Generating tree amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SG

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- arXiv:0808.1720 w/ Michael Kiermaier and Dan Freedman
- arXiv:0805.0757 w/ Massimo Bianchi and Dan Freedman
- arXiv:0710.1270 w/ Dan Freedman

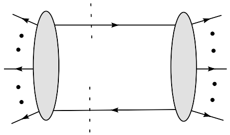
1. Motivation

Is $\mathcal{N} = 8$ supergravity perturbatively finite?

Explicit calculations of loop amplitudes:

Use generalized unitarity cuts [Bern, Dixon, Kosower, ...]
to construct loop amplitudes from products of on-shell tree amplitudes.

Example:



$\rightarrow \sum_{\text{intermediate states}} A^{\text{tree}}_{n_1} \times A^{\text{tree}}_{n_2}$

Our work focuses on developing efficient calculational methods for explicit construction of *any* on-shell n-point *tree* amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory and $\mathcal{N} = 8$ supergravity.

$\rightarrow$ Generating functions.

Applications to intermediate state sums in unitarity cuts.

How to calculate on-shell tree level scattering amplitudes

- Feynman rules $\longleftarrow$ very many, very complicated diagrams
- On-shell recursion relations $\longleftarrow$ very useful

  Get n-point amplitudes from k-point amplitudes with $k < n$.
- Generating functions $\longleftarrow$ very efficient

  Idea: all n-point tree amplitudes of $\mathcal{N} = 4$ SYM encoded in a set of simple Grassmann functions Z_n^{MHV}, Z_n^{NMHV}, . . . , $Z_n^{\overline{\mathrm{MHV}}}$:

  $$A_n(X_1, X_2, ..., X_n) = D_{X_1} D_{X_2} \cdots D_{X_n} Z_n$$

  with differential operators D_{X_i} in 1-1 correspondence with the states X_i.

  Advantage: obtain amplitude directly without having to first compute set of lower-point amplitudes.

MHV sector and beyond

SUSY $\implies$ helicity violating n-gluon amplitudes vanish:

$$A_n(+,+,\dots,+) = A_n(-,+,\dots,+) = 0.$$

- The *simplest* amplitudes are MHV (maximally helicity violating)
  $\rightarrow$ n-gluon amplitude $A_n(-,-,+,\dots,+)$
  MHV sector: amplitudes related to A_n via SUSY Ward identities.

- The *next-to-simplest* amplitudes are Next-to-MHV
  $\rightarrow$ n-gluon amplitude $A_n(-,-,-,+,\dots,+)$
  NMHV sector: SUSY related (but much harder to solve SUSY Ward identities).

...

Properties of the generating function

$\longrightarrow$ Generating functions developed for MHV, NMHV amplitudes + for anti-MHV and anti-NMHV.

$\longrightarrow$ Precise characterization of MHV and NMHV sectors, e.g. $A_6(\lambda_+ \, \lambda_+ \, \lambda_+ \, \lambda_+ \, \phi \, \phi)$ is MHV in $\mathcal{N} = 4$ SYM.

$\longrightarrow$ Counts distinct processes in each sector:

	MHV	NMHV
$\mathcal{N} = 4$:	15	34
$\mathcal{N} = 8$:	186	919

counting $\leftrightarrow$ partitions of integers!

$\longrightarrow$ Simple relationship $Z_n^{\mathcal{N}=8} \propto Z_n^{\mathcal{N}=4} \times Z_n^{\mathcal{N}=4}$ (MHV) clarifies SUSY and global symmetries in map $[\mathcal{N} = 8] = [\mathcal{N} = 4]_L \otimes [\mathcal{N} = 4]_R$ of states and KLT relations $M_n = \sum (k_n \, A_n \, A'_n)$.

$\longrightarrow$ Evaluation of state sums in unitarity cuts of loop amplitudes.

