Thermodynamic Bubble Ansatz

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IAS

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arXiv:0705.0303,..., arXiv:0904.0663, L.F.A & J. Maldacena;
arXiv:0911.4708, L.F.A, D. Gaiotto & J. Maldacena

Motivations

We will be interested in gluon scattering amplitudes of planar $\mathcal{N}=4$ super Yang-Mills.

Motivation: It can give non trivial information about more realistic theories but is more tractable.

- Weak coupling: Perturbative computations are easier than in QCD. In the last years a huge technology was developed.
- The strong coupling regime can be studied, by means of the gauge/string duality, through a weakly coupled string sigma model.

Our aim

Learn about scattering amplitudes of planar $\mathcal{N}=4$ super Yang-Mills by means of the AdS/CFT correspondence.

- AdS/CFT: Scattering amplitudes $\rightarrow$ minimal surfaces in AdS

What we will do in practice

Compute the area of minimal surfaces in AdS

Our main tool

AdS/CFT duality

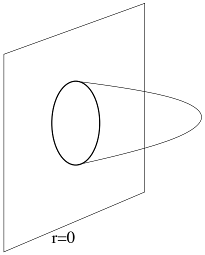
Four dimensional maximally SUSY Yang-Mills $\Leftrightarrow$ Type IIB string theory on $AdS_5 \times S^5$.

$$\sqrt{\lambda} \equiv \sqrt{g_{YM}^2 N} = \frac{R^2}{\alpha'} \qquad \frac{1}{N} \approx g_s$$

- The AdS/CFT duality allows to compute quantities of $\mathcal{N} = 4$ SYM at strong coupling by doing geometrical computations on AdS.

- Remember a similar problem: Expectation value of Wilson loops at strong coupling (Maldacena, Rey)

$$ds^2 = R^2 \frac{dx_{3+1}^2 + dr^2}{r^2}$$



- We need to consider the minimal area ending (at $r = 0$) on the Wilson loop.

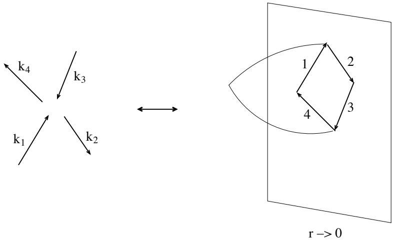
$$\langle W \rangle \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- Problem: Scattering amplitude of gluons with momenta k_1, k_2, ... k_N at strong coupling.

$$\Downarrow$$

- The problem reduces to a minimal area problem in AdS_5.

What is now the boundary of our world-sheet?



- Draw a polygon whose segments are $\Delta y^\mu = k^\mu$
- Polygon of light-like edges.
- Look for the minimal surface ending in such polygon.

- Vev of a Wilson-Loop given by a sequence of light-like segments!

Prescription

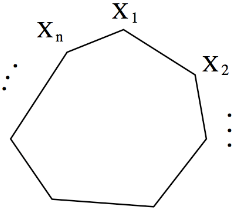
$$\mathcal{A}_N \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

$\mathcal{A}_N$: Leading exponential behavior of the $n-$point scattering amplitude.
$A_{min}(k_1^\mu, k_2^\mu, ..., k_N^\mu)$: Area of a minimal surface that ends on a sequence of light-like segments on the boundary.

- Minimal surfaces are described by classical strings.
- The full problem involves strings on AdS_5.

Mathematical problem

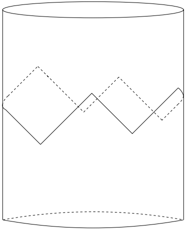
- Minimal area surface ending in the boundary of AdS at a polygon parametrized by X_i.



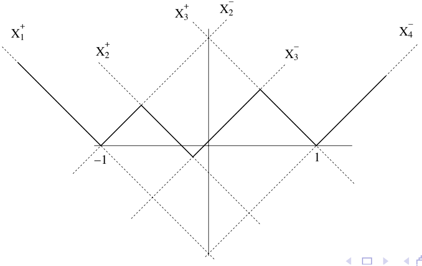
- X_i: Location of the cusps; $X^2_{i,i+1} = 0$.

