Symmetry constraints on counterterms in $N = 8$ supergravity

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Based on

arXiv:1009.1643 w/ Niklas Beisert, Dan Freedman, Michael Kiermaier, Alejandro Morales, Stephan Stieberger

arXiv:1007.4813 w/ Michael Kiermaier

arXiv:1003.5018 w/ Dan Freedman, Michael Kiermaier

Is $\mathcal{N} = 8$ supergravity UV finite in 4d?

Perturbative structure of $\mathcal{N} = 8$ supergravity in 4d

L-loop divergence $\leftrightarrow$ counterterm of mass dimension $(2L + 2)$

for example: R^4 at 3-loop order

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Candidate counterterms are

- local operators
- $\mathcal{N} = 8$ SUSY
- $SU(8)_R$-invariant
- $E_{7(7)}$-compatible

Chart of potential counterterms

Pure supergravity finite at 1- and 2-loop order.

Purely gravitational operators are contractions of Riemann tensors $R_{\mu\nu\rho\sigma}$ and covariant derivatives D_μ. Here's the chart:

L	$n=4$	5	6			
3	R^4					
4	D^2R^4	R^5				
5	D^4R^4	D^2R^5	R^6			
6	D^6R^4	D^4R^5	D^2R^6	R^7		
7	D^8R^4	D^6R^5	D^4R^6	D^2R^7	R^8	
8	$D^{10}R^4$	D^8R^5	D^6R^6	D^4R^7	D^2R^8	R^9

Must require $\mathcal{N}=8$ SUSY and $SU(8)$.

Analysis of potential counterterms

Instead of studying the operators, we analyze their matrix elements:

operator	$\leftrightarrow$	*matrix elements*
local	$\leftrightarrow$	polynomial in momenta and polarizations
	$\leftrightarrow$	polynomial in $\langle ij \rangle$ and $[ij]$.
L-loop	$\leftrightarrow$	$\langle ij \rangle$, $[ij]$ polynomial has degree $2L+2$.
$\mathcal{N}=8$ SUSY	$\leftrightarrow$	SUSY Ward identities.
$SU(8)$-invariant	$\leftrightarrow$	$SU(8)$ Ward identities.
$E_{7(7)}$-compatible	$\leftrightarrow$	low-energy theorems

no such matrix elements $\leftrightarrow$ *no such operator* $\leftrightarrow$ *no such counterterm.*

If matrix elements do exist: determine multiplicities of such operators.

Outline

1. PART 1: $\mathcal{N} = 8$ SUSY and $SU(8)$.
2. PART 2: $E_{7(7)}$ constraints.
3. THE END: "Landscape" of candidate counterterms.

Tool kit

- **"Little group scaling":**

For each external state $i = 1, \ldots, n$,

$$|i\rangle \to t_i|i\rangle \text{ and } |i] \to t_i^{-1}|i], \quad \Longrightarrow \quad A_n \to t_i^{-2h_i} A_n$$

where h_i is the helicity.

- **Dimensional analysis:**

Each $\langle ij\rangle$ and $[ij]$ has mass dimension 1.

- $\mathcal{N} = 4, 8$ **SUSY Ward identities:**

MHV: $\langle + + - - + + \ldots\rangle = \frac{\langle 34\rangle^{\mathcal{N}}}{\langle 12\rangle^{\mathcal{N}}}\langle - - + + + + \ldots\rangle$.

Example: 4-gluon MHV amplitude

$$A_n(1^-2^-3^+4^+ \ldots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle \cdots \langle n1\rangle}$$

has mass dim. $4 - n$.

Example of how we exclude operators as candidate counterterms.

4-loops: R^5 (mass dim. $2L + 2 = 10$)

10 derivatives in R^5 $\longrightarrow$ leading 5-point interaction has 10 power of momentum

$\longrightarrow$ 5-pt matrix element has mass dim. 10

and is polynomial of degree 10 in $\langle .. \rangle$'s and $[..]$'s.