Outline

1. Motivation
2. MHV generating functions in $\mathcal{N} = 4$ SYM
3. Intermediate State Spin Sums
4. Recursion relations $\leftrightarrow$ MHV vertex expansion
5. Next-to-MHV generating functions in $\mathcal{N} = 4$ SYM
6. From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ SG
7. Outlook

Notation

I will use *spinor helicity* formalism:

- If momentum p_μ null, i.e. $p^2 = 0$, then

$$p_{\alpha\dot{\beta}} \;=\; p_\mu(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \;=\; |p\rangle^{\dot{\alpha}}\,[p|^{\beta}$$

  with bra and kets being 2-component commuting spinors which are solutions to the massless Dirac eqn, $p_{\alpha\dot{\beta}}|p\rangle^{\dot{\beta}} = 0$.

- Spinor products $\langle 12\rangle \equiv \langle p_1|_{\dot{\alpha}}\,|p_2\rangle^{\dot{\alpha}}$ and $[12] = [p_1|^{\alpha}|p_2]_{\alpha}$ are just $\sqrt{|s_{12}|} = \sqrt{|2p_1 \cdot p_2|}$ up to a complex phase.

- Note $[i\,j] = -[j\,i]$ and $\langle i\,j\rangle = -\langle j\,i\rangle$.

2. MHV generating function — $\mathcal{N} = 4$ SYM

$$\begin{array}{ccc} \text{States } X_i & \leftrightarrow & \text{differential operators } D_{X_i} \\ \downarrow & & \downarrow \\ \text{Amplitude } A_n(X_1\, X_2 \ldots X_n) & = & D_{X_1}\, D_{X_2} \cdots D_{X_n}\, Z_n \end{array}$$

First need (state $\leftrightarrow$ diff op) correspondence.

$\mathcal{N} = 4$ SYM has 2^4 massless states:

$a, b = 1, 2, 3, 4 \;\in SU(4)$ global sym

1+1 gluons B^-, B_+

4+4 gluini F_a^-, F_+^a

6 self-dual scalars $B^{ab} = \frac{1}{2}\epsilon^{abcd} B_{cd}$

4 supercharges $\tilde{Q}_a = \epsilon_{\dot\alpha} \tilde{Q}_a^{\dot\alpha}$ and $Q^a = \tilde{Q}_a^*$ act on annihilation operators:

$$
\begin{aligned}
\big[\tilde{Q}_a, B_+(p)\big] &= 0\,, \\
\big[\tilde{Q}_a, F_+^b(p)\big] &= \langle \epsilon\, p\rangle\, \delta_a^b\, B_+(p)\,, \\
\big[\tilde{Q}_a, B^{bc}(p)\big] &= \langle \epsilon\, p\rangle\, \big(\delta_a^b\, F_+^c(p) - \delta_a^c\, F_+^b(p)\big)\,, \\
\big[\tilde{Q}_a, B_{bc}(p)\big] &= \langle \epsilon\, p\rangle\, \epsilon_{abcd}\, F_+^d(p)\,, \\
\big[\tilde{Q}_a, F_b^-(p)\big] &= \langle \epsilon\, p\rangle\, B_{ab}(p)\,, \\
\big[\tilde{Q}_a, B^-(p)\big] &= -\langle \epsilon\, p\rangle\, F_a^-(p)
\end{aligned}
$$

(consistent with crossing sym. and self-duality)

$\mathcal{N} = 4$ SYM (state $\leftrightarrow$ diff op) correspondence

Introduce auxiliary Grassman variable η_{ia}

i momentum label p_i, $\quad a = 1, \ldots, 4$ is $SU(4)$ index.

Associate to each state Grassman diff ops $\partial_i^a = \frac{\partial}{\partial \eta_{ia}}$:

$$
\begin{aligned}
B_+(p_i) \quad &\leftrightarrow \quad 1 \\
F_+^a(p_i) \quad &\leftrightarrow \quad \partial_i^a \\
B_+^{ab}(p_i) \quad &\leftrightarrow \quad \partial_i^a \, \partial_i^b \\
F_a^-(p_i) \quad &\leftrightarrow \quad -\tfrac{1}{3!} \, \epsilon_{abcd} \, \partial_i^b \, \partial_i^c \, \partial_i^d \\
B^-(p_i) \quad &\leftrightarrow \quad \partial_i^1 \, \partial_i^2 \, \partial_i^3 \, \partial_i^4
\end{aligned}
$$

This is our (state $\leftrightarrow$ diff op) correspondence.