- The area will depend on cross-ratios $\frac{X^2_{ij}X^2_{kl}}{X^2_{ik}X^2_{jl}}$.
- Given the cross-ratios we would like to compute $A_{min}(\frac{X^2_{ij}X^2_{kl}}{X^2_{ik}X^2_{jl}})$
- Start with the simpler case: strings on AdS_3.

Strings on AdS_3: world-sheet ending on a $2D$ polygon, *e.g.* in the cylinder.



- Consider a zig-zagged Wilson loop of $2n$ sides
- Parametrized by n X_i^+ coordinates and n X_i^- coordinates.
- We can build $2n-6$ invariant cross ratios.



Classical strings on AdS_3

Strings on AdS_3 : $\vec{Y}.\vec{Y} = -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1$

$$Eoms : \partial\bar{\partial}\vec{Y} - (\partial\vec{Y}.\bar{\partial}\vec{Y})\vec{Y} = 0, \qquad Virasoro : \partial\vec{Y}.\partial\vec{Y} = \bar{\partial}\vec{Y}.\bar{\partial}\vec{Y} = 0$$

Pohlmeyer kind of reduction $\longrightarrow$ generalized Sinh-Gordon

$$\alpha(z,\bar{z}) = \log(\partial\vec{Y}.\bar{\partial}\vec{Y}), \qquad p^2 = \partial^2\vec{Y}.\partial^2\vec{Y}$$

$$\downarrow$$

$$p = p(z), \qquad \partial\bar{\partial}\alpha - e^{\alpha} + |p(z)|^2 e^{-\alpha} = 0$$

- $\alpha(z,\bar{z})$ and $p(z)$ invariant under conformal transformations.
- Area of the world sheet: $\mathcal{A} = \int e^{\alpha} d^2 z$

Generalized Sinh-Gordon $\rightarrow$ Strings on AdS_3?

- From α, p construct flat connections $B_{L,R}$ and solve two linear auxiliary problems.

$$(\partial + B^L)\psi^L_a = 0 \qquad (\partial + B^R)\psi^R_{\dot{a}} = 0 \qquad B^L_z = \begin{pmatrix} \partial\alpha & e^{\alpha} \\ e^{-\alpha}p(z) & -\partial\alpha \end{pmatrix}$$

Space-time coordinates

$$Y_{a,\dot{a}} = \begin{pmatrix} Y_{-1} + Y_2 & Y_1 - Y_0 \\ Y_1 + Y_0 & Y_{-1} - Y_2 \end{pmatrix} = \psi^L_a M \psi^R_{\dot{a}}$$

One can check that Y constructed that way has all the correct properties.

Relation to Hitchin equations

Consider self-dual YM in 4d reduced to 2d

- $A_{1,2} \to A_{1,2}$: 2d gauge field, $A_{3,4} \to \Phi, \Phi^*$: Higgs field.

Hitchin equations

$$F^{(4)} = *F^{(4)} \qquad \to \qquad \begin{aligned} D_{\bar{z}}\Phi = D_z\Phi^* &= 0 \\ F_{z\bar{z}} + [\Phi, \Phi^*] &= 0 \end{aligned}$$

- We can decompose $B = A + \Phi$.
- $dB + B \wedge B = 0$ implies the Hitchin equations.
- We have a particular solution of the $SU(2)$ Hitchin system.
- Nice relation: $\mathcal{A} = \int Tr\Phi\Phi^*$.

Standard form of the sinh-Gordon equation: go to the $w-$plane

$$dw = \sqrt{p(z)}dz, \quad \hat{\alpha} = \alpha - \frac{1}{4}\log p\bar{p} \rightarrow \partial_w \bar{\partial}_{\bar{w}} \hat{\alpha} = sinh 2\hat{\alpha}$$

- Simpler equation in a more complicated space.
- Convenient to understand some features of the solution.

$$\mathcal{A} = \int e^{\alpha} d^2 z = \int e^{\hat{\alpha}} d^2 w$$

Feature of α and $p(z)$ for solutions corresponding to scattering amplitudes?

