Note Title

Exponent of cubic matrixes and non-abelian tensor fields

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hep-th/0503234
hep-th/0504160 (with V. Dolotin & A. Morozov)
hep-th/0506032

Work in progress

We start with the mathematical question: How to exponentiate qubic matrices?

$$\hat{B} = \|B_{ijk}\| \qquad i = \overline{1,N}$$

$$\begin{matrix} B_{311} & B_{312} & B_{313} & \dots \end{matrix}$$

$$\begin{matrix} B_{211} & B_{212} & B_{213} & \dots \end{matrix}$$

$$\begin{matrix} B_{111} & B_{112} & B_{113} & \dots \\ B_{121} & B_{122} & B_{123} & \dots \\ B_{131} & B_{132} & B_{133} & \dots \end{matrix}$$

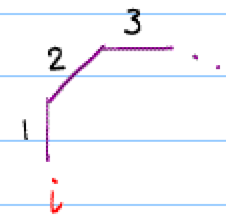
$$\boxed{e^{\hat{B}} = ?}$$

Let us recall exponent of a quadr

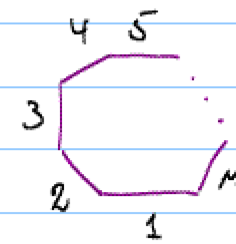
matrix

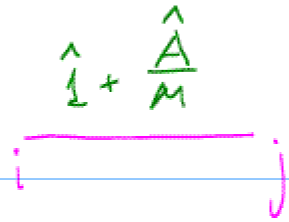
$$\hat{A} = \| A_{ij} \| \quad i = \overline{1, N}$$

$$\left[e^{\hat{A}} \right]_i^{\ j} = \lim_{M \to \infty} \Big[\prod_{\substack{\text{connec.} \\ \text{chain}}}^{M} \left(\hat{1} + \frac{\hat{A}}{M} \right) \Big]_i^{\ j}$$



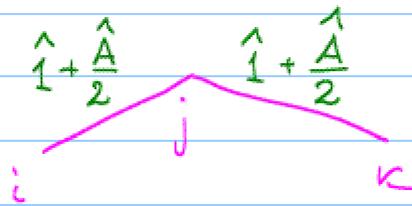
$$\mathrm{Tr}\, e^{\hat{A}} = \lim_{M \to \infty} \prod_{\substack{\underline{\text{closed}} \\ \text{connec.} \\ \text{chain}}}^{M} \left(\hat{1} + \frac{\hat{A}}{M} \right)$$