SUSY generators $\tilde{Q}_a = \sum_{i=1}^n \langle \epsilon i \rangle \, \eta_{ia}$ and $Q^a = \sum_{i=1}^n [i \, \epsilon] \, \frac{\partial}{\partial \eta_{ia}}$ give correct SUSY algebra

$$[Q^a, \tilde{Q}_b] = \delta^a_b \sum_{i=1}^n [\epsilon_1 i] \langle i \epsilon_2 \rangle = \delta^a_b \sum_{i=1}^n \epsilon_1^\alpha \, p_{i\,\alpha\dot{\beta}} \, \tilde{\epsilon}_2^{\dot{\beta}} \rightarrow 0 \quad \text{(mom. cons.)},$$

and

$$[\tilde{Q}, \text{diff op}] = \langle \epsilon p \rangle (\text{diff op})'$$

produces correct algebra on states.

The **MHV generating function** is

$$Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-, 2^-, 3^+, \ldots, n^+)}{\langle 12\rangle^4}\, \delta^{(8)}\Big(\sum_i |i\rangle \eta_{ia}\Big)\,,$$

where $\delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big) = 2^{-4}\prod_{a=1}^4 \sum_{i,j=1}^n \langle i\, j\rangle\, \eta_{ia}\,\eta_{ja}\,.$

[Nair (1988)] [GGK (2004)]

(δ-function of Grassman variables θ_a is $\prod \theta_a$)

η_{ia}	—	auxilliary Grassman variables
$a = 1, 2, 3, 4$	—	$SU(4)$ indices
$i, j = 1, 2, \ldots, n$	—	momentum labels

Claim: any 8th order derivative operator built from (state $\leftrightarrow$ diff op) correspondence gives an MHV amplitude when applied to $Z_n^{\mathcal{N}=4}$:

$$A_n^{\mathrm{MHV}}(X_1, \ldots, X_n) = D_{X_1} \cdots D_{X_n}\, Z_n^{\mathcal{N}=4}\,.$$

Let's prove this!

Proof: $Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-,2^-,3^+,\ldots,n^+)}{\langle 12\rangle^4}\;\delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big)$

- $Z_n^{\mathcal{N}=4}$ reproduces pure MHV gluon amplitude $A_n(1^-, 2^-, 3^+, \ldots, n^+)$ correctly:

$$
\begin{aligned}
&(\partial_1^1\partial_1^2\partial_1^3\partial_1^4)(\partial_2^1\partial_2^2\partial_2^3\partial_2^4)\,\delta^{(8)}\big(\textstyle\sum_i |i\rangle\eta_{ia}\big)\\
&= (\partial_1^1\partial_1^2\partial_1^3\partial_1^4)(\partial_2^1\partial_2^2\partial_2^3\partial_2^4)\,\big(2^{-4}\textstyle\prod_{a=1}^4\sum_{i,j=1}^n\langle i\,j\rangle\,\eta_{ia}\,\eta_{ja}\big)\\
&= \langle 12\rangle^4.
\end{aligned}
$$

Proof: $Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-,2^-,3^+,\dots,n^+)}{\langle 12\rangle^4}\ \delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big)$

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$$= (\partial_1^1\partial_1^2\partial_1^3\partial_1^4)(\partial_2^1\partial_2^2\partial_2^3\partial_2^4)\,\big(2^{-4}\textstyle\prod_{a=1}^4\sum_{i,j=1}^n\langle i\,j\rangle\,\eta_{ia}\,\eta_{ja}\big)$$
$$= \langle 12\rangle^4.$$

- $\tilde{Q}_a\, Z_n^{\mathcal{N}=4} \propto \big(\sum_{i=1}^n |i\rangle\,\eta_{ia}\big)\,\delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big) = 0.$

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- $[\tilde{Q}_a, D^{(9)}]\,Z_n^{\mathcal{N}=4} = 0$

  encode the MHV SUSY Ward identities:

$$0 = [\tilde{Q}_a, D^{(9)}]\,Z_n^{\mathcal{N}=4} = \textstyle\sum_t D_{X_1}\cdots[\tilde{Q}_a, D_{X_t}]\cdots D_{X_n}\,Z_n^{\mathcal{N}=4},$$

$$0 = \langle[\tilde{Q}_a, X_1\dots X_n]\rangle = \textstyle\sum_t\langle X_1\dots[\tilde{Q}_a, X_t]\dots X_n\rangle\,.$$

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- MHV SUSY Ward identities have *unique* solutions.