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Little grp scaling $\longrightarrow$ $\langle 1^- 2^- 3^+ 4^+ 5^+ \rangle_{R^5}$ contains $\left\{ \begin{array}{l} |1\rangle^4, |2\rangle^4 \\ |3]^4, |4]^4, |5]^4 \end{array} \right.$

unique: $\langle 1^- 2^- 3^+ 4^+ 5^+ \rangle_{R^5} = \langle 12 \rangle^4 [34]^2 [45]^2 [53]^2$

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SUSY Ward Id.s $\longrightarrow$ $\langle 1^+ 2^+ 3^- 4^- 5^+ \rangle_{R^5} = \frac{\langle 34\rangle^8}{\langle 12\rangle^8} \langle 1^- 2^- 3^+ 4^+ 5^+ \rangle_{R^5}$ i.e.

$$\langle 34\rangle^4 [12]^2 [25]^2 [51]^2 = \frac{\langle 34\rangle^8}{\langle 12\rangle^8} \langle 12\rangle^4 [34]^2 [45]^2 [53]^2$$

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local = non-local conflict

$\Longrightarrow$ No $\mathcal{N} = 8$ SUSY matrix elements. So R^5 is not indep. supersymmetrizable.

Analysis

Carry out an analysis of matrix elements at MHV and NMHV level.
[HE, Freedman, Kiermaier, 1003.5018]

- Use superamplitudes.
- Use solution to SUSY Ward identities.
  [HE, Freedman, Kiermaier, 0911.3169]
- Use Gröbner basis.
  [Beisert, HE, Freedman, Kiermaier, Morales, Stieberger, 1009.1643]

Chart of potential counterterms

The matrix elements of a prospective counterterm must respect $\mathcal{N} = 8$ SUSY and $SU(8)$ Ward identities.

If *no*: excluded. If *yes*: we find multiplicities of such operators.

Explicit 4-pt calc. shows finite (→ L = 3, 4)

L	$n=4$	5	6			
3	R^4	None →				
4	~~D^2R^4~~	~~R^5~~	None →			
5	D^4R^4	~~D^2R^5~~	~~R^6~~	None →		
6	D^6R^4	~~D^4R^5~~	~~D^2R^6~~	~~R^7~~	None →	
7	D^8R^4	~~D^6R^5~~	D^4R^6	~~D^2R^7~~	R^8	
8	$D^{10}R^4$	D^8R^5	D^6R^6	D^4R^7	D^2R^8	R^9

"None $\longrightarrow$":
we proved no MHV and no NMHV, and conjectured no N^kMHV for $L < 7$ in [HE, Freedman, Kiermaier, 1003.5018] .
Conjecture proven by [Howe, Heslop, Drummond, 1008.4939]

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Let's now include $E_{7(7)}$

L	$n=4$	5	6			
3	R^4	None →				
4	~~D^2R^4~~	~~R^5~~	None →			
5	D^4R^4	~~D^2R^5~~	~~R^6~~	None →		
6	D^6R^4	~~D^4R^5~~	~~D^2R^6~~	~~R^7~~	None →	
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Is R^4 compatible with $E_{7(7)}$?

R^4

To test $E_{7(7)}$ we will need a 6-point matrix element of R^4 with two scalars and four gravitons:

$$\langle \varphi \, \overline{\varphi} + + - - \rangle_{R^4}$$

Very hard to calculate from Feynman diagrams 

We use a trick to extract the 6-point R^4 matrix elements from the closed string theory tree amplitude.

Outline

1. PART 1: $\mathcal{N} = 8$ SUSY and $SU(8)$. ✓
2. PART 2: $E_{7(7)}$ constraints.
   - From open string amplitudes to closed string amplitudes via KLT.
   - String tree amplitudes and their symmetries.
   - R^4 and $E_{7(7)}$.
   - Matching with automorphic function.
   - $E_{7(7)}$ at higher loop order.
3. "Landscape" of candidate counterterms.