$$= \delta_i{}^j + \frac{A_i{}^j}{M}$$

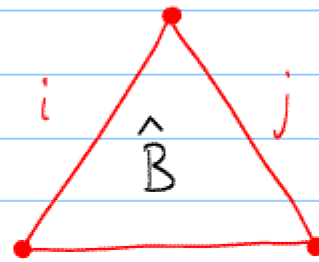
$\sum_{j=1}^{N}$



$$= \left(\delta_i{}^j + \frac{A_i{}^j}{2}\right)\left(\delta_j{}^k + \right.$$

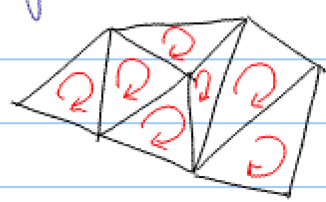
 — 1D simplex $\Longrightarrow$

Natural generalization of the quadratic case



— 2D s

Oriented triangulated Rieman surface consisting of n triangles



$\Longleftrightarrow$

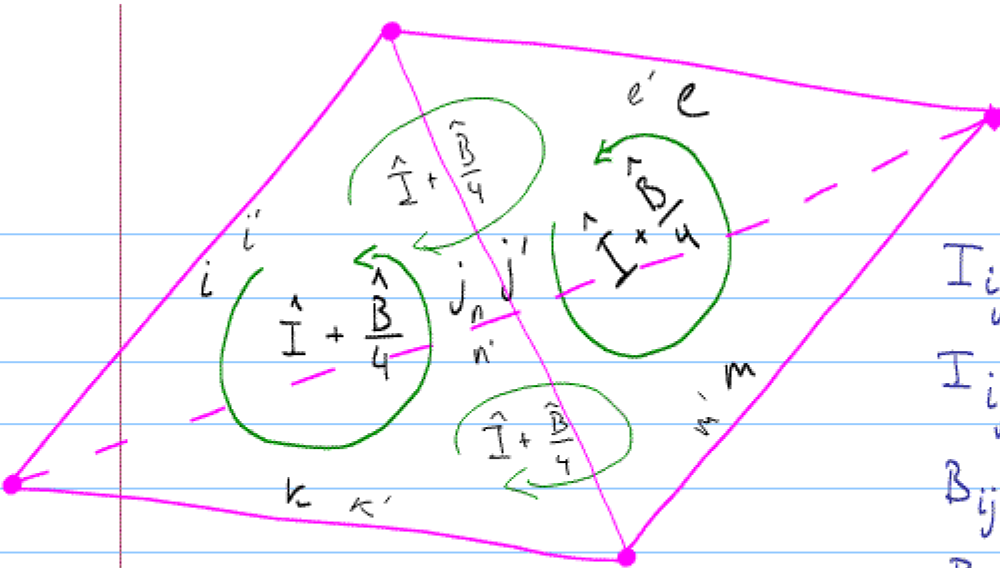
n-th power of a

cyclicly sym. matrix

$\hat{B}^n$ if $B_{ijk} = B_{k}$

Then, the definition of the exponent

$$\mathrm{Tr}\left[E^{\hat{B}}\right] = \lim_{M\to\infty} \prod_{\substack{\text{connected}\\ \text{closed}\\ \text{triang.}\\ \text{Riemann}\\ \text{surface}}}^{M} \left(\hat{I} + \frac{\hat{B}}{M}\right)_{\varkappa}$$



$I_{ijk} = I_{kij} = I_{jki}$

$I_{ijk} \neq I_{jik}$

$B_{ijk} = B_{kij} = B_{jki}$

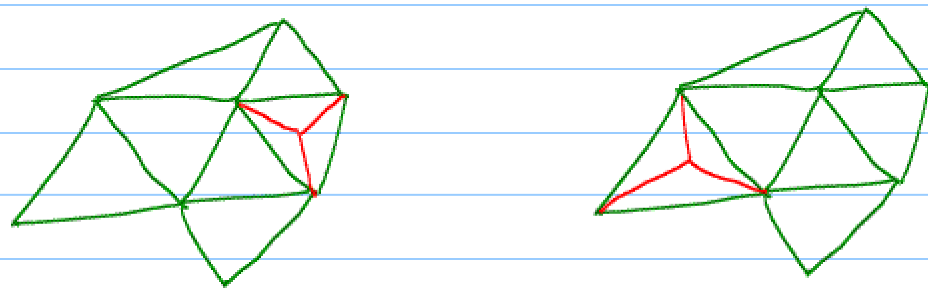
$B_{ijk} \neq B_{jik}$

$$\prod \left(\hat{I} + \frac{B}{4} \right) =$$

$$= \left(I + \frac{B}{4} \right)_{ijk} \left(I + \frac{B}{4} \right)_{j'em} \left(I + \frac{B}{4} \right)_{i'e'n'} \left(I + \frac{B}{4} \right.$$

$$x^{ii'} x^{jj'} x^{kk'} x^{ee'} x^{mm'} x^{nn'}$$

As $M \to \infty$



Many various sequences of graphs!!!

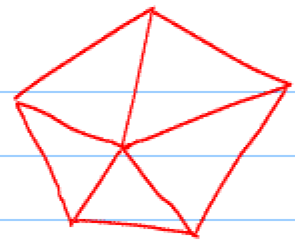
Which one to choose?

The answer for the limit in question does not depend on the choice of the sequence of graphs if:

0) $I_{ijk} = I_{kij} = I_{jki}$ & the same for

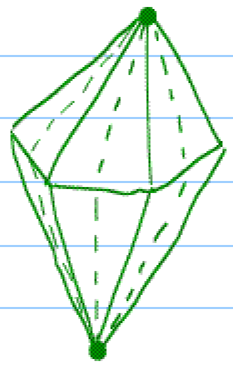
1) Genus (g) of the graphs in the sequence fixed. Real triangulations.

