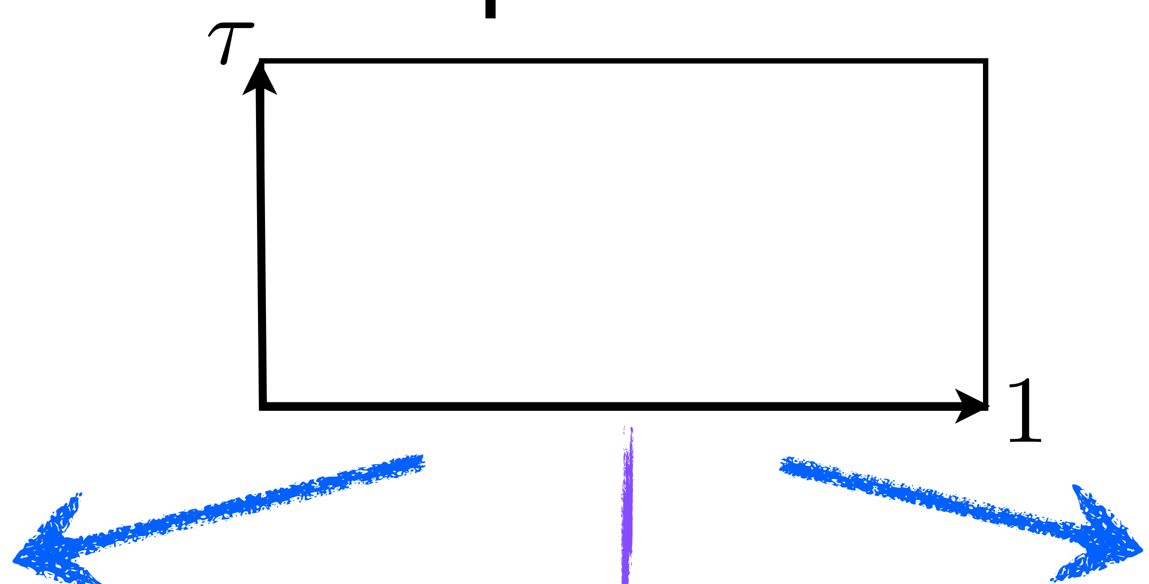
$$Z_{\mathcal{M}_n} = \sum_\gamma \exp\left(-S^\gamma_{\text{grav}} + \mathcal{O}(c^0)\right) \qquad G_N \propto c^{-1}$$

Classical gravity limit: only need least action solution

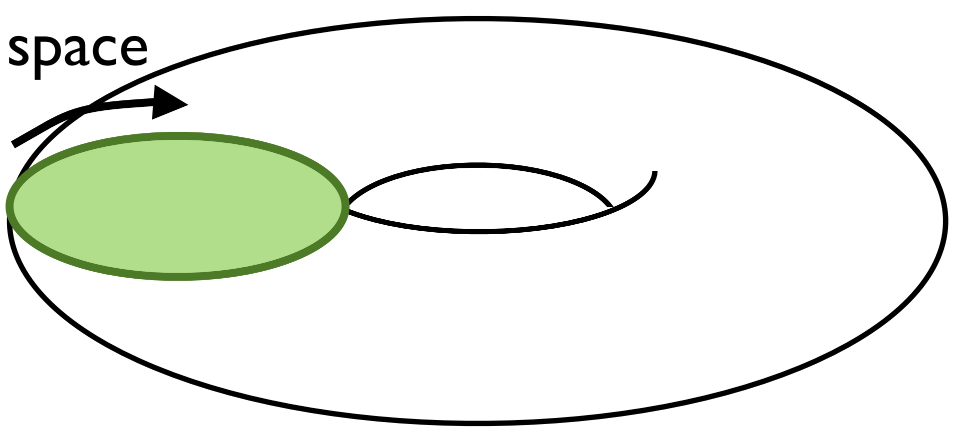
The case $n = 2$ is easy:

Double cover gives a simple torus, and $Z_{\mathcal{M}_2}$ is the thermal partition function.

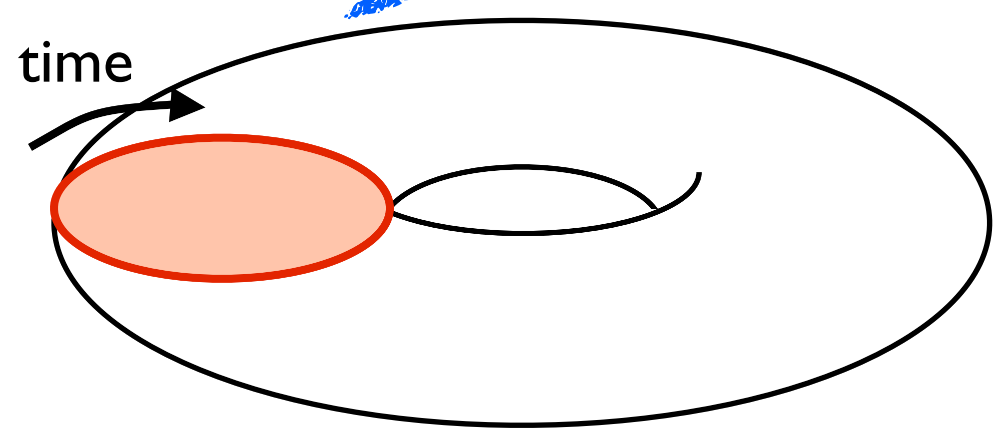
Headrick `10



Solid torus:



Thermal AdS



BTZ black hole

Hawking Page phase transition!

