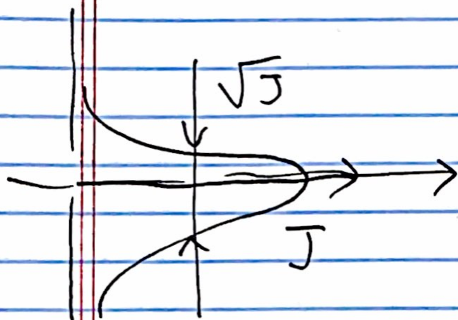
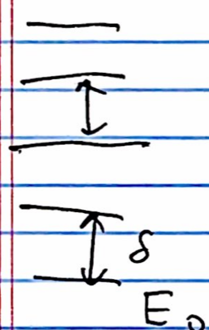
Then, can no longer replace $\vec{J} \to \langle \vec{J} \rangle$



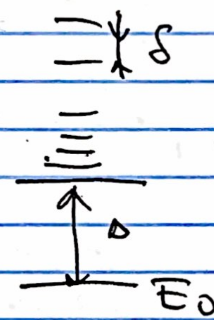
Need $\frac{\sqrt{J}}{J} \ll 1 \Rightarrow J \gg 1$

$J = \frac{N}{2}$ Need $N \gg 1$

normal state (Fermi liquid)



sc



Current carrying state $c^+_{-\vec{k}\uparrow} c^+_{\vec{k}\downarrow} \to$

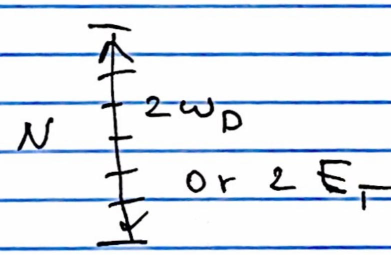
$\to c^+_{-\vec{k}+\vec{q}\uparrow} c^+_{k\downarrow}$