2)



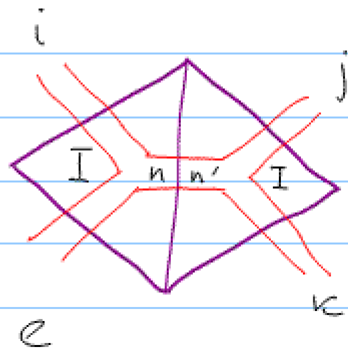
$$\frac{n_v(\text{graph})}{n} \longrightarrow 0 \text{ as } n$$

E.g.



— this sequence contradicts the condition (2)

3) $\hat{I}$ & $\hat{æ}$ should obey:



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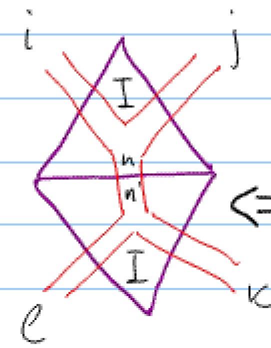
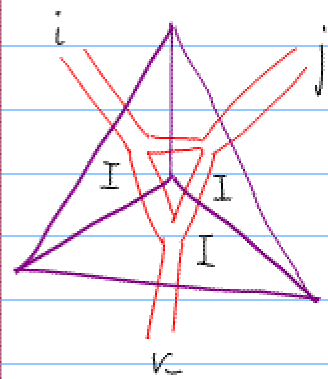


$\Longleftrightarrow$ $I_{in\ell} I_{n'jk}$ …

$= I_{ijn} I_{n'k\ell}$ …



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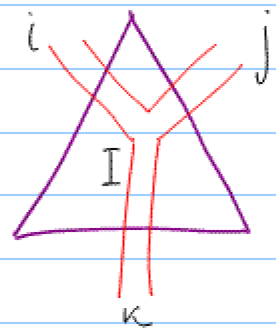
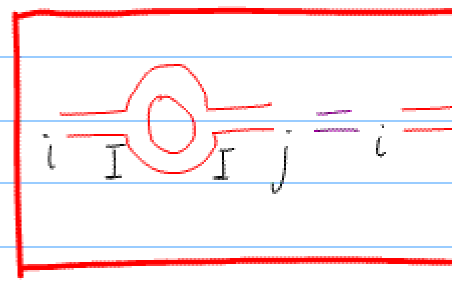
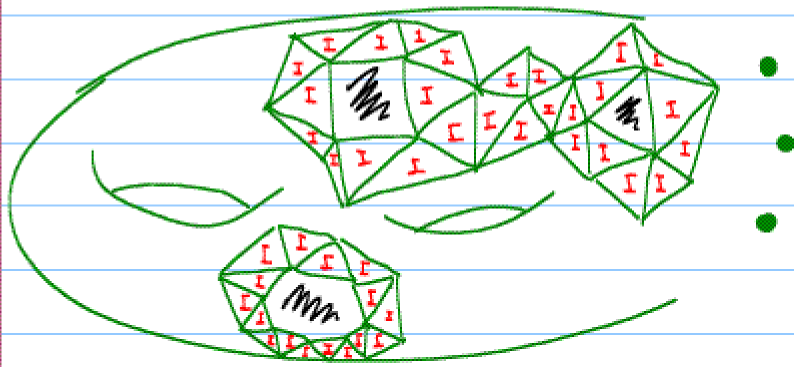
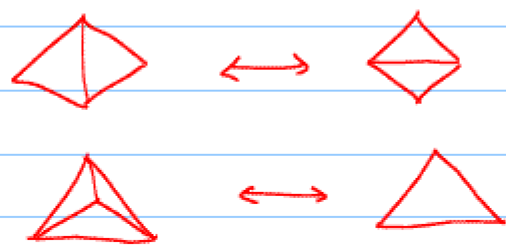




Meaning of these conditiones:





Alexandrov moves

E.g., Fukuma, Hosono & Kawai

Solutions of the conditiones for $\hat{I}$ &

$e_i e_j = I_{ij}{}^{k} e_k$ – product in an algeb

$e_i\,,\ i=\overline{1,N}$ – generators of the algeb

If the product is associatve and

$\boxed{\varkappa_{ij} \equiv I_{ij}{}^{l} I_{lk}{}^{k}}$ is non-degenerate,

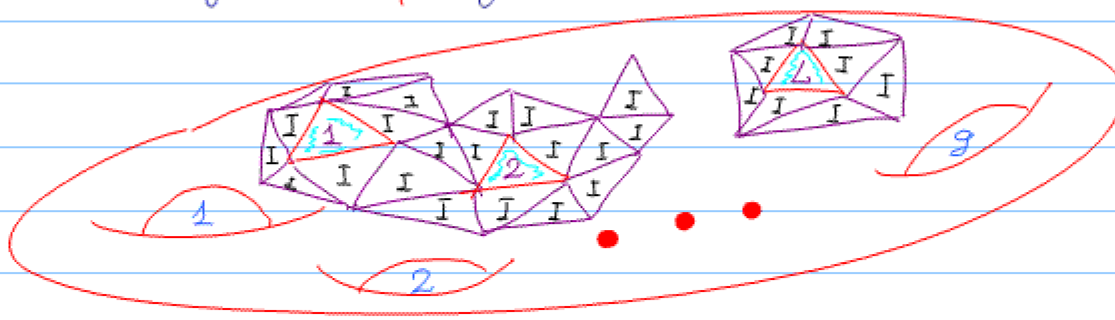
Then $\varkappa_{ij}$ and

$\boxed{I_{ijk} \equiv I_{ij}{}^{l} \varkappa_{lk}}$ Solve the conditi

Then

$$\operatorname{Tr} E^{\hat{B}}_{g,\hat{I},\hat{\varkappa}} = \sum_{L=0}^{\infty} \frac{1}{L!} B^{i_1 j_1 k_1} \cdots B^{i_L j_L k_L} \cdot$$

$$\cdot \mathbb{I}^{g}_{k_1 j_1 i_1 | \cdots | k_L j_L i_L}$$



To obtain the exponent with indeces one has to use open surfaces:

$$\left[E^{\hat{B}}_{g,I,\varkappa}\right]_{\bullet\bullet\bullet} = \lim_{M\to\infty}\left[\prod_{\substack{\text{connected}\\ \text{triang.}\\ \text{Riemann}\\ \text{surface}}}^{M}\left(\hat{I}+\frac{\hat{B}}{M}\right)\right]$$

fix the topology

$$= \sum_{L=0}^{\infty}\frac{1}{L!}\,B^{i_1j_1k_1}\cdots B^{i_Lj_Lk_L}\times \mathbb{I}^{g}_{\bullet\bullet\bullet}\;|k_1j_1i_1|$$

Instead of that one can use the building block for the exponent with indices:

$$\left[E^{\hat{B}}_{g=0,I,\varkappa}\right]_{m_1 m_2 m_3} \equiv \lim_{M\to\infty}\left[\prod_{\substack{\text{connected}\\ \text{triang.}\\ \text{disc}}}^{M}\left(\hat{I}+\frac{\hat{B}}{M}\right)\right]_{m}$$