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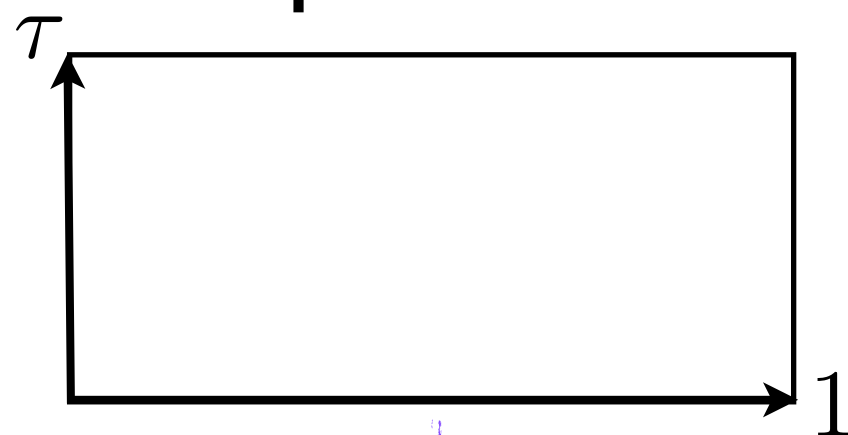
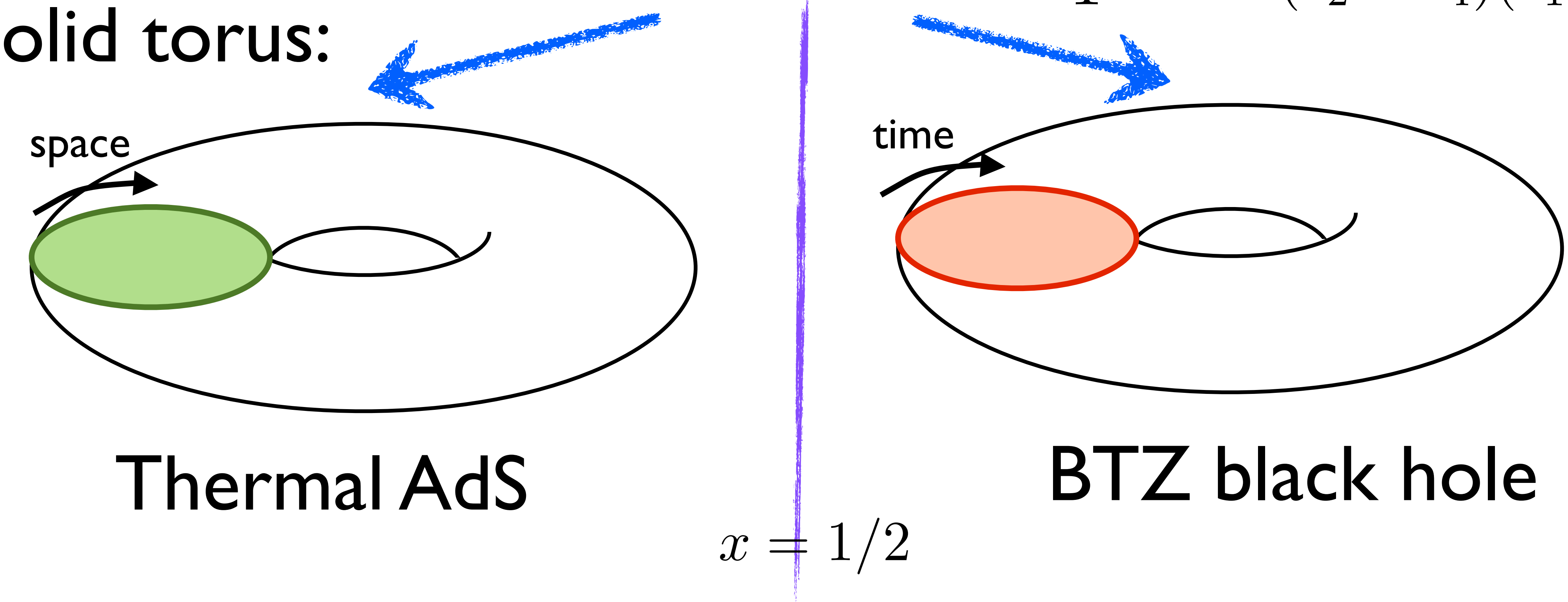


$$\tau = i \frac{K(x-1)}{K(x)}$$

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_4)(z_1 - z_3)}$$

Solid torus:



Hawking Page phase transition!

But to find EE need solutions for all integer n??

Simplifying assumptions:

1. Least action solution is a handlebody
2. This handlebody preserves the boundary symmetries:

$\mathbb{Z}_n$ replica symmetry not spontaneously broken

← cyclic permutations of the replicas

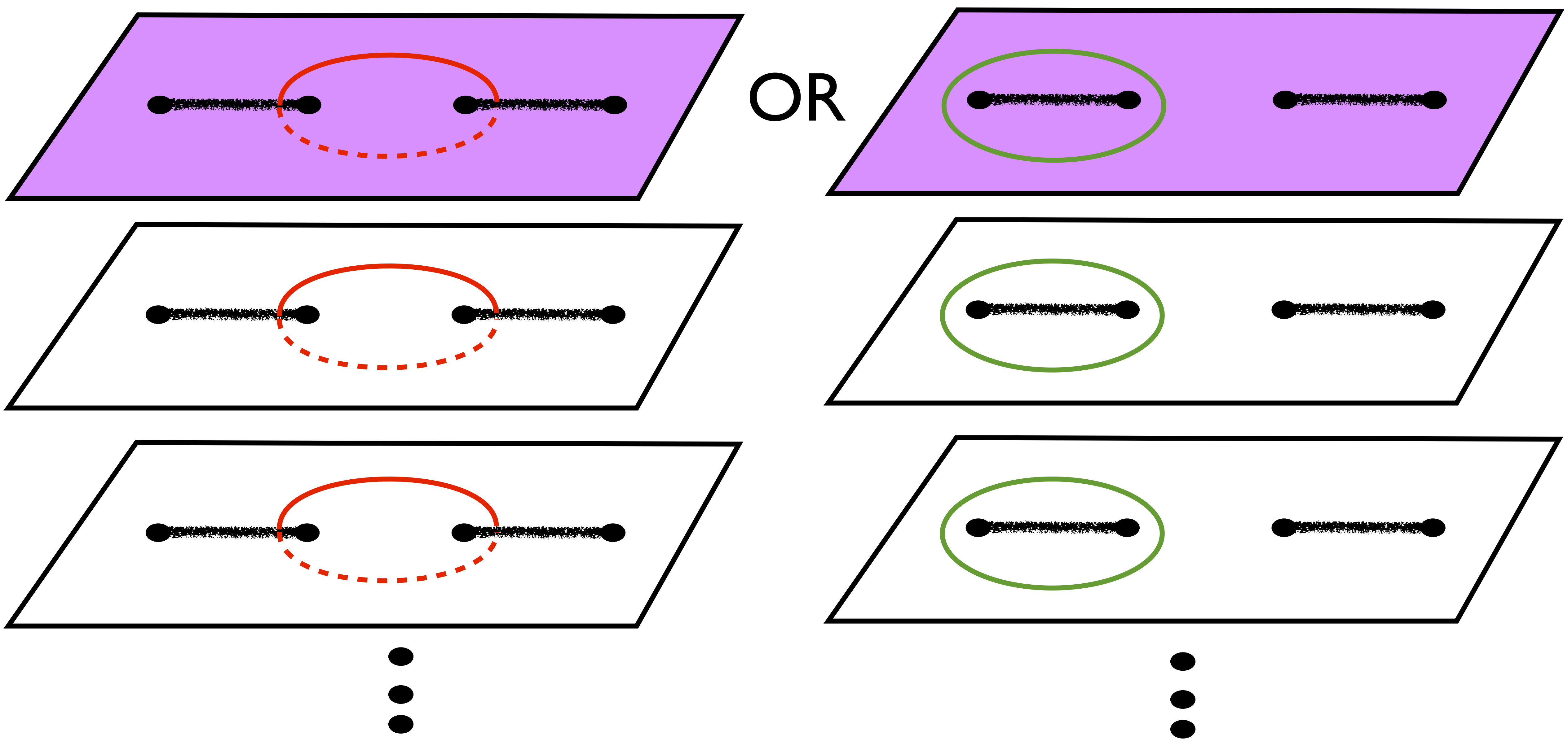
We found two solutions satisfying these assumptions