Proof: $Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-,2^-,3^+,\dots,n^+)}{\langle 12\rangle^4}\;\delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big)$

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$$0 = \langle[\tilde{Q}_a, X_1\dots X_n]\rangle = \textstyle\sum_t\langle X_1\dots[\tilde{Q}_a, X_t]\dots X_n\rangle\,.$$

- MHV SUSY Ward identities have *unique* solutions.

$\Rightarrow$ $Z_n^{\mathcal{N}=4}$ produces all MHV amplitudes correctly.

Characterizing amplitudes in the MHV sector of $\mathcal{N}=4$ SYM:

$$D^{(8)}\, Z_n^{\mathcal{N}=4} = \text{MHV amplitude}$$

hence

$$\#\ \text{MHV amplitudes} \;=\; \#\ \text{partitions of 8 with } n_{\max} = 4.$$

MHV amplitudes:

$$\begin{aligned}
8 \;&=\; 4+4 &&\leftrightarrow\;\; \langle B^- \, B^- \, B_+ \ldots B_+ \rangle \\
&=\; 4+3+1 &&\leftrightarrow\;\; \langle B^- \, F_a^- \, F_+^a \, B_+ \ldots B_+ \rangle \\
&\ldots \\
&=\; 1+\cdots+1 &&\leftrightarrow\;\; \langle F_+^{a_1} \ldots F_+^{a_8} \, B_+ \ldots B_+ \rangle
\end{aligned}$$

Total of 15 MHV amplitudes in $\mathcal{N}=4$ SYM.

Example:

Calculate $\langle B^-(p_1)\, F^1_+(p_2)\, F^2_+(p_3)\, F^3_+(p_4)\, F^4_+(p_5)\, B^+(p_6)\rangle$

$$(\partial^1_1\partial^2_1\partial^3_1\partial^4_1)(\partial^1_2)(\partial^2_3)(\partial^3_4)(\partial^4_5)\ \delta^{(8)}\Big(\sum_i |i\rangle\eta_{ia}\Big)$$

$$= (\partial^1_1\partial^1_2)(\partial^2_1\partial^2_3)(\partial^3_1\partial^3_4)(\partial^4_1\partial^4_5)\ \delta^{(8)}\Big(\sum_i |i\rangle\eta_{ia}\Big)$$

$$= \langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle$$

using $\delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big) = \big(2^{-4}\prod_{a=1}^4 \sum_{i,j=1}^n \langle i\, j\rangle\, \eta_{ia}\, \eta_{ja}\big)$,

so

$$\langle B^-(p_1)\, F^1_+(p_2)\, F^2_+(p_3)\, F^3_+(p_4)\, F^4_+(p_5)\, B^+(p_6)\rangle$$

$$= \frac{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle}{\langle 12\rangle^4} A_n(1^-, 2^-, 3^+, 4^+, 5^+, 6^+).$$

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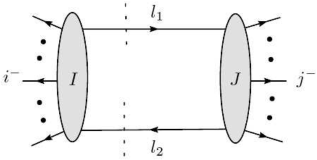
3. Intermediate State Spin Sums
4. Recursion relations $\leftrightarrow$ MHV vertex expansion
5. Next-to-MHV generating functions in $\mathcal{N} = 4$ SYM
6. From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ SG
7. Outlook

Example: One-loop MHV amplitude

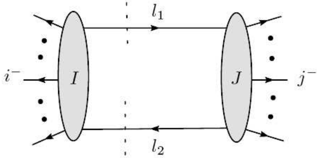


Use MHV generating function to compute intermediate state sum of unitarity cut:

$$D^{(4)}_{l_1}\, D^{(4)}_{l_2} \left[\delta^{(8)}(I)\, \delta^{(8)}(J)\right]$$

D_{l_1} and D_{l_2} distribute themselves between $\delta^{(8)}(I)$ and $\delta^{(8)}(J)$. This automatically takes care of the intermediate state sum.