No dissipation bc of the gap

Normal st. and sc indistinguishable when

$\Delta \sim \delta$

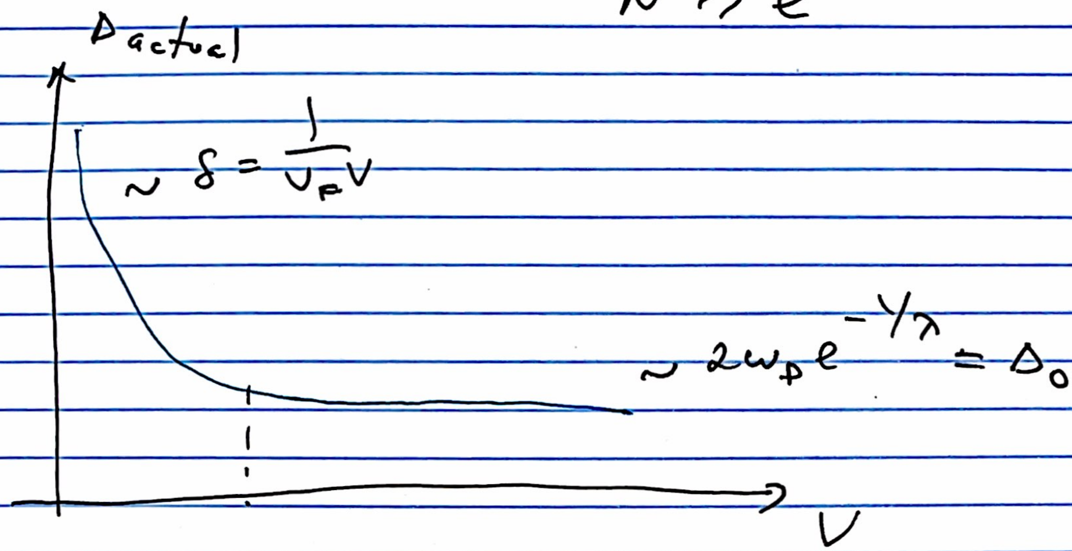
$$2\omega_D e^{-1/\lambda} = \delta$$



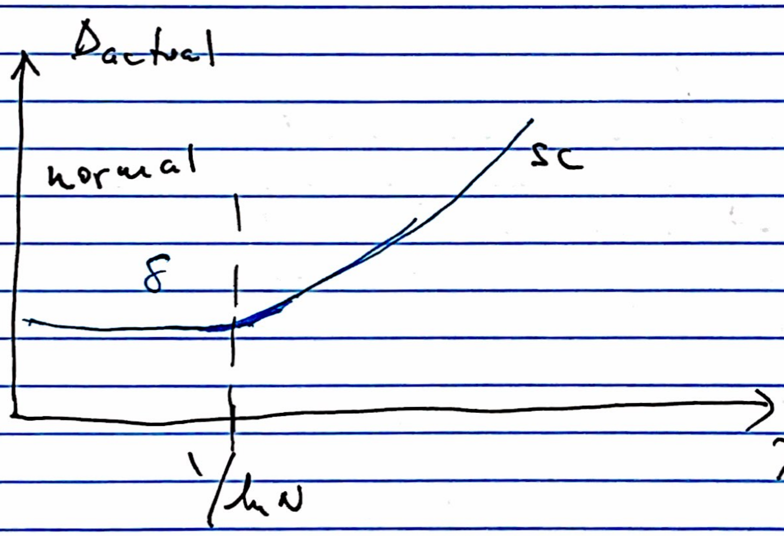
$\frac{2\omega_D}{\delta} = N$ $\qquad N e^{-1/\lambda} = 1$

$\lambda = \frac{1}{\ln N}$ Need $\lambda \gg \frac{1}{\ln N}$

$N \gg e^{1/\lambda}$



$\frac{1}{v_F V} = \Delta_0$ $V = \frac{1}{v_F \Delta_0}$



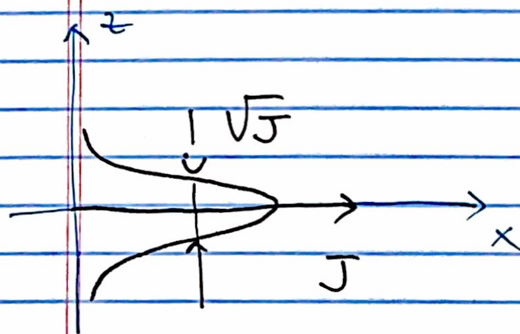
$N \gg 1$

We made an approximation $\sum_j \hat{\vec{k}}_j \to \langle \sum_j \hat{\vec{k}}_j \rangle$

Mean field. When is this approximation accurate?

When quantum fluctuations of $\hat{\vec{J}} = \sum_j \hat{\vec{k}}_j$ are significant.

Consider large quantum spin $\hat{\vec{J}}$ in a coherent state

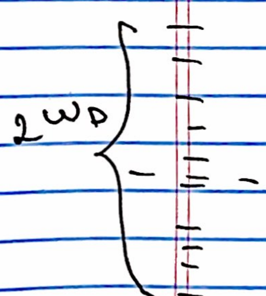


$$\langle \hat{\vec{J}} \rangle = J\hat{x}$$

$$\delta J_z \sim \sqrt{J}$$

Need $\frac{\sqrt{J}}{J} \ll 1 \Rightarrow J \gg 1$

In our case $J_{max} = \frac{N}{2}$

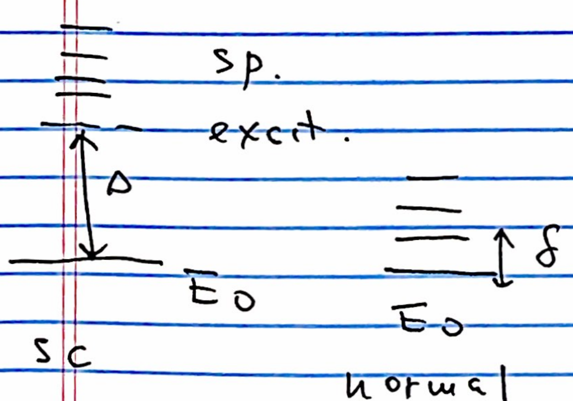


$$N = \frac{2\omega_D}{\delta} \qquad J_{max} = \frac{N}{2} = \frac{\omega_D}{\delta}$$

Need $N \gg 1$, i.e. $\omega_D \gg \delta$

Mf breaks down for $\omega_D \lesssim \delta$

Different approach (Anderson criterion)



sc & normal indistinguishable when $\Delta \approx \delta$

BCS mf breaks down for $\Delta \lesssim \delta$

Need $\Delta \gg \delta$

Second criterion more stringent bc $\omega_D \gg \Delta$

But both fullfilled when $N \to \infty$.