Exchange dominance at $x = 1/2$ for all n

Analytically continue the action to $n = 1$ - RT

A Handlebody is a 3 manifold which fills in the Riemann surface in such a way that there are g=(n-1) contractible cycles in the bulk (analog of solid torus)

Pick these cycles symmetrically to preserve replica symmetry:



Constructed as follows:

Find a flat $SL(2,\mathbb{C})$ connection living on $\mathcal{M}_n$. This can then be extended to a 3d solution of Einstein's equations (a`la Witten's $SL(2,\mathbb{C})$ CS description of gravity.)

In particular contractible cycles must necessarily have zero $SL(2,\mathbb{C})$ monodromy and this uniquely specifies the flat connection
(e.g. we can find it numerically)

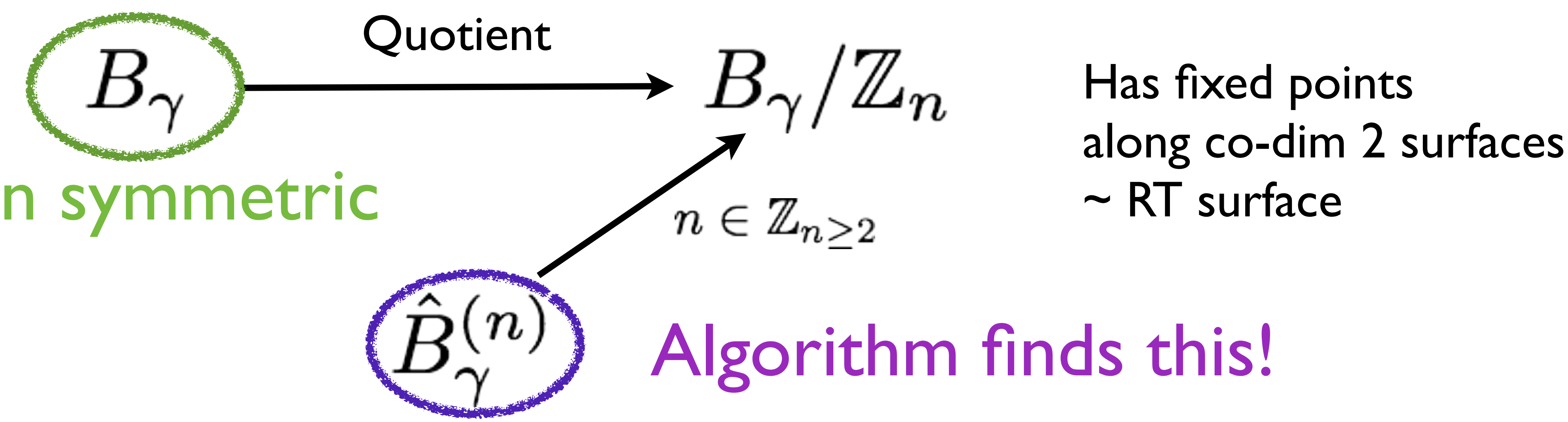
Extract Renyi Entropy!
Large-N phase transition at x=1/2

Bulk action is then easy to compute (numerically.)
Actually this algorithm works for non-integer n.
Why?

General Lessons:

Maldacena Lewcowycz

Why can we analytically continue in n?



- $\hat{B}_\gamma^{(n)}$ has conical deficit singularities, opening angle: $2\pi/n$
- For $n \approx 1$ regain original bulk + tensionless cosmic string
- Equations for cosmic string fixed by Einstein's equations - RT answer

General Lessons:

Universality at large-c

(Riemann $\propto$ Ricci)

Solutions we construct are locally AdS_3 which is maximally symmetric and thus remains a solution including higher derivative corrections

Expect Renyi Entropy to be universal for large-c CFTs

$$S_n = c f_{\text{universal}} + \mathcal{O}(c^0)$$

Recall $S_{\text{grav}} \propto c$

Like universality of thermodynamics at large-c

Dijkgraaf, Maldacena, Moore, Verlinde `00 Keller `11

Additional constraint on spectrum: density of states is $\mathcal{O}(c^0)$ for $h < \mathcal{O}(c)$

CFT derivation

Hartman 13

- Exact same prescription can be arrived at in a completely different way for large-c CFTs

$$Z(\mathcal{M}_n) = \langle \sigma_+\sigma_-\sigma_+\sigma_- \rangle = \sum_p C^p_{+-} C^p_{+-} F(h_n, h_p, c; x)$$

Twist operators in $(CFT)^n/\mathbb{Z}_n$

primaries

OPE coefficients

conformal blocks

"Classical conformal blocks"

Zamolodchikov `87

- At large-c the relevant F's are computed by the same monodromy problem as for the handlebodies
- Assuming nice behavior of the spectrum of primaries as well as for the OPE coefficients one arrives at the same result

Plan:

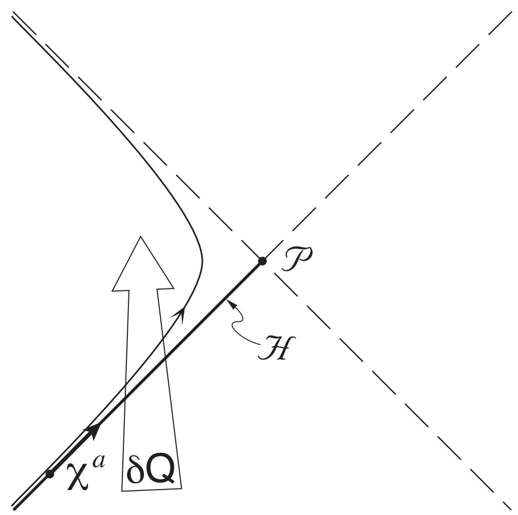
Two interesting consequences/lessons of this formula:

- Large-N phase transitions in Entanglement Entropy
- Spacetime dynamics (Einstein's Equations) from the Entanglement First Law

Dynamics & Entanglement

If geometry emerges, what about the dynamics of this geometry? eg Eintein's Equations

Many Hints - Thermodynamic in Nature



Jacobson `95

Padmanabhan; Verlinde

Recent precise statement: linearized Einstein's Equations from "First Law of Entanglement"

Lashkari, McDermott, Van Raamsdonk

Now: discuss a simple proof of this result and extension to higher derivative gravity

First Law of Entanglement Entropy

$$|0\rangle \in (\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}) \longrightarrow \rho_A = \mathrm{tr}_{A^c}|0\rangle\langle 0|$$

$$S_A(\rho_A) = -\mathrm{tr}\rho_A \log \rho_A$$

Modular Hamiltonian (Entanglement Hamiltonian):

$$H_A^{(|0\rangle)} \equiv -\log \rho_A + C \qquad (T = 1)$$

State dependent operator (always vacuum for this talk)

Calculate expectation in another state: Modular Energy

$$\rho'_A = \mathrm{tr}_{A^c}|\psi\rangle\langle\psi| \qquad E_A(\rho'_A) = \mathrm{tr}\rho'_A H_A^{(|0\rangle)}$$

"Small" variation in state

$$E_A(\rho'_A) - E_A(\rho_A) = \boxed{\delta E_A \approx \delta S_A} = S_A(\rho'_A) - S_A(\rho)$$

Some Comments:

- "Small" change in state:

$$|\psi(\lambda)\rangle = |0\rangle + \lambda|\phi\rangle + \dots$$

- Can be understood as a consequence of the positivity of relative entropy:

Casini `08
a form of the
Bekenstein Bound

$$S(\rho_A|\rho'_A) = \delta E - \delta S \geq 0$$

Blanco, Casini, Hung, Myers `13

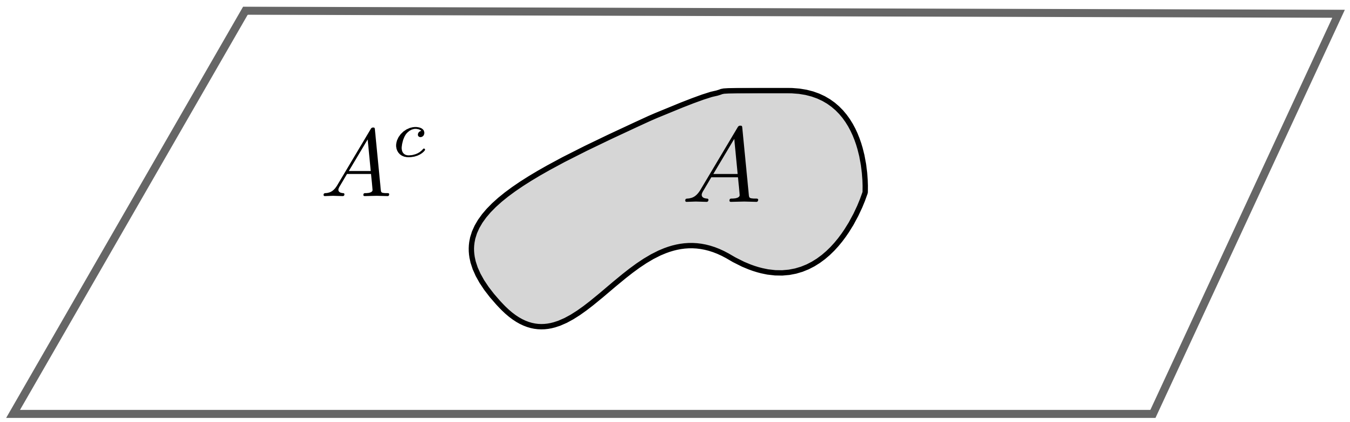
- For a ρ_A a thermal density matrix: $\exp(-\beta H)$ an exact quantum statement of the first law allowing for arbitrary first order variations:

$$T\delta S = \delta E$$

Energy Energy

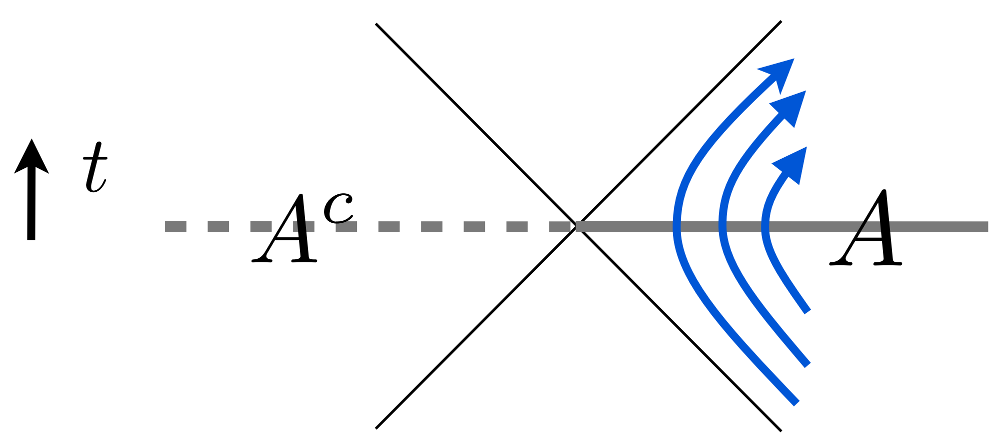
Modular Hamiltonian

Consider a local QFT; take A to be a subregion in a constant time slice of the QFT:



In general H_A will be some horrible non local operator

Well known example of a Modular Hamiltonian:



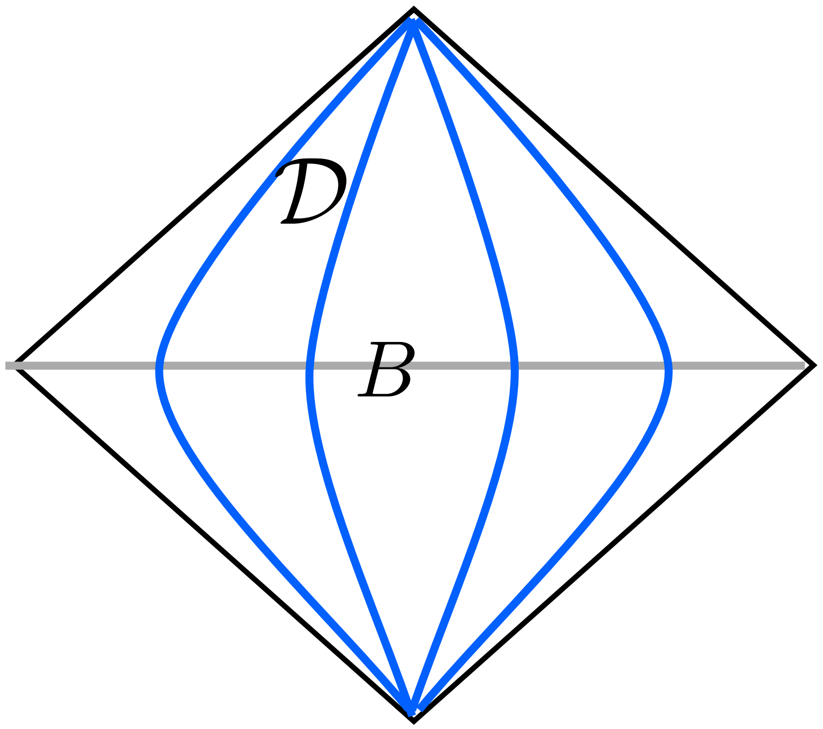
Half space/Rindler wedge

$H_A =$ boost generator

Modular Hamiltonian for a Ball in CFT

In a CFT can conformally map half space to a ball and the Rindler wedge to $\mathcal{D}$

Casini, Huerta, Myers `11



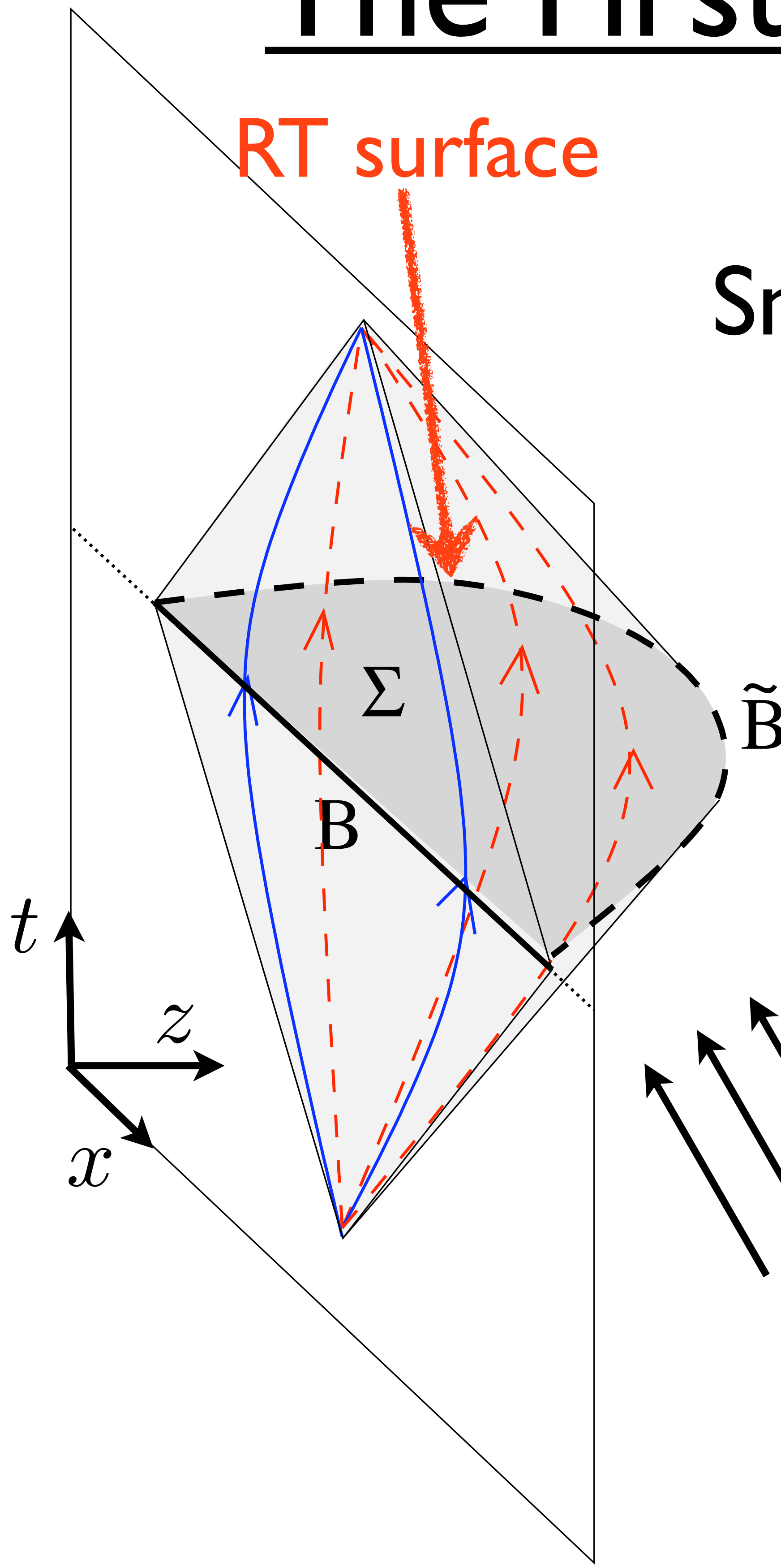
Boost generator maps to:
Conformal Killing Vector

$$\zeta_B = \frac{\pi}{R}\left((R^2 - t^2 - \vec{x}^2)\partial_t - 2tx^i\partial_i\right)$$

$$H_B = 2\pi \int_B d^{d-1}x \, \frac{R^2 - \vec{x}^2}{2R} \, T_{tt}$$

This explicit expression for H will allow us to understand the consequences of first law in AdS/CFT!