How to evaluate the spin sum: $D^{(4)}_{l_1}\, D^{(4)}_{l_2} \left[\delta^{(8)}(I_a)\, \delta^{(8)}(J_a)\right]$



$$\begin{aligned} I_a &= |l_1\rangle \eta_{1a} - |l_2\rangle \eta_{2a} + \textstyle\sum_{\text{ext } i} |i\rangle \eta_{ia} \\ J_a &= -|l_1\rangle \eta_{1a} + |l_2\rangle \eta_{2a} + \textstyle\sum_{\text{ext } j} |j\rangle \eta_{ja} \end{aligned}$$

Use δ-function identity $\delta^{(8)}(I_a)\, \delta^{(8)}(J_a) = \delta^{(8)}(I_a + J_a)\, \delta^{(8)}(J_a)$ and note that

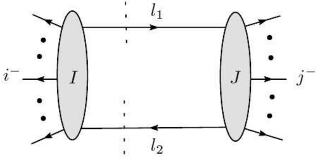
- $\delta^{(8)}(I_a + J_a) = \delta^{(8)}(\text{ext})$ is independent of loop momenta.
- $\delta^{(8)}(J_a) = 2^{-4} \prod_{a=1}^{4} \sum_{j,j' \in J} \langle j j' \rangle \eta_{ja} \eta_{j'a} = \prod_{a=1}^{4} \left(\langle l_1 l_2 \rangle \eta_{1a} \eta_{2a} + \dots \right)$.

So

$$D^{(4)}_{l_1}\, D^{(4)}_{l_2} \left[\delta^{(8)}(I_a)\, \delta^{(8)}(J_a)\right] = \delta^{(8)}(\text{ext})\, D^{(4)}_{l_1}\, D^{(4)}_{l_2}\, \delta^{(8)}(J_a) = \delta^{(8)}(\text{ext})\, \langle l_1 l_2 \rangle^4 .$$

Include prefactors and you have a *generating function* for the cut amplitude!

Example: One-loop MHV amplitude



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D_{l_1} and D_{l_2} distribute themselves between $\delta^{(8)}(I)$ and $\delta^{(8)}(J)$.
This automatically takes care of the intermediate state sum.

Have done 1-, 2-, 3-, and 4-loop state sums involving MHV, NMHV, $\overline{\text{MHV}}$, and $\overline{\text{NMHV}}$ generating functions in $\mathcal{N} = 4$.

Outline

4. Recursion relations $\leftrightarrow$ MHV vertex expansion

- **Recursion relations**: express on-shell n-point amplitude in terms of k-point on-shell sub-amplitudes with $k < n$.

- Even better if sub-amplitudes are MHV
  $\rightarrow$ **MHV vertex expansion**.

For gluons:
[Britto, Cachazo, Feng (2004)] [Britto, Cachazo, Feng, Witten (2005)] [Cachazo, Svrcek, Witten (2004)] [Risager (2005)]

For general $\mathcal{N} = 4$ external state:
[Bianchi, Freedman, HE (May 2008)]
[Freedman, Kiermaier, HE (Aug 2008)]

[Cheung (2008)] $[-, \text{anything}\rangle$-shift OK
[Arkani-Hamed, Cachazo, Kaplan (2008)] new 2-line SUSY shift.
[Brandhuber, Heslop, Travaglini (2008)]
[Drummond, Henn (2008)]

3-line shift recursion relations

- Analytically continue amplitudes to complex values by *shifts* of 3 external momenta:

$$p_i^\mu \to \hat{p}_i^\mu = p_i^\mu + z\, q_i^\mu\,, \qquad \text{for} \quad i = 1, 2, 3.$$

  where

$$q_1^\mu + q_2^\mu + q_3^\mu = 0 \qquad \leftrightarrow \qquad \text{momentum conservation}$$

$$q_i^2 = 0 = q_i \cdot p_i \qquad \leftrightarrow \qquad \text{on-shell} \quad \hat{p}_i^2 = 0.$$

  Achieved by $|1] \to |\hat{1}] = |1] + z\langle 23\rangle |X]$ (+ cyclic) with $|X]$ arbitrary "reference spinor".