KLT relations in string theory

Kawai-Lewellen-Tye (KLT) relations:

$$\big(\text{closed string tree amplitude}\big) = \sum f(s) \, \big(\text{open string tree amplitude}\big)_L \times \big(\text{open string tree amplitude}\big)_R$$

for example for 5-point amplitudes

$$M_5(1,2,3,4,5) = -\frac{\sin(\alpha'\pi s_{12})\sin(\alpha'\pi s_{34})}{\alpha'^2\pi^2} A_5(1,2,3,4,5)\,\widetilde{A}_5(2,1,4,3,5) + (2 \leftrightarrow 3)\,.$$

The decomposition of states is "closed string = L and R movers".

In the following:

- Toroidally compactified Type II superstring theory in $D = 4$.
- Allow ONLY massless external states.

  open string states $\leftrightarrow$ 16 states of $\mathcal{N} = 4$ SYM

  closed string states $\leftrightarrow$ 256 states of $\mathcal{N} = 8$ supergravity

$\mathcal{N} = 4$ SYM

$2^4 = 16$ massless states

state	helicity	
1 gluon	$+1$	g^+
4 gluinos	$+\frac{1}{2}$	λ^a
6 scalars	0	z^{ab}
4 gluinos	$-\frac{1}{2}$	λ^{abc}
1 gluon	-1	$g^{1234} = g^-$

3 pairs of complex scalars are self-conjugate: $\bar{z}_{ab} = \frac{1}{2}\epsilon_{abcd} z^{cd}$.

Global $SU(4)$ R-symmetry: $A_n(z^{12}, g^-, z^{34}, \ldots) = 0$ unless $SU(4)$-singlet.

$\mathcal{N} = 8$ supergravity

$2^8 = 256$ massless states

state	helicity	
1 graviton	$+2$	h^+
$\vdots$		
70 scalars	0	φ^{abcd}
$\vdots$		
1 graviton	-2	$h^- = h^{12345678}$ $(a, b, \ldots = 1, \ldots, 8)$

35 pairs of complex scalars are self-conjugate: $\overline{\varphi}_{abcd} = \frac{1}{4!}\epsilon_{abcdefgh}\varphi^{efgh}$.

$\mathcal{N} = 8$ supergravity has global $SU(8)$ R-symmetry:
$M_n^{\text{SUGRA}}(v^{12}, \varphi^{1245}, \ldots) = 0$ unless $SU(8)$-singlet.

$[\mathcal{N}=8 \text{ supergravity}] = [\mathcal{N}=4 \text{ SYM}]^2$

All 2^8 $\mathcal{N}=8$ states decompose into $2^4 \times 2^4$ $\mathcal{N}=4$ SYM states.

For example, gravitons = gluon^2: $h^\pm = g^\pm \otimes g^\pm$

Where the 35 pairs of complex **scalars** come from:

Decompose $SU(8) \rightarrow SU(4) \times SU(4)$ as $\{1,\ldots,8\} \rightarrow \{1,2,3,4\} \otimes \{5,6,7,8\}$

1) 1 pair is $SU(4) \times SU(4)$-singlet

$$\varphi \equiv \varphi^{1234} = g^{1234} \otimes g^+ = g^- \otimes g^+$$
$$\overline{\varphi} \equiv \varphi^{5678} = g^+ \otimes g^{5678} = g^+ \otimes g^- .$$

2) 16 pairs $\bar{\mathbf{4}} \otimes \mathbf{4}$: $\varphi_f = \lambda^- \otimes \lambda^+$ ex. $\varphi^{123|5}$

3) 18 pairs $\mathbf{6} \otimes \mathbf{6}$: $\varphi_s = z \otimes z$ ex. $\varphi^{12|56}$

back to KLT

KLT relations, e.g. with $h^{\pm} = g^{\pm} \otimes g^{\pm}$

$$M_5(1^-,2^-,3^+,4^+,5^+) = -\frac{\sin(\alpha'\pi s_{12})\sin(\alpha'\pi s_{34})}{\alpha'^2\pi^2} A_5(1^-,2^-,3^+,4^+,5^+)\,\widetilde{A}_5(2^-,1^-,4^+,3^+,5^+) + (2 \leftrightarrow 3)\,.$$