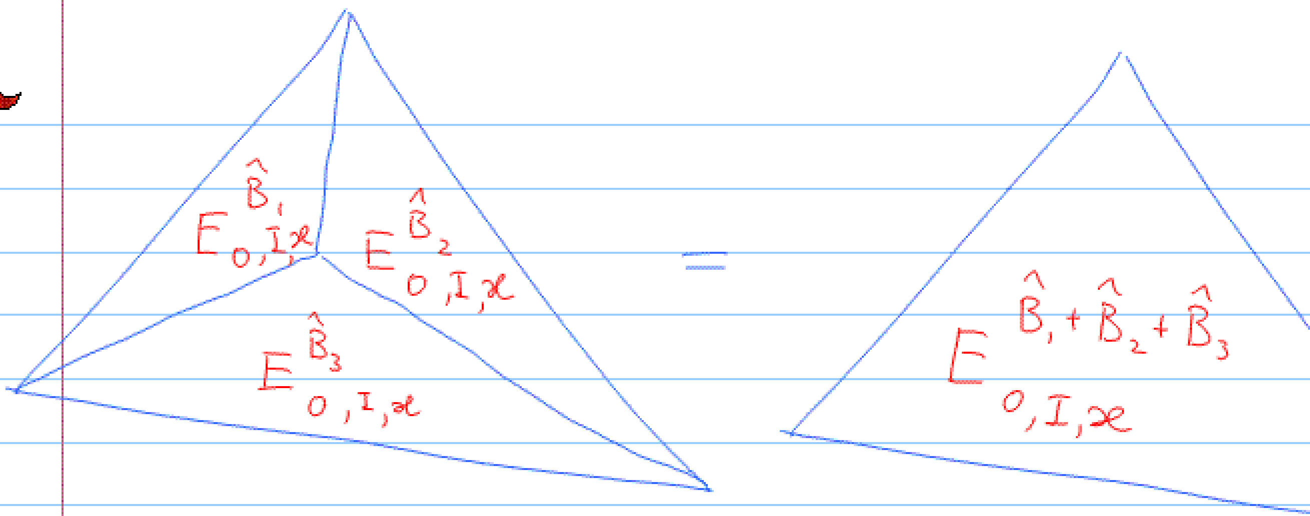
$$=\sum_{L=0}^{\infty}\frac{1}{L!}B^{i_1j_1k_1}\cdots B^{i_Lj_Lk_L}\times \mathbb{I}^{g=0}_{m_1m_2m_3|k_1j_1i_1|}$$

Properties of the exponent:

$$e^{t_1 \hat{A}} \, e^{t_2 \hat{A}} = e^{(t_1 + t_2)\hat{A}}$$

But

$$e^{\hat{A}_1} \, e^{\hat{A}_2} \neq e^{\hat{A}_1 + \hat{A}_2}$$



$\boxed{\text{Generalizations to the exponents of ma}}$

Generalizations to the exponents of ma
with more than three indexes is obvious

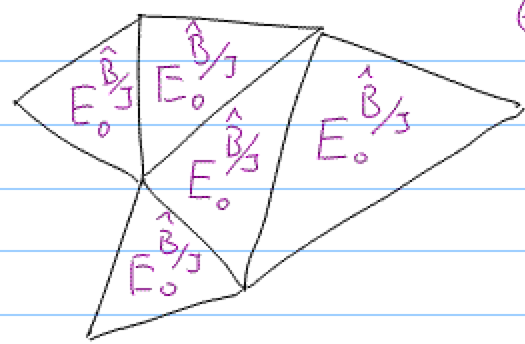
Thus, to obtain an exponent w

indices take a triangulated c

Riemann surface consisting of

J triangles and place $E_{0,I,\varkappa}^{\hat{B}/J}$ at

each triangle.

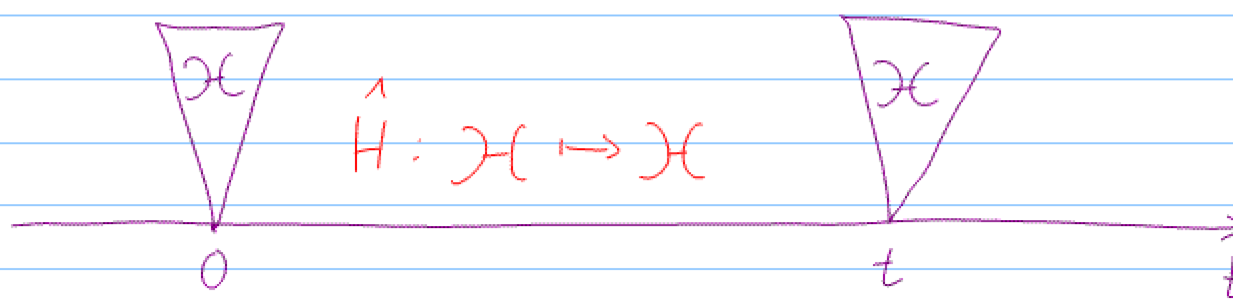


What for do we need this exponen

Quantum mechanics:

$$i \frac{\partial}{\partial t} \Psi(x,t) = \hat{H}\left(x, \tfrac{\partial}{\partial x}\right) \Psi(x,t) \; ; \; \Psi(x, t=0) =$$

Solution: $\Psi(x,t) = e^{-i \hat{H}\left(x, \frac{\partial}{\partial x}\right) t} \varphi$