But do corrections go as δ/Δ or δ/ω_D

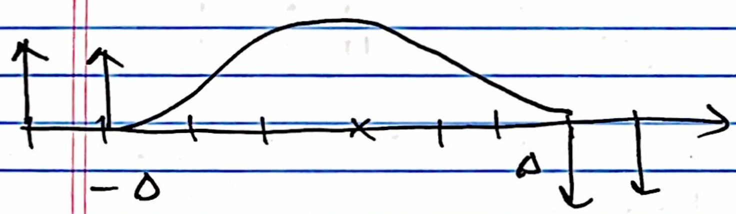
If δ/ω_D, BCS theory would be exact $\forall\, \delta$

bc (recall) we need to take the limit $\omega_D \to \infty$

Keeping δ/ω_D corrections would anyway go beyond

Recall $\langle k_j^x \rangle = \dfrac{\Delta}{2\sqrt{\varepsilon_j^2 + \Delta^2}}$

Only spins w $|\varepsilon_j| \sim \Delta$ can be $\parallel x$ and form a large spin



$$N_{eff} = \frac{2\Delta}{\delta} \qquad J_{eff} = \frac{\Delta}{\delta} \Rightarrow \text{Need } \Delta \gg \delta$$

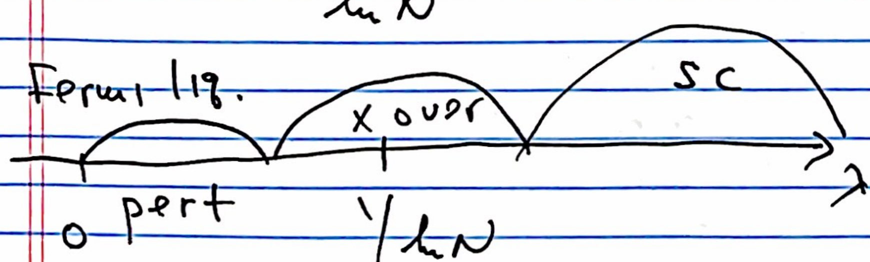
not $\omega_D \gg \delta$

Anderson criterion $\Delta \simeq \delta$ Anderson (1959) J. Phys. Chem. Solids

$$\lambda = \frac{1}{\ln N} \qquad 2\omega_D e^{-1/\lambda} = \delta$$

$$e^{1/\lambda} = \frac{2\omega_D}{\delta} = N$$

Need $\lambda \gg \dfrac{1}{\ln N}$



How to quantitavily access the accuracy of BCS mean field?

How not to do it.

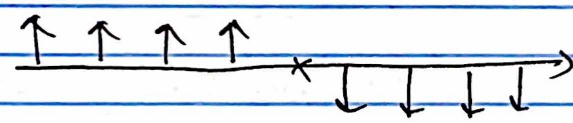
Schechter, Delft, Imry, Levinson, PRB (2003)

"Two pairing parameters in SC grains"

Condensation energy! $E_{cond} = E_0(\lambda) - E_0(0)$

1) BCS mean field $\quad E_{cond} = -\dfrac{\Delta^2}{2\delta}$

2) Perturbation theory around Fermi gr. st $= |F\rangle$

$|F\rangle =$ 

$$E_0(\lambda) - E_0(0) = \langle F | H_{BCS}^{1st} | F \rangle$$

$$H_{BCS}^{1st} = -g \sum_j b_j^{+} b_i^{-}$$

$$E_{cond} = -g \frac{N}{2} = -g \frac{1}{2} \frac{2\omega_D}{\delta} = -\lambda \omega_D$$

Mean field breaks down when

$$\lambda \omega_D \sim \frac{\Delta^2}{2\delta}$$

$$\Delta^2 \sim \omega_D \delta$$

Two parameters

$$\delta_1 \sim \Delta \qquad\qquad \delta_2 \sim \frac{\Delta^2}{\omega_D} = \Delta \frac{\Delta}{\omega_D} << \Delta$$

Anderson

much larger size

$$\lambda \sim \frac{2}{\ln N}$$

1st param 2nd param

$$\Delta_1 = \Delta \quad \text{vs.} \quad \sqrt{\omega_D \delta} = \Delta_2$$

This is wrong bc Δ_2 doesn't survive $\omega_D \to \infty$ limit

E_{cond} cannot be reliably evaluated within BCS theory beyond the mean field result