- KLT makes $SU(4) \times SU(4)$ a manifest global symmetry of the $D = 4$ closed string tree amplitudes M_n with massless external states.
- But closed string theory has no global continuous symmetries!
- $SU(4) \times SU(4) \subset$ T-duality group $SO(6,6)$.
  Global symmetry only in this sector, only at tree level.
- Classification needs two integers k and $\tilde{k}$: $\mathrm{N}^{(k,\tilde{k})}$MHV.

Example of $SU(8)$-violating amplitude

$M_5(1^- \, 2^- \, 3^+ \, 4^+ \, \varphi_5^{1234})$ classification $\mathrm{N}^{(1,0)}\mathrm{MHV} \equiv$ "$\sqrt{\mathrm{N}}$MHV"

$$= \; -\frac{\sin(\alpha' \pi s_{12}) \sin(\alpha' \pi s_{34})}{\alpha'^2 \pi^2} \; A_5(1^- \, 2^- \, 3^+ \, 4^+ \, \mathbf{5}^-) \, \widetilde{A}_5(2^- \, 1^- \, 4^+ \, 3^+ \, 5^+) \; + \; (2 \leftrightarrow 3)$$

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$$= \; \alpha'^3\; 6\,\zeta(3)\,\langle 12\rangle^4\,[34]^4 + O(\alpha'^5)\,.$$

This amplitude violates $SU(8)$!! but vanishes for $\alpha' = 0$ as required by $SU(8)$ in supergravity.

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Let's try to understand this better:

Note the α'^3 matrix element has no poles

. . . comes from a local operator with 8 derivatives

. . . candidate: $\varphi\, R^4$

Closed string effective action

The first operator in the closed string effective action is (in Einstein frame)

$$\alpha'^3 \sqrt{-g}\, e^{-6\phi} R^4 = \alpha'^3 \sqrt{-g}\,(1 - 6\phi + \dots) R^4 \,,$$

where ϕ is the dilaton.

Its 4- and 5-point matrix elements are

$$\begin{aligned} \big\langle 1^- \, 2^- \, 3^+ \, 4^+ \big\rangle_{e^{-6\phi}R^4} &= -\,\alpha'^3 \, 2\,\zeta(3)\, \langle 12\rangle^4 \, [34]^4 \,, \\ \big\langle 1^- \, 2^- \, 3^+ \, 4^+ \, \phi \big\rangle_{e^{-6\phi}R^4} &= \alpha'^3 \, 12\,\zeta(3)\, \langle 12\rangle^4 \, [34]^4 \,. \end{aligned}$$

Closed string effective action $\alpha'^3 \sqrt{-g}\, e^{-6\phi} R^4$

How to identity the dilaton among the 70 scalars of the $\mathcal{N} = 8$ spectrum?

It is $SU(4) \times SU(4)$-invariant and respects L/R exchange:

Recall: 1 pair is $SU(4) \times SU(4)$-singlet

$$\varphi \equiv \varphi^{1234} = g^- \otimes g^+ , \qquad \overline{\varphi} \equiv \varphi^{5678} = g^+ \otimes g^- .$$

This identifies: $\phi = \frac{1}{2}(\varphi^{1234} + \varphi^{5678})$.

Then $\langle 1^-\, 2^-\, 3^+\, 4^+\, \phi \rangle_{e^{-6\phi}R^4} = M_5(1^-\, 2^-\, 3^+\, 4^+\, \varphi^{1234})\big|_{\alpha'^3} + M_5(1^-\, 2^-\, 3^+\, 4^+\, \varphi^{5678})\big|_{\alpha'^3}.$

$$12\,\zeta(3)\,\langle 12\rangle^4\,[34]^4 \quad = \quad 6\,\zeta(3)\,\langle 12\rangle^4\,[34]^4 \quad + \quad 6\,\zeta(3)\,\langle 12\rangle^4\,[34]^4$$

So the dilaton is 'responsible' for the $SU(8)$-violation.