$\hat{H}$ is a connection of the fiber bu

time line is the base of the bu

$\mathcal{H}$ (Hilbert space) is the fiber

What is the generalization of all

to the Quantum Field theor

Standard (2D case):

$$i \frac{\partial}{\partial \tau} \Psi[X(\sigma), \tau] = \hat{H}\left[\frac{\delta}{\delta X(\sigma)}, X(\sigma)\right] \Psi[X(\sigma),$$

$$\Psi[X(\sigma), 0] = \varphi[X(\sigma)]$$

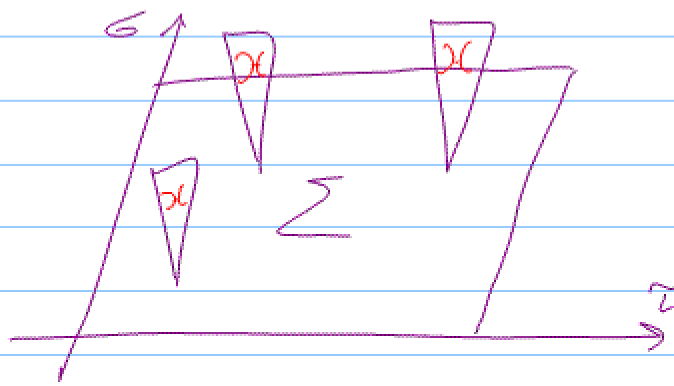
Green function:

$$K[X(\sigma), Y(\sigma) | T] = e^{-i \hat{H}\left[\frac{\delta}{\delta X(\sigma)}, X(\sigma)\right] T} \delta[X(\sigma) -$$

$$= \delta[X(\sigma) - Y(\sigma)] - i \hat{H}\left[\frac{\delta}{\delta X(\sigma)}, X(\sigma)\right] T \, \delta[X(\sigma) -$$

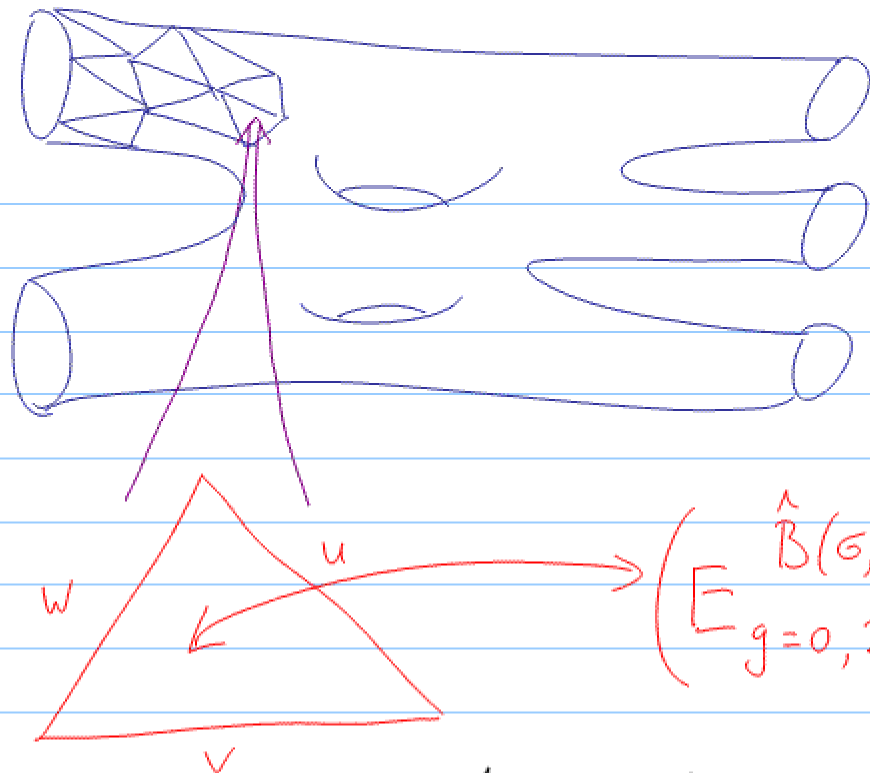
We propose a generalization of the Schrödinger equation for the non-point objects

$$\frac{\delta^2}{\delta\sigma\delta\tau}\left[E_{I,\varkappa}^{\Sigma}\iint\hat{B}\right]_{W(s)} = B^{uvz}(\sigma,\tau)\left[E_{I,\varkappa}^{\Sigma}\iint\hat{B}\right]_{W(s)}$$