$p(z) = 1, \quad \hat{\alpha} = 0 \rightarrow$ trivial inverse problem $\rightarrow$ four cusps solution!

- Near the boundary we approach the cusps, so $\hat{\alpha}$ decays at infinity.
- Assume $p(z)$ is a polynomial.
- Define a "regularized" area $A_{reg} = \int d^2w(e^{\hat{\alpha}} - 1)$.

Consider a generic polynomial of degree $n-2$

$$p(z) = z^{n-2} + c_{n-4} z^{n-4} + \dots + c_1 z + c_0$$

- We have used translations and re-scalings in order to fix the first two coefficients to one and zero.
- For a polynomial of degree $n-2$ we are left with $2n-6$ (real) variables.
- This is exactly the number of invariant cross-ratios in two dimensions for the scattering of $2n$ gluons!

Null Wilsons loops of $2n$ sides $\Leftrightarrow p^{(n-2)}(z)$ and $\hat{\alpha}(z,\bar{z}) \to 0$

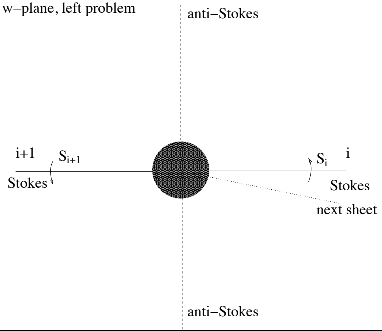
- Degree of the polynomial $\rightarrow$ number of cusps.
- Coefficients of the polynomial $\rightarrow$ shape of the polygon.

Regular polygons

- Simplest case: $p(z) = z^{n-2} \rightarrow w = \frac{n}{2} z^{n/2}$
- In the w plane we go around $n/2$ times.

Boundary: $|w| \gg 1 \rightarrow \hat{\alpha} \approx 0$. Gral solution of the linear problem:

$$\psi_a^L = c_a^+ \eta^+ + c_a^- \eta^-, \qquad \eta^+ = \begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix}, \quad \eta^- = \begin{pmatrix} 0 \\ e^{-(w+\bar{w})} \end{pmatrix}$$



- Focused in the left problem.
- $w-$plane divided into two regions (anti-Stokes sectors), $\pm Re(w) > 0$
- In each sector, one of the two solutions dominates.
- In the anti-Stokes lines, both are of the same order.

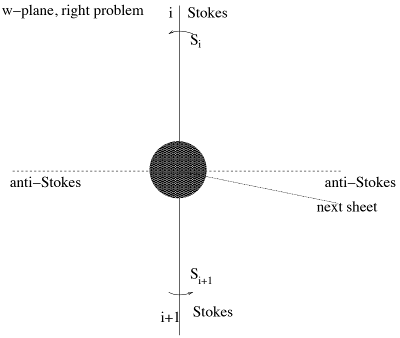
There is a Stokes phenomenon going on...

- The large solution is only defined up to a multiple of the small solution.
- Actually, as we cross the (*e.g.* the first) Stokes line...

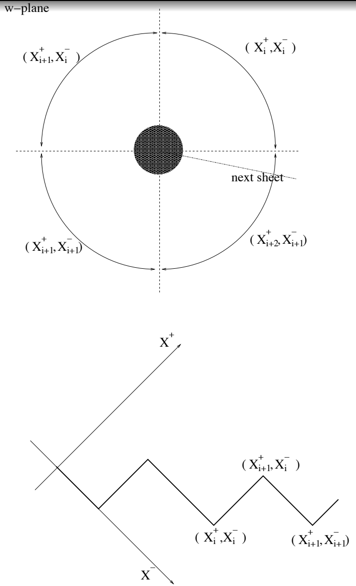
$$\begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ e^{-(w+\bar{w})} \end{pmatrix}$$

- This jump in the small component of the large solution is characterized by the Stokes matrix $S_a^b = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$
- This small component becomes important as we cross to the other anti-Stokes region.