Lessons (so far)

- The α'^3-correction to the closed string tree amplitude are encoded in the supersymmetrization of

$$\alpha'^3 \sqrt{-g}\, e^{-6\phi} R^4$$

This preserves only $SU(4) \times SU(4)$.

- The α'-corrections explicitly break $SU(8) \rightarrow SU(4) \times SU(4)$ because the dilaton singles out a special "direction" in $SU(8)$.

- We cannot use the closed string tree amplitude directly to explore the 3-loop R^4 candidate counterterm of $\mathcal{N} = 8$ supergravity, because it has to be an $SU(8)$-invariant supersymmetrization.

Symmetries

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  Low-energy theorems:

  In $\mathcal{N} = 8$ supergravity, single soft scalar limits vanish,

$$M_n(\varphi(p), \dots) \to 0 \quad \text{as} \quad p \to 0.$$

  [Bianchi, HE, Freedman '0805; Arkani-Hamed, Cachazo, Kaplan '0808; Kallosh, Kugo '0811]

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- Counterterms:

  $E_{7(7)}$ compatible? Test if the single soft scalar limits of their matrix elements vanish.

  Specifically, for R^4 we would like to calculate

  $$\lim_{p_1 \to 0} \left\langle \varphi\ \overline{\varphi}\ 3^-\ 4^-\ 5^+\ 6^+ \right\rangle_{R^4} = ?$$

  to test if it vanishes or not. Earlier work w/ $e^{-6\phi} R^4$ [Brödel & Dixon, 2009]

  Single soft limits of the MHV 4-, 5- and 6-pt matrix elements trivially vanish

From $e^{-6\phi}R^4$ to R^4

How to obtain R^4 matrix elements from α'^3 of the string amplitude:

'Average' the α'^3 contributions of the string amplitude over $SU(8)$

$\Longrightarrow$

'Average' the matrix elements of $e^{-6\phi}R^4$ over $SU(8)$

$\Longrightarrow$

matrix elements of an $SU(8)$-invariant supersymmetric 8-derivative operator.

There is only ONE such operator, namely the desired R^4.

[Freedman, Kiermaier, H.E. (March 2010)]

Average of $SU(8)$

Product of two scalars ϕ^{abcd} contains one singlet: $(\varphi\,\overline{\varphi})_{\text{sing}} = \frac{1}{8!}\,\epsilon_{abcdefgh}\,\varphi^{abcd}\,\varphi^{efgh}$.

Thanks to $SU(4)\times SU(4)$, we get

$$\langle \varphi\,\overline{\varphi} + + - - \rangle_{R^4} \;=\; \frac{1}{35}\langle \varphi^{1234}\varphi^{5678} + + - - \rangle_{e^{-6\phi}R^4} - \frac{16}{35}\langle \varphi^{123|5}\varphi^{4|678} + + - - \rangle_{e^{-6\phi}R^4} + \frac{18}{35}\langle \varphi^{12|56}\varphi^{34|78} + + - - \rangle_{e^{-6\phi}R^4}\,.$$

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We calculate these 3 matrix elements from the α'-expansion of the closed string NMHV amplitudes, obtained via KLT

(α'-expansion of open string amplitude from Stieberger & Taylor)

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$$\lim_{p_1\to 0}\,\langle \varphi^{1234}\varphi^{5678} + + - - \rangle_{e^{-6\phi}R^4} \;=\; -12\,\zeta(3)\,\times [34]^4\langle 56\rangle^4,$$

$$\lim_{p_1\to 0}\,\langle \varphi^{123|5}\varphi^{4|678} + + - - \rangle_{e^{-6\phi}R^4} \;=\; -6\,\zeta(3)\,\times [34]^4\langle 56\rangle^4,$$

$$\lim_{p_1\to 0}\,\langle \varphi^{12|56}\varphi^{34|78} + + - - \rangle_{e^{-6\phi}R^4} \;=\; 0\,.$$

hence

$$\boxed{\lim_{p_1\to 0}\,\langle \varphi\,\overline{\varphi} + + - - \rangle_{R^4} \;=\; 2\zeta(3)\,\frac{6}{5}\,[34]^4\langle 56\rangle^4 \;\neq\; 0\,.}$$