Discrete $i \longrightarrow$

Continuous $u \in$



$$\left(E_{g=0,I,\varkappa}^{\hat{B}(\sigma,\tau)\,\Delta\sigma\,\Delta\tau} \right)$$

More general than the standard c

Green function in the new constructi

$$\hat{K} = \left[E^{\,i \iint_{\text{cylinder}} \hat{B}(\sigma,\tau)\, d\sigma\, d\tau} \right]_{W(\sigma_1)}^{U(\sigma_2)} = \mathbb{I}_{W(\sigma_1)}^{U(\sigma_2)}$$

$$+ \; i \iint_{\text{cylinder}} B^{v_1 v_2 v_3}(\sigma,\tau)\, d\sigma\, d\tau \;\; \mathbb{I}_{v_1 v_2 v_3 |\, W(\sigma_1)}^{U(\sigma_2)}$$

Losev $\mathbb{I}_{W(G)}^{U(G')}$ is a projector:

$$\int DU(G_2)\ \mathbb{I}_{W(G_1)}^{U(G_2)}\ \mathbb{I}_{U(G_2)}^{V(G_3)} = \mathbb{I}_{W(G_1)}^{V}$$

Fukuma, Hosono & Kawai

$|X\rangle$ - forms a basis in the cent

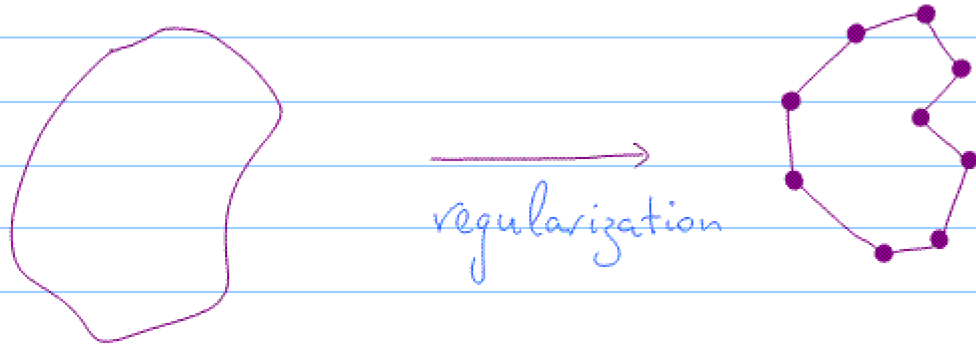
of the algebra with the basis $|W$

I.e. there should be a generalization

of the Hilbert space.

Let me clarify this point.

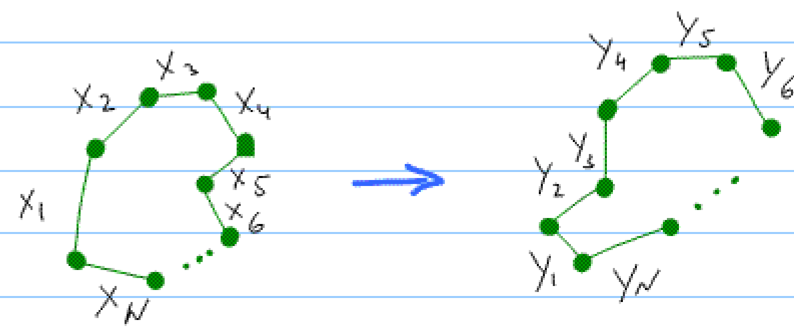
Consider broken lines instead of strings