The right-problem is similar: $\psi_a^R = c_a^+ \begin{pmatrix} e^{\frac{w+\bar{w}}{i}} \\ 0 \end{pmatrix} + c_a^- \begin{pmatrix} 0 \\ e^{-\frac{w+\bar{w}}{i}} \end{pmatrix}$



But now the Stokes and anti-Stokes lines are rotated.



- The $w-$plane is divided into quadrants.
- At each quadrant, a pair of solutions (η^L and η^R) is dominant.
- The whole region corresponds to a single point in space-time, a cusp.
- As we cross one of the anti-Stokes lines, the dominant solution L or R changes and we jump to the next cusp.
- At each step only one changes $\rightarrow$ in $R^{1,1}$ only the X^+ or X^- coordinate changes
- As we go around the $w-$plane $n/2$ times, we get the $2n$ cusps!

- The general case is less symmetric but works similarly.

Strings on AdS_5:

- Still a holomorphic quantity $P(z) = \partial^2 \vec{Y}.\partial^2 \vec{Y}$.
- AdS_3 limit: $P(z) \to p(z)^2$.
- Two more physical fields: $\alpha(z,\bar{z}) = \log(\partial \vec{Y}.\bar{\partial} \vec{Y})$ but also $\beta(z,\bar{z})$ and $\gamma(z,\bar{z})$.

For N gluons, how does the counting of cross-ratios work?

$$P(z) = z^{N-4} + c_{N-6} z^{N-6} ... + c_0 \to 2N - 10 \text{ real coefficients.}$$

- For AdS_4 we have exactly $2N-10$ cross ratios, so this is the whole picture (α and β are unique once you have fixed $P(z)$)
- For AdS_5 there are $N-5$ extra degrees of freedom coming from the boundary conditions, giving the expected $3N-15$ cross-ratios in 4d scattering!

What about the Hitchin equations?

- Not anymore Left+Right factorization but we still have a Hitchin system!
- We obtain a particular case of $SU(4)$ Hitchin system.

$$\Phi_z = \begin{pmatrix} 0 & e^{-1/2\alpha} v_I \tau^I \\ e^{1/2\alpha} 1_2 & 0 \end{pmatrix}, \quad A_z = \begin{pmatrix} -\partial\alpha + d_{IJ}\tau^{IJ} & 0 \\ 0 & \partial\alpha + d_{IJ}\tau^{IJ} \end{pmatrix}$$

Not generic but fixed by the following projection

$$C\Phi^T C^{-1} = i\Phi, \quad CA^T C^{-1} = -A, \quad \rightarrow Tr(x-\Phi) = x^4 - P(z)$$

Space time coordinates:

$$(\partial + A_z + \Phi_z)\psi_a = 0, \quad (\bar{\partial} + A_{\bar{z}} + \Phi_{\bar{z}})\psi_a = 0, \quad Y \approx \psi_a \Gamma_{ab} \psi_b$$

General prescription for polynomial $p(z)$ in AdS_3 or $P(z)$ on AdS_5

- Compute the space-time cross-ratios in terms of the coefficients of $P(z)$.
- Compute the area in terms of the coefficients of $P(z)$.
- Write the area in terms of the space-time cross-ratios.

First non trivial cases:

- On AdS_3: $p(z) = z^2 - m$, the "octagon".
- On AdS_5: $P(z) = z^2 - U$, the "hexagon".

but its very hard to proceed...

Ask someone else!

- Gaiotto, Moore and Neitzke have found the same Hitchin equations in a completely different context.
- They have developed a technology very useful for solving the problem at hand!

Idea: Use integrability to promote the Hitchin system to a family of flat connections (introduce a spectral parameter)

$$B_z^{(\zeta)} = A_z + \frac{\Phi_z}{\zeta}, \quad B_{\bar{z}}^{(\zeta)} = A_{\bar{z}} + \zeta \Phi_{\bar{z}}$$

Why is this useful?