Conclusion: the unique $SU(8)$-invariant supersymmetrization of R^4 is NOT $E_{7(7)}$-compatible.

Chart of potential counterterms

Candidate counterterm operators must be $\mathcal{N}=8$ SUSY and $SU(8)$-invariant and have $E_{7(7)}$ symmetry.

L	$n=4$	5	6			
3	~~R^4~~ $E_{7(7)}$	None $\longrightarrow$				
4	~~D^2R^4~~	~~R^5~~	None $\longrightarrow$			
5	D^4R^4	~~D^2R^5~~	~~R^6~~	None $\longrightarrow$		
6	D^6R^4	~~D^4R^5~~	~~D^2R^6~~	~~R^7~~	None $\longrightarrow$	
7	D^8R^4	~~D^6R^5~~	D^4R^6	~~D^2R^7~~	R^8	
8	$D^{10}R^4$	D^8R^5	D^6R^6	D^4R^7	D^2R^8	R^9

Observation 1

$(\star)$ Why $\lim_{p_1 \to 0} \langle \varphi^{12|56} \varphi^{34|78} ++--\rangle_{e^{-6\phi} R^4} = 0$?

- $\mathcal{N} = 8$ supergravity:
  Global $E_{7(7)}$ symmetry spontaneously broken to $SU(8)$.
  The $133 - 63 = 70$ scalars are the Goldstone bosons, which transform in the **70**.

- For $\alpha' > 0$:
  Global $SO(6,6)$ spontaneously broken to $SU(4) \times SU(4)$.
  There are $66 - 30 = 36$ Goldstone bosons. They transform in the $\mathbf{6} \otimes \mathbf{6}$.

- These are type 3) of list we constructed early in the talk:

  3) $\varphi_s = z \otimes z$ ex. $\varphi^{12|56}$

- Eq. $(\star)$ holds to all orders in α'. have checked explicit up to and incl. α'^7.

Observation 2: Duality and supersymmetry

Green, Miller, Russo, and Vanhove (GMRV) have shown that duality and supersymmetry requires the SUSY operator R^4 to have a non-linear completion of the form $f_{R^4} R^4$, where f_{R^4} is a moduli-dependent automorphic function which satisfies

$$\Delta f_{R^4} = -42 f_{R^4} \quad \text{for} \quad D = 4$$

Here Δ is the Laplacian on $E_{7(7)}/SU(8)$.

Compare:

Let's compare GMRV to our result:

$$\lim_{p_1\to 0} \langle \varphi\,\overline{\varphi} + + - - \rangle_{R^4} \;=\; 2\zeta(3)\frac{6}{5}\,[34]^4\langle 56\rangle^4 \;\neq\; 0\,.$$

Must come from local operator $(\varphi\overline{\varphi})_{\rm sing}R^4$, so that must be part of the non-linear completion of R^4, i.e. $f_{R^4}R^4$ with

$$f_{R^4} \propto -2\zeta(3)\Big[1 - \frac{6}{5}\big(\varphi^{1234}\varphi^{5678} + 34 \text{ others}\big) + \ldots\Big]$$

The Laplacian on $E_{7(7)}/SU(8)$ is

$$\Delta = \Big(\frac{\partial}{\partial\varphi^{1234}}\frac{\partial}{\partial\varphi^{5678}} + 34 \text{ inequivalent perms}\Big) \;+\; \ldots$$

Indeed we find

$$\Delta\, f_{R^4} + 42\, f_{R^4} \;=\; -2\zeta(3)\Big(-\frac{6}{5}\times 35 + 42\Big) + O(\varphi\overline{\varphi}) \;=\; 0 + O(\varphi\overline{\varphi})$$

so our result matches GMRV!