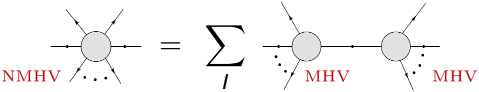
- The tree amplitude $A_n(z)$ has only simple poles, so **if** $A_n(z) \to 0$ for $z \to \infty$, then

$$0 = \oint \frac{A_n(z)}{z} \qquad \to \qquad A_n(0) = -\sum_{z \neq 0} \mathrm{Res} \frac{A_n(z)}{z}$$

- Result is on-shell recursion relation

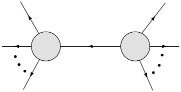
$$A_n(0) = \sum_I A_{n_1} \frac{1}{P_I^2} A_{n_2}\,, \qquad n_1 + n_2 = n + 2$$

The special 3-line shift ensures that the sub-amplitudes are both MHV if A_n is NMHV. [Risager (2005)]



→ Now use this to get NMHV gen func.

5. Next-to-MHV generating functions — $\mathcal{N} = 4$ SYM

- Consider a single MHV vertex diagram: 

- Apply MHV gen func to each vertex to derive (details omitted)

$$\Omega_{n,I}^{\mathcal{N}=4} \quad = \quad \frac{A_{n,I}^{\text{gluons}}}{\langle m_1 P_I\rangle^4 \langle m_2 m_3\rangle^4}\delta^{(8)}(L_a + R_a)\prod_{a=1}^{4}\langle P_I\, L_a\rangle$$

where $L_a = \sum_{i\in L} |i\rangle\eta_{ia}$ and $R_a = \sum_{j\in R} |j\rangle\eta_{ja}$.

[Georgio, Glover and Khoze (2004)]

- Each term in $\Omega_{n,I}^{\mathcal{N}=4}$ is order 12 in η_{ia}'s.
- Value of diagram is $D^{(12)}\,\Omega_{n,I}^{\mathcal{N}=4}$ with $D^{(12)}$ built from the external states.
- Sum all diagram gen func's to get full NMHV gen func:

$$\Omega_n^{\mathcal{N}=4} = \sum_I \Omega_{n,I}^{\mathcal{N}=4}$$

Example:
NMHV gluon amplitude

$$A_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+) = D_1^{(4)} D_2^{(4)} D_3^{(4)}\ \Omega_n^{\mathcal{N}=4}$$

Partition $12 = 4+4+4$.

$\mathcal{N} = 4$ SYM:

$$\#\text{ NMHV amplitudes} \;=\; \#\text{ partitions of 12 with } n_{\max} = 4.$$

Total of 34.

But. . .

We used MHV vertex expansion from 3-line shift recursion relations, which *assumed*

$$A_n(z) \to 0 \quad \text{for} \quad z \to \infty.$$

Is this OK?

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Is this OK?

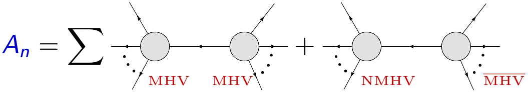
YES! [Freedman, Kiermaier, HE (Aug 2008)] .

— *provided the three lines share a common (upper)* $SU(4)$ *index.*

In $\mathcal{N} = 4$ SYM, $A_n(\hat{1}, \dots, \hat{i}, \dots, \hat{j}, \dots) \to 0$ for $z \to \infty$ when the 3 shifted states $1, i, j$ share a common (upper) $SU(4)$ index.