Consider

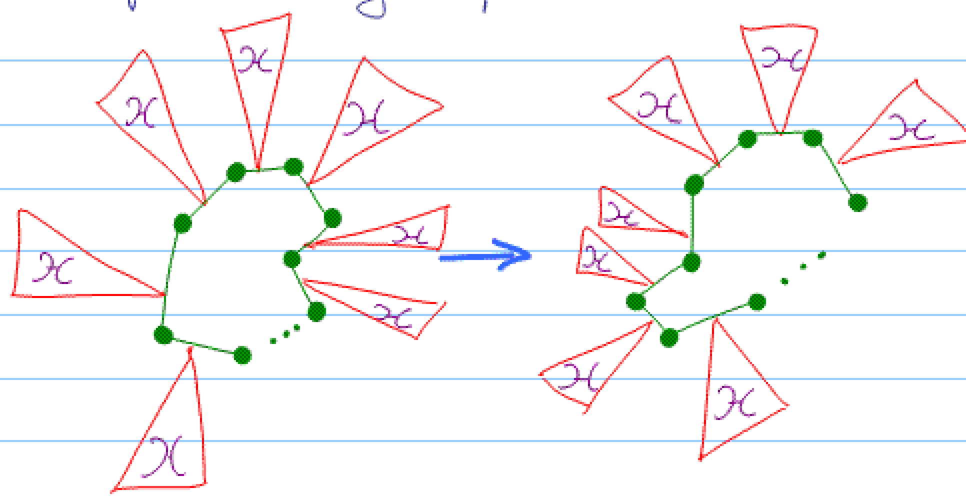
$$K_N(x,y) \equiv e^{-i\hat{H}\left[\left\{\frac{\partial}{\partial x_n}\right\}, x_n\right]T} \prod_{k=1}^{N} \delta(x,$$

K_N :



$$\prod_{n=1}^{N} \delta(x_n - y_n)$$ is the unity oper

on the following space



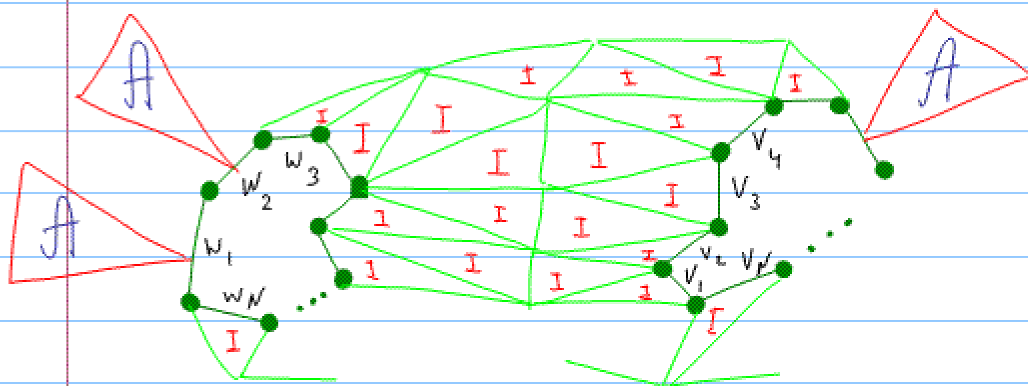
In general we have:

$$\int\cdots\int_{A} \prod_{k=1}^{N} dv_k \; \mathbb{I}_{w_1\ldots w_M}^{\;v_1\ldots v_N} \; \mathbb{I}_{v_1\ldots v_N}^{\;u_1\ldots u_L} =$$

$$= \mathbb{I}_{w_1\ldots w_M}^{\;u_1\ldots u_L} \;, \quad \forall\, M,\, N \;\&\; L$$

At the same time $\mathbb{I}_{w_1 \ldots w_N}^{v_1 \ldots}$

is the projection operator on :



w & v e

X & Y c

It can be shown that at $N=1$

$\mathbb{I}_w^{\,v}$ projects on the center o

I.e. $\mathcal{H}$ should be the center of

On what does the operator

$\mathbb{I}_{w_1 \ldots w_N}^{\,v_1 \ldots v_N}$ projects ?

Many questions remain:

1) What is $\mathcal{A}$ (i.e. I & $\varkappa$) ?

2) What is the operator Ω such that: $\Omega \, \boxed{\Pi} \, \Omega^{-1} = \left(\begin{array}{c|c} \Pi\,\delta(\ldots) & 0 \\ \hline 0 & 0 \end{array}\right)$

3) When Ω & $\mathcal{A}$ are found we have to calculate $\hat{B}$ from $\hat{H}$ is