$\mathcal{N} = 8$ supergravity

The R^4 operator in $D = 4$:

- $\mathcal{N} = 8$ SUSY and $SU(8)$ invariant.
- NOT $E_{7(7)}$ invariant.
- Explains why R^4 is not a candidate counterterm...
- ...and why the 3-loop 4-point amplitude is finite.

[Bern, Carrasco, Dixon, Johansson, Kosower, Roiban '07]

L	$n=4$	5	6			
3	~~R^4~~ $E_{7(7)}$	None $\longrightarrow$				
4	~~D^2R^4~~	~~R^5~~	None $\longrightarrow$			
5	D^4R^4	~~D^2R^5~~	~~R^6~~	None $\longrightarrow$		
6	D^6R^4	~~D^4R^5~~	~~D^2R^6~~	~~R^7~~	None $\longrightarrow$	
7	D^8R^4	~~D^6R^5~~	D^4R^6	~~D^2R^7~~	R^8	
8	$D^{10}R^4$	D^8R^5	D^6R^6	D^4R^7	D^2R^8	R^9

Next up: $D^4 R^4$

Closed string effective action

$$S_{\text{eff}} = S_{\text{SG}} - 2\,\alpha'^3 \zeta(3)\, e^{-6\phi} R^4 - \alpha'^5\,\zeta(5)\, e^{-10\phi} D^4 R^4 + \tfrac{2}{3}\,\alpha'^6\,\zeta(3)^2\, e^{-12\phi} D^6 R^4 - \tfrac{1}{2}\alpha'^7 \zeta(7)\, e^{-14\phi} D^8 R^4 + \ldots .$$

$SU(8)$ average procedure gives unique $D^4 R^4$ matrix elements from α'^5 of closed string amplitude.

- NOT $E_{7(7)}$ invariant.
- Single soft limit shows SUSY operator is $f_{D^4 R^4} D^4 R^4$ with $f_{D^4 R^4} \propto -\zeta(5)\Big[1 - \tfrac{6}{7}\big(\varphi^{1234}\varphi^{5678} + 34 \text{ others}\big) + \ldots\Big]$
- Satisfies Green et al's $\Delta\, f_{D^4 R^4} = -60\, f_{D^4 R^4}$
- Conclude: $D^4 R^4$ is not a candidate counterterm.
- $\mathcal{N} = 8$ SG finite at 5-loops.

Next up: D^4R^4 and D^6R^4

Closed string effective action

$$S_{\rm eff} = S_{\rm SG} - 2\,\alpha'^3\zeta(3)\,e^{-6\phi}R^4 - \alpha'^5\,\zeta(5)\,e^{-10\phi}D^4R^4 + \tfrac{2}{3}\,\alpha'^6\,\zeta(3)^2\,e^{-12\phi}D^6R^4 - \tfrac{1}{2}\alpha'^7\zeta(7)\,e^{-14\phi}D^8R^4 + \ldots .$$

Matrix elements from α'^6 of closed string amplitude are polluted by pole terms R^4—R^4 from $\alpha'^3 \times \alpha'^3$.

- We calculate fully $\mathcal{N}=8$ SUSY'ize R^4—R^4.
- Extract $\langle \varphi\,\overline{\varphi}++--\rangle_{R^4-R^4}$ and subtract it from $\langle \varphi\,\overline{\varphi}++--\rangle_{e^{-12\phi}D^6R^4}$.
- $SU(8)$ average then gives $\langle \varphi\,\overline{\varphi}++--\rangle_{D^6R^4}$, which has non-vanishing single soft scalar limit.
- Satisfies Green et al's $\Delta f_{D^6R^4} = -60\, f_{D^6R^4} - (f_{R^4})^2$.