Outline of proof:

- Consider first amplitude A_n with state 1 a $-$ve helicity gluon.
- Use [Cheung (2008)]'s result that $[1^-, k\rangle$-shift gives valid BCFW 2-line shift recursion relations



- Perform subsequent $[1, i, j|$-shift: The as $z \to \infty$:
  diagrams MHV $\times$ MHV $\to O(\frac{1}{z})$
  diagrams $\text{NMHV}_{n-1} \times \overline{\text{MHV}}_3 \to O(\frac{1}{z})$ using inductive assumption.
- Basis of induction established by careful examination of $n = 6$ cases.
- So $A_n(\hat{1}^-, \dots, \hat{i}, \dots, \hat{j}, \dots) \to 1/z$ for large z.
- Use SUSY Ward identities to generalize state 1 to any $\mathcal{N} = 4$ state sharing a common index with i and j.

Summary — $\mathcal{N} = 4$ SYM

This proves the validity of the NMHV generating function in $\mathcal{N} = 4$ SYM. It also shows that the MHV vertex expansion is valid for all external states.

Also, the generating function is unique: once established, it does not know which valid 3-line shift it came from!

Anti-(N)MHV: The generating function for $\overline{\text{(N)MHV}}$ can be obtained from that of (N)MHV by a Grassman Fourier transform.

We have succesfully applied our generating functions to the evaluation of several 1-, 2-, 3-, and 4-loop intermediate state sums.

Outline

6. From $\mathcal{N}=4$ SYM to $\mathcal{N}=8$ SG

- $\mathcal{N}=8$ SG has 2^8 massless states:
  1 graviton$^\pm$, 8 gravitinos$^\pm$, 28 gravi-photons$^\pm$,
  56 gravi-photinos$^\pm$, 70 self-dual scalars ϕ_{abcd}.
  Global $SU(8)$ symmetry.

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- MHV generating function generalizes directly.
  $\longrightarrow$ Useful for testing map
  $[\mathcal{N}=4] \times [\mathcal{N}=4] = [\mathcal{N}=8]$
  at tree level

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  $SU(4) \times SU(4) \leftrightarrow SU(8)$
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  → Relationship between global symmetries
  $SU(4) \times SU(4) \leftrightarrow SU(8)$
  included in map and generating functions.

- Natural implementation of NMHV generating function.
  → but it doesn't work for all possible external states of $\mathcal{N} = 8$ SG!
  → because the MHV vertex expansion fails in these cases!

From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ SG (cont'd)

Large z for pure graviton n-point amplitude:

$$M_n(\hat{1}^-, \hat{2}^-, \hat{3}^-, 4^+, \ldots, n^+) \to z^{n-12} \quad \text{for} \quad z \to \infty$$

Numerically verified for $n = 5, \ldots, 11$.

- When the $M_n(z)$ does not vanish for large z the $O(1)$-term contributes as the residue of the pole at infinity. No (known) amplitude factorization that allows systematic calculation of this part.
- Also "bad" large z behavior for lower point amplitudes, for instance no good 3-line shifts for $\langle \phi^{1234}\,\phi^{1358}\,\phi^{1278}\,\phi^{5678}\,\phi^{2467}\,\phi^{3456} \rangle$.
- Intermediate state sums in unitarity cuts of $\mathcal{N} = 8$ SG loop amplitudes performed in terms of $\mathcal{N} = 4$ SYM via the KLT (Kawai-Lewellen-Tye) relations $M_n \sim \sum (k.f.) A_n A'_n$.

7. Outlook

Loops in $\mathcal{N} = 8$ supergravity

Is there are connection between "bad" large z behavior in supergravity tree amplitudes and potential UV divergencies?

Role of $E_{7,7}$?

- 70 scalars of $\mathcal{N} = 8$ SG are Goldstone bosons of spontaneously broken $E_{7,7} \to SU(8)$.
- How will $E_{7,7}$ reveal itself?
  $\to$ soft-scalar limits of amplitudes
  (analogous to soft-pion low-energy theorems of Adler).
- We find that 1-soft-"pion" limits of $\mathcal{N} = 8$ tree amplitudes *vanish*.
- Note that in pion physics 1-pion soft limits do not necessarily vanish, even in models with pions and nucleons both massless.
- Since our May paper: new results by [Arkani-Hamed, Cachazo, Kaplan (2008)]