How to do it properly

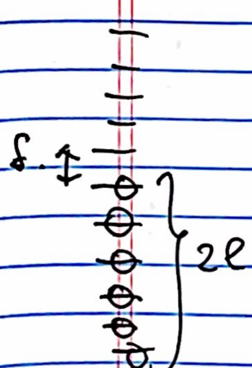
Parity effect. Matveev-Larkin parameter (PRL, 1997)

$$\Delta_p = E_{2\ell+1} - \frac{1}{2}\left(E_{2\ell} + E_{2\ell+2}\right)$$

E_{N_e} – gr. st. energy with N_e electrons

Fermi liquid

$$E_{2\ell+1} = E_{2\ell} + \delta$$



$$E_{2\ell+2} = E_{2\ell} + 2\delta$$

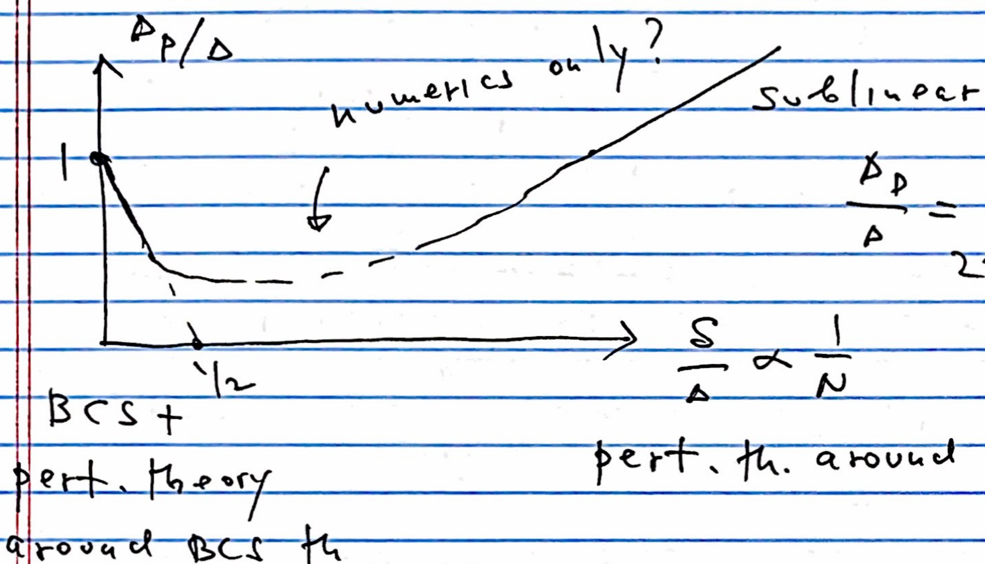
$$\Rightarrow \Delta_p = 0$$

BCS theory ($N \gg 1$)

$$E_{2e+1} = E_{2e} + \Delta \qquad E_{2e+2} = E_{2e} + O\left(\frac{1}{N}\right)$$

$$\delta \sim \frac{1}{N}$$

$$\Delta_p \simeq \Delta = 2\omega_D e^{-1/\lambda}$$



$$\frac{\Delta_p}{\Delta} = \frac{\delta/\Delta}{2\ln(\delta/\Delta)} = \frac{x}{2\ln x}$$

$$\frac{\delta}{\Delta} \propto \frac{1}{N}$$

pert. th. around Fermi liquid

$\Delta \gg \delta$ large grain $\qquad \Delta \ll \delta$ ultrasmall

$$\Delta_p = \Delta - \delta/2 \qquad \Delta_p = \frac{\delta}{2\ln \delta/\Delta} \ll \delta$$

$\frac{\delta}{\Delta} \to 0$ (BCS limit) $\Delta_p \to \Delta$

$\frac{\delta}{\Delta} \to \infty$ (FL limit) $\Delta_p \to 0$

$\frac{\Delta_p}{\delta} \to 0$ but $\frac{\Delta_p}{\Delta} \to \infty$ bc $\Delta \to 0$

How to treat the intermediate regime

$\delta \sim \Delta$?

Both perturbation theories break down!

Nonperturbative regime. Nothing but numerics unless the model is integrable.

What does it mean <u>integrable</u>?