  The inhom. term is from R^4—R^4.
- NOT $E_{7(7)}$ invariant.
- Conclude: D^6R^4 is not a candidate counterterm.
- $\mathcal{N}=8$ SG finite at 6-loops.

$\mathcal{N} = 8$ SUSY and $SU(8)$-invariant candidate counterterm operators.

L	$n = 4$	5	6			
3	~~R^4~~ ($E_{7(7)}$)	None →				
4	~~$D^2 R^4$~~	~~R^5~~	None →			
5	~~$D^4 R^4$~~ ($E_{7(7)}$)	~~$D^2 R^5$~~	~~R^6~~	None →		
6	~~$D^6 R^4$~~ ($E_{7(7)}$)	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	None →	
7	$D^8 R^4$	~~$D^6 R^5$~~	$D^4 R^6$	~~$D^2 R^7$~~	R^8	
8	$D^{10} R^4$	$D^8 R^5$	$D^6 R^6$	$D^4 R^7$	$D^2 R^8$	R^9

What do we know about $L \geq 7$ loops?

$\mathcal{N} = 8$ SUSY and $SU(8)$-singlet candidate counterterm operators and $SU(8)$ **70** operators for their single soft scalar limits.

7-loop	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt	11-pt	12-pt	13-pt	14-pt	15-pt	16-pt
singlet	D^8R^4 $1\times$MHV	~~D^6R^5~~	D^4R^6 $2\times$NMHV	~~D^2R^7~~	R^8 $3\times N^2$MHV	~~$\varphi^2D^2R^7$~~	φ^2R^8 $4\times N^3$MHV	~~$\varphi^4D^2R^7$~~	φ^4R^8 $6\times N^4$MHV	~~$\varphi^6D^2R^7$~~	φ^6R^8 $8\times N^5$MHV	~~$\varphi^8D^2R^7$~~	φ^8R^8 $10\times N^6$MHV
		↙ soft		↙ soft		↙ soft		↙ soft		↙ soft		↙ soft	
70		φD^8R^4 $2\times$		φD^4R^6 $4\times$		φR^8 $6\times$		φ^3R^8 $9\times$		φ^5R^8 $14\times$		φ^7R^8 $19\times$	

8-loop	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt	11-pt	12-pt	13-pt	14-pt
singlet	$D^{10}R^4$ $1\times$MHV	D^8R^5 $1\times$MHV	D^6R^6 $3\times$NMHV	D^4R^7 $3\times$NMHV	D^2R^8 $8\times N^2$MHV	R^9 $8\times N^2$MHV	$\varphi^2D^2R^8$ $25\times N^3$MHV	φ^2R^9 $22\times N^3$MHV	$\varphi^4D^2R^8$ $66\times N^4$MHV	φ^4R^9 $51\times N^4$MHV	$\varphi^6D^2R^8$ $153\times N^5$MHV
		↙	↙	↙	↙	↙	↙	↙	↙	↙	
70		$\varphi D^{10}R^4$ $3\times$	φD^8R^5 $4\times$	φD^6R^6 $17\times$	φD^4R^7 $16\times$	φD^2R^8 $81\times$	φR^9 $63\times$	$\varphi^3D^2R^8$ $232\times$	φ^3R^9 $211\times$	$\varphi^5D^2R^8$ $1033\times$	

Multiplicities found using $SU(2,2|8)$.

[Beisert, HE, Freedman, Kiermaier, Morales, Stieberger, 1009.1643]

For $n > 4$ none of the $L = 7$ operators respect $E_{7(7)}$ compatible.
This means that the 4-graviton amplitude determines whether theory finite or not at $L = 7$.

Standings

SUSY, $SU(8)$, $E_{7(7)}$ $\Longrightarrow$ $\mathcal{N} = 8$ supergravity in 4d finite up to 7-loop order.

First divergence at $L = 7$?

Candidate full superspace integral — but does is vanish?

First divergence at $L = 8$?

Candidate full superspace integral available [Kallosh (1981), Howe & Lindstrom (1981)]