The First Law in AdS/CFT



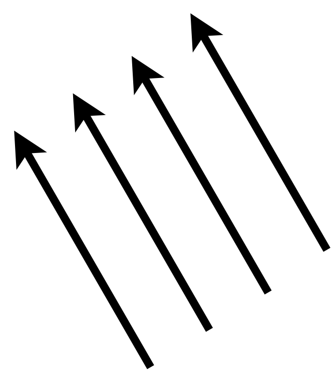
Basic setup:

Small perturbation to vacuum:

$$|\psi(\lambda)\rangle = |0\rangle + \lambda|\phi\rangle + \dots$$

$$g_{ab} = g_{ab}^{AdS} + h_{ab} \qquad h \ll 1$$

Consider gravity waves on AdS



Entanglement Entropy:

$$\delta S = \frac{1}{4G_N}\int_{\tilde{B}} \delta\sqrt{g_{\tilde{B}}}$$

Modular Energy in AdS/CFT

Assymptotically AdS: *

$$ds^2 = z^{-2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) + z^{d-2}h^{(d)}_{\mu\nu}dx^\mu dx^\nu + \ldots$$

Stress tensor constructed from assymptotic expansion:

Balasubramanian, Kraus ...

$$\langle T_{\mu\nu}\rangle \equiv \frac{d}{16\pi G_N}\, h^{(d)}_{\mu\nu}$$ (Einstein Gravity)

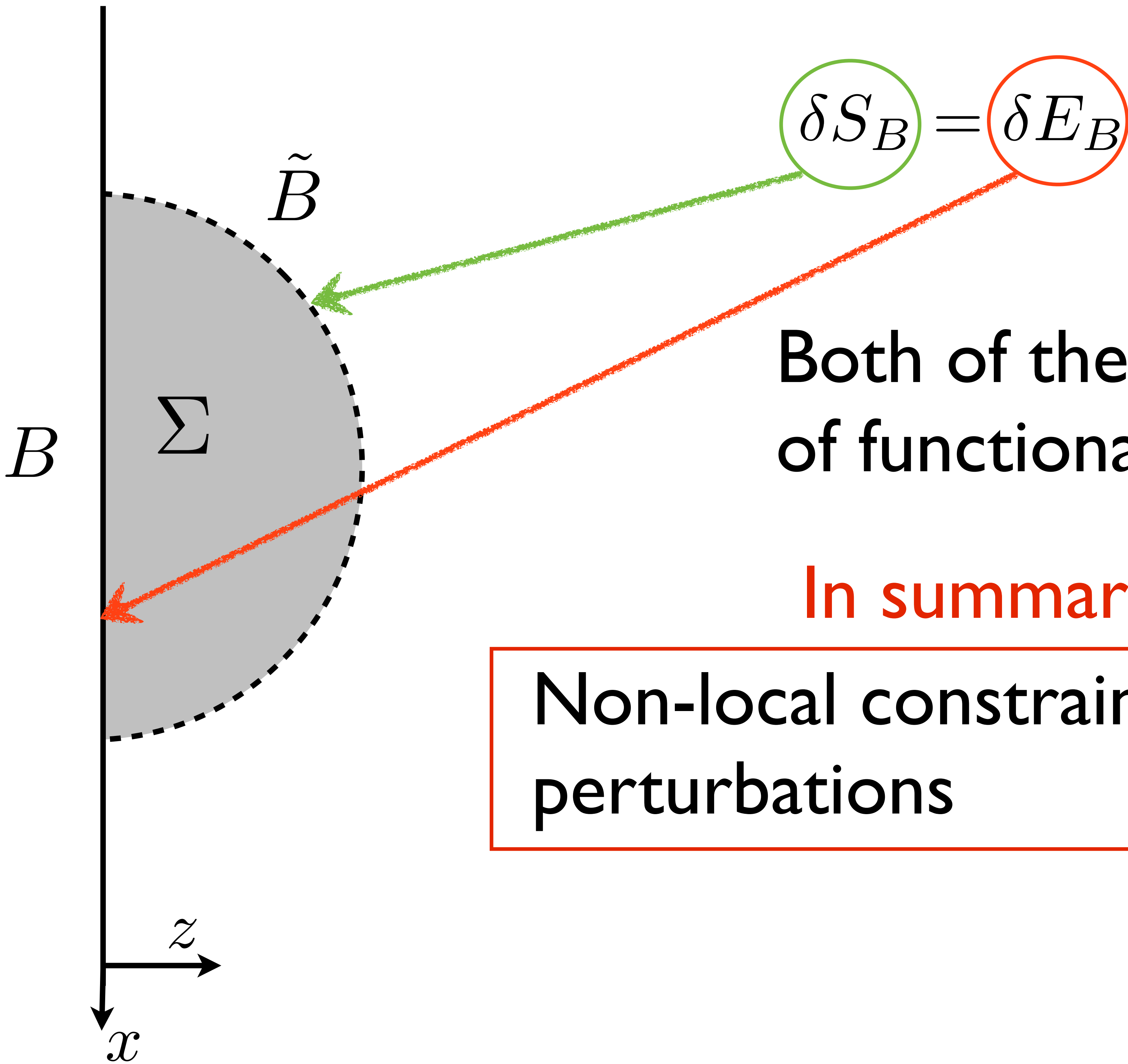
Modular Energy:

$$\langle H_B\rangle \equiv E_B = \frac{d}{8G_N}\int_B \frac{(R^2 - \vec{x}^2)}{2R} h^{(d)}_{tt}$$

$$B : \vec{x}^2 < R^2$$

*(expansion found by solving EOM. We are soon to discuss *deriving* the EOM - might worry this is circular. Never fear: first law can also be used to directly derive this assymptotic behavior.)