CM first. Consider Hamiltonian sys w n degrees of freedom $H(q, p)$ $\quad q = q_1, \dots, q_n$

2n-dim phase space

<u>Def</u> $H(q,p)$ is said to be integrable if it possesses ~~has~~ n functionally indep integrals of motion $I_k(q,p)$ in involution: $\{I_k(q,p), I_j(q,p)\} = 0$

1) Integral of motion

$$\frac{dI_k}{dt} = 0$$

$$\frac{dI_k}{dt} = \{H, I_k\}$$

2) Poisson bracket $\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$

Obvious integral: $I_1 = H(q,p)$ energy conserv.

Functional independence: $I_k \neq f(I_{j \neq k})$

Ex. $I_2 = I_3^2 I_1 + I_4$ – functionally dependent

Liouville – Arnold theorem:

If $H(p, q)$ is integrable, then

1) if motion is bounded, trajectory $\in$ manifold $\equiv$ $\equiv$ (difeomorphic) n-torus

In other words,

$$\text{Manifold}\{I_k(q, p) = \underset{\overset{\parallel}{c_k}}{\text{const}_k}\} = n\text{-torus}$$

2) In ~~proper~~ appropriate ~~canonical~~ coord EOM reduce to

$$\frac{d\varphi_i}{dt} = \omega_i(c_1, \dots, c_n)$$

φ_i – angles on the torus

torus = circle$_1$ × circle$_2$ × ... × circle$_n$

3) EOMs are exactly solvable – integrable in quadratures. Solving EOMs reduce to taking 1D integrals and inverting functions

$\Rightarrow$ Integrable ($\exists I_n$) $\Rightarrow$ exactly solvable

Moreover 0) Integrability and properties 1), 2), & 3) are equivalent, i.e., from any one of them the rest follow

Autonomous, Hamiltonian system, classical

$$\frac{\partial H}{\partial t} = 0 \qquad H(q, p)$$

$$q = q_1, \dots, q_n$$

n degrees of freedom

Def $H(q,p)$ is integrable if $\exists$ n functionally indep. integrals of motion $I_k(q,p)$ $k=1,\dots,n$ in involution

$$\{I_k, I_j\} = 0 \quad \forall\, k, j$$

Recall: $\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right)$

Another way to define Poisson brackets is axiomatically

1) $\{A, B\} = -\{B, A\}$ 2) $\{AB, C\} = A\{B, C\} + \{A, C\}B$

3) Jacobi identity 4) ...

Then, introduce fundamental Poisson brackets

$$\{q, p\} = 1 \;/\; \{q, q\} = \{p, p\} = 0$$ Follows from antisym

$$\{q^2, p\} = 2q \qquad \{f(q), p\} = \frac{\partial f}{\partial q}$$

$$\{A(q,p), B(q,p)\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

i.e., all other brackets follow.

- Advantages of this approach: more general

Ex. classical spin: $\vec{S} = (S_x, S_y, S_z)$

Fundamental brackets $\{S_x, S_y\} = S_z$ & cyclic permut.

Many spins: $\{S_i^x, S_j^y\} = \delta_{ij} S_i^z$

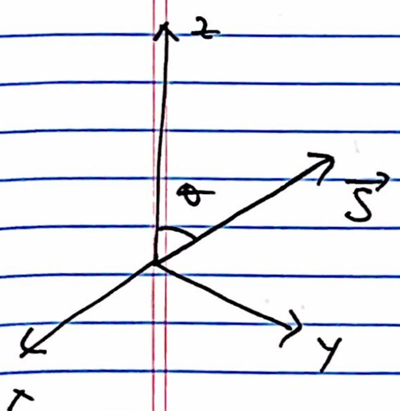
All other brackets follow, e.g., $\{S_x, \vec{S}^2\} = 0$

Same as angular momentum $\vec{L} = m\vec{r} \times \vec{p}$

$\{L_x, L_y\} = L_z \leftarrow$ can derive from $\{x, p_x\} = 1 \ldots$

- spin = 1 3D particle = 3 degrees of freedom?

Not so! 1 spin = 1 degree of freedom



$S_z = S\cos\theta$

$S_x = \cancel{S\sin\theta}\; S\sin\theta \cos\varphi$

$S_y = S\sin\theta \sin\varphi$

Let $q = \varphi/\sqrt{S}$ $p = \cos\theta/\sqrt{S}$

Fundamental bracket: $\{\varphi, \cos\theta\} = 1/S$

$$\{S_x, S_z\} = S^2\{\sin\theta\cos\varphi, \cos\theta\} =$$

$$= S^2 \sin\theta \{\cos\varphi, \cos\theta\} = S^2 \sin\theta(-\sin\varphi)\frac{1}{S} = -S_y$$

Functional independence is crucial.