- Consider the deformed auxiliary linear problem leading to $Y[\zeta]$ (such that $Y[1]$ is the physical solution).

$$(\partial + A_z + \frac{\Phi_z}{\zeta})\psi(\zeta) = 0$$

- Consider the cross-ratios as a function of ζ (such that at $\zeta = 1$ we obtain the physical cross-ratios).
- For $\zeta \to 0$ or $\zeta \to \infty$ the connections simplify and we can solve such inverse problem in a WKB approximation!

$$\psi \approx e^{\frac{1}{\zeta}\int P(z)^{1/4}dz}$$

Setting up a Riemann-Hilbert problem.

- We expect the cross-ratios as a function of ζ to be analytic away from $\zeta = 0, \infty$.
- The cross-ratios have different asymptotic behaviors as $\zeta \to 0, \infty$ depending on the phase of ζ, so they display Stokes phenomenon in the ζ plane.

Alternatively

- We can find some functions $\chi_a[\zeta]$ of the cross-ratios, with uniform small/large ζ behavior on the whole $\zeta-$plane.
- Due to Stokes, the price to pay is that there are discontinuities in the $\chi_a[\zeta]$ along some rays in the $\zeta-$plane.
- GMN: This defines a Riemann-Hilbert problem which can be rewritten as an integral equation for the cross-ratios! (as a function of ζ)

For the case of the Hexagon $P(z) = z^2 - U^{3/4}$

$$\begin{aligned}
\epsilon(\theta) = & \quad 2|U|\cosh\theta + \frac{\sqrt{2}}{\pi}\int d\theta' \frac{\cosh(\theta-\theta')}{\cosh 2(\theta-\theta')}\log(1+e^{-\tilde{\epsilon}}) + \\
& + \frac{1}{2\pi}\int d\theta' \frac{1}{\cosh(\theta-\theta')}\log(1+\mu e^{-\epsilon})(1+\frac{e^{-\epsilon}}{\mu}) \\
\tilde{\epsilon}(\theta) = & \quad 2\sqrt{2}|U|\cosh\theta + \frac{1}{\pi}\int d\theta' \frac{1}{\cosh(\theta-\theta')}\log(1+e^{-\tilde{\epsilon}}) + \\
& + \frac{\sqrt{2}}{\pi}\int d\theta' \frac{\cosh(\theta-\theta')}{\cosh 2(\theta-\theta')}\log(1+\mu e^{-\epsilon})(1+\frac{e^{-\epsilon}}{\mu})
\end{aligned}$$

- Exactly the form of TBA equations!

- What is the regularized area?

$$A_{reg} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta 2|U|\cosh\theta \log\left(1+e^{-\epsilon}\mu\right)\left(1+\frac{e^{-\epsilon}}{\mu}\right) + \\ +\frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta 2\sqrt{2}|U|\cosh\theta\log\left(1+e^{-\tilde{\epsilon}}\right)$$

- Exactly the free energy of the TBA system!

Some exact results...

- High temperature/Conformal limit of the TBA equations
  $U=0 \rightarrow u_1=u_2=u_3$

Hexagonal Wilson loop in AdS_5 in $U\rightarrow 0$ limit

$$R(u,u,u)=\frac{\phi^2}{3\pi}+\frac{3}{8}(\log^2 u+2Li_2(1-u)),\qquad u=\frac{1}{4\cos^2(\phi/3)}$$

Some more exact results...

Eight sided Wilson loop in AdS_3 (the first non trivial)

$$R(m,\bar{m})=\frac{1}{2}\int dt\frac{\bar{m}e^t-me^{-t}}{\tanh 2t}\log\left(1+e^{-\pi(\bar{m}e^t+me^{-t})}\right)$$

What have we done and what needs to be done

- We have given a further step towards the computation of classical solutions relevant to scattering amplitudes at strong coupling.
- Integrability is the key ingredient of the computation

For the future...

- Could we compute these amplitudes at all values of the coupling?!
- What about other kind of solutions? *e.g.* correlations functions?
- Include fermions and understand non MHV amplitudes?