The First Law in AdS/CFT

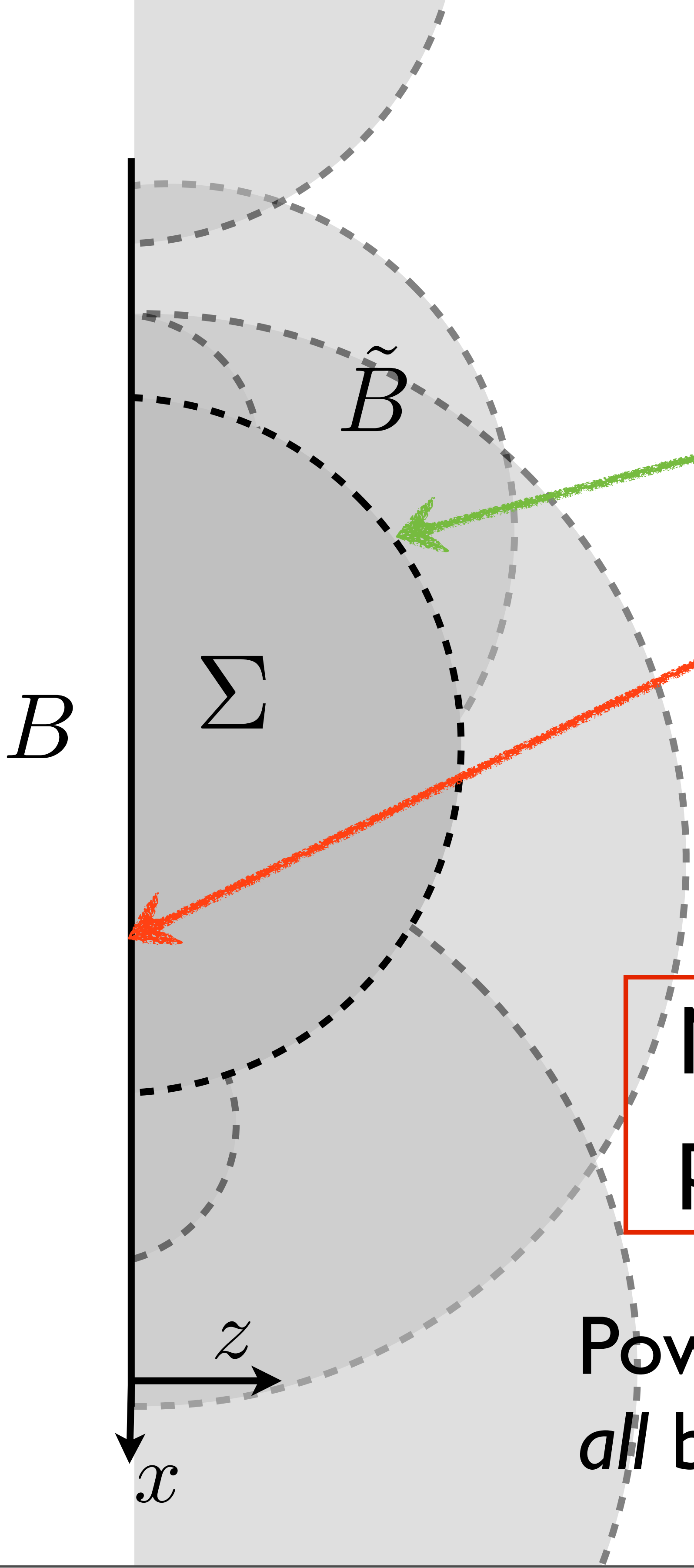


Both of these are integrals of functionals of h

In summary:

Non-local constraint on metric perturbations

The First Law in AdS/CFT



$$\delta S_B = \delta E_B$$

Both of these are integrals of functionals of h

In summary:

Non-local constraint on metric perturbations

Power: applies to *all* sizes of balls centered at *all* boundary points, in *all* Lorentz frames

EOM ⟷ First Law

Claim: this set of non-local constraints on h, equivalent to Einstein's Equations

To show this we constructed a (D-2) form: $\boldsymbol{\chi}(h)$ with the following properties:

$$\int_B \boldsymbol{\chi} = \delta E_B \qquad \int_{\tilde{B}} \boldsymbol{\chi} = \delta S_B$$

$$d\boldsymbol{\chi} \propto (\mathrm{EOM})\mathbf{v}_\Sigma$$

Linearized metric EOM (tt component)

Simple application of Stokes:

$$0 = \delta S_B - \delta E_B \propto \int_\Sigma (\mathrm{EOM})$$

Since this should be true for all balls of all sizes etc:

$$(\mathrm{EOM})_{tt} = 0$$

EOM $\longleftrightarrow$ First Law

In all Lorentz frames labelled by a 4-vector: u^μ

$$u^\mu u^\nu (\mathrm{EOM})_{\mu\nu} = 0 \quad \Longrightarrow \quad (\mathrm{EOM})_{\mu\nu} = 0$$

Does not work for the z-components of the EOM

Appeal to initial value formulation of gravity on radial slices where these equations are constraint equations. Just need to show they are satisfied at the boundary $(z = 0)$

Then this is preserved under radial evolution.

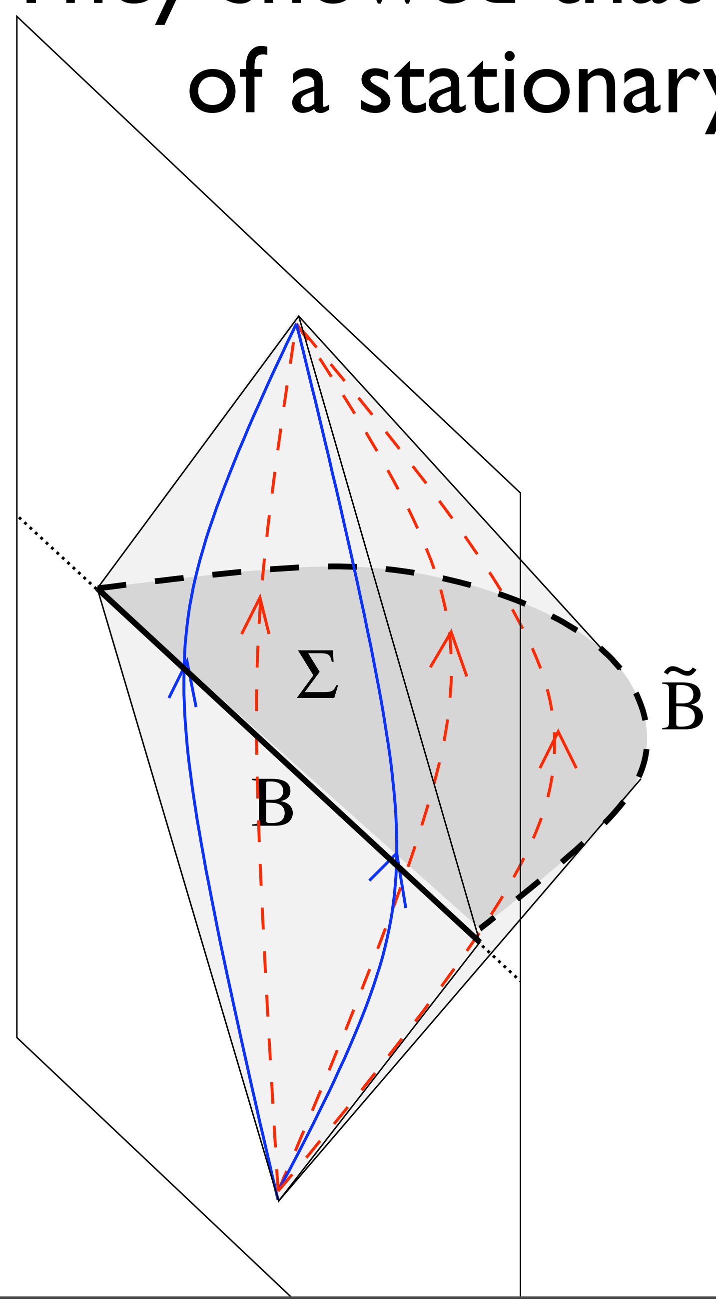
Follows from conservation + tracelessness of $T^{\mu\nu}$ $\Box$

EOM ⟶ First Law

But how did we construct this magical form $\chi(h)$?

Short answer: looked up Iyer & Wald `94

They showed that all **on-shell** linearized perturbations of a stationary black hole with a killing horizon satisfy a first law.



The region of interest to us can be thought of as a Rindler wedge with a killing horizon:

$$\xi_B = -\frac{2}{R}t[z\partial_z + x^i\partial_i] + \frac{1}{R}[R^2 - z^2 - t^2 - \vec{x}^2]\,\partial_t$$

$$\xi_B|_{z=0} \to \zeta_B$$

Killing energy = modular energy

EOM $\longleftrightarrow$ First Law

Iyer & Wald `94

Constructed a closed (D-2) form $\boldsymbol{\chi}(h, \xi_B)$
for on-shell perturbations.

We generalized to

$$d\boldsymbol{\chi} \propto (\mathrm{EOM})\mathbf{v}_\Sigma$$

Bonus: TF, Guica, Hartman, Myers, Van Raamsdonk `13

Their construction applies to arbitrary theories of higher derivative gravity. Extend our proof:

RT (area law) $\longrightarrow$ S_{wald}

Einstein's Equations $\longrightarrow$ Equations of Higher derivative gravity

EOM ⟷ First Law

Iyer & Wald `94

Constructed a closed (D-2) form $\chi(h, \xi_B)$ for on-shell perturbations.

(Covariant phase space formalism)

$$\chi = \delta Q[\xi_B] - \xi_B \cdot \Theta(h)$$

Noether Charge

Symplectic potential

We generalized to

$$d\chi \propto (\mathrm{EOM})\mathbf{v}_\Sigma$$

TF, Guica, Hartman, Myers, Van Raamsdonk `13

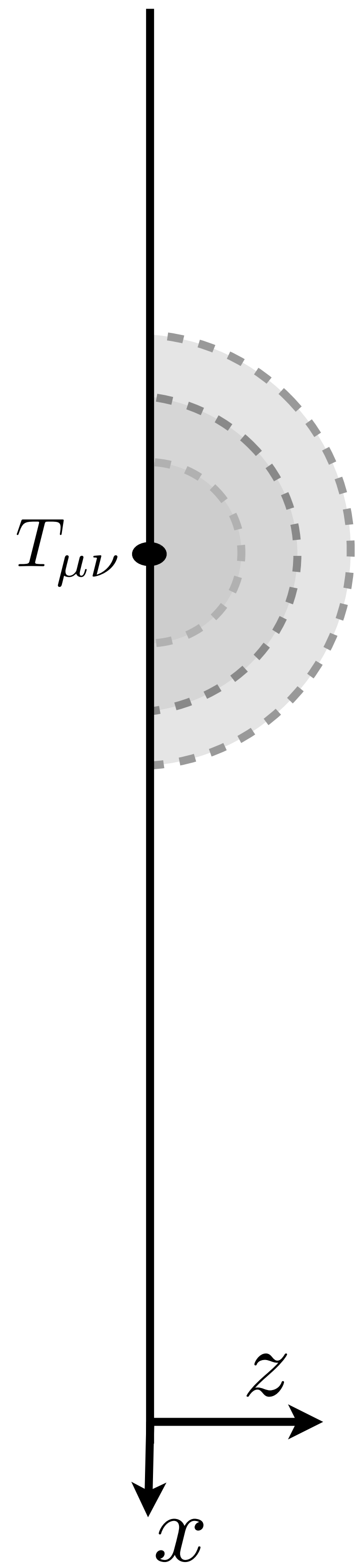
Bonus:

Their construction applies to arbitrary theories of higher derivative gravity. Extend our proof:

RT (area law) ⟶ S_{wald}

Einstein's Equations ⟶ Equations of Higher derivative gravity

Holographic Dictionary from First Law



Remedy a gap in the proof:

Take size of ball to be vanishingly small:

$$\lim_{R\to 0} R^{-d}\delta E_B = \frac{2\pi\Omega_{d-2}}{d^2-1}\,\delta\langle T_{tt}(x_0)\rangle$$
$$\delta E_B = \delta S_B = \delta S_{Wald}$$

First law along with the Wald functional allows us to read off the stress tensor from the asymptotic metric!

Allows us to derive the full Fefferman-Graham expansion.

Conclusions

- First Law for Entanglement Entropy:
  - Non-local constraint on dual spacetime
  - Equivalent to linearized metric EOM
  - Also gives us the holographic dictionary
- Further work:
  - Non-linear equations?
  - More precise relationship to Jacobson?