Note: $f(I_1, \dots, I_n)$ is also an integral

Involution isn't crucial I believe

EOM for dynamical variable $A(q,p)$

$$\frac{dA}{dt} = \{H, A\}$$

Integral of motion

$$\frac{dI_k}{dt} = 0 \iff \{H, I_k\} = 0$$

$$I_1 \equiv H$$

Liouville - Arnold thm

Suppose $H(q,p)$ is integrable. Then

1) Bound trajectories $\in$ manifold $\equiv$ n-torus

In other words

Manifold $\{I_k(q,p) = C_k, \ k=1,\dots,n\} \equiv$ n-torus

2) EOM reduce to

$$\frac{d\varphi_i}{dt} = \omega_i(C_1, \dots, C_n) \qquad i = 1, \dots, n$$

$$\varphi_i - \text{n-torus} = \underbrace{C \times C \times \dots \times C}_{\text{n circles}}$$

3) EOM are exactly solvable in quadratures.
(Solving EOM reduces to taking 1D integrals and inverting functions)

Consequence: Integrable ($\exists I_\alpha$) $\Longleftrightarrow$ Exactly solvable

Ex 1: 1D motion in a pot'l $V(x)$

1 degree of freedom, 1 integral $H(x,p)$

$\Rightarrow$ integrable

$$\frac{m\dot{x}^2}{2} + V(x) = E$$

$$\int \frac{dx}{\sqrt{\frac{2}{m}(E - V(x))}} = t$$

1D integral (quadrature). Need to 1) take the integral $F(x) = t + \text{const}$

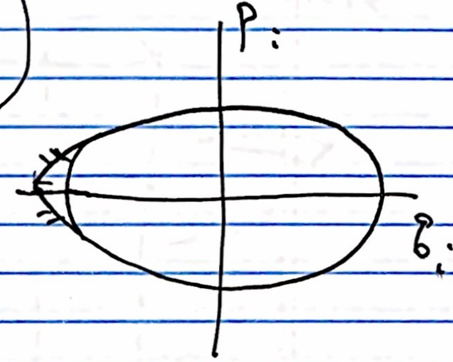
2) invert $x(t) = F^{-1}(t + \text{const})$

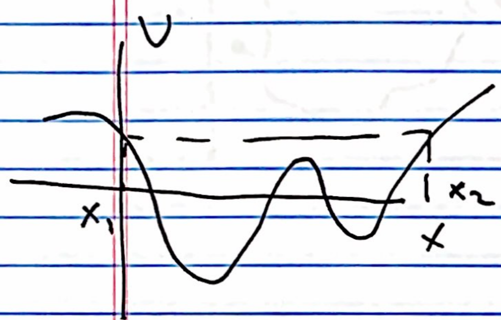
In the many-particle case need to invert ~~inert~~ multivariable functions. Very difficult and not practicle. Resort to particular solutions. But integrability tells us the general features of the motion (quasiperiodic etc.)

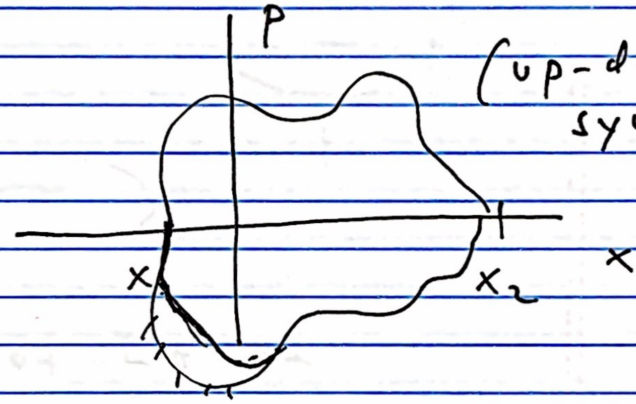
Eeach integral effectively provides one circle.

Decoupled example: n harmonic oscill

$$H = \sum_{i=1}^{n} \underbrace{\left(\frac{p_i^2}{2m_i} + \frac{m_i \omega_i^2 q_i^2}{2}\right)}_{I_i}$$







(up-down symm)

<u>Ex 2</u> Central pot'l $V(r)$

$H, \vec{L}$ 4 integrals?

Only 3 in involution $H, \vec{L}^2, L_z$ - integrable

$$\frac{m\dot{r}^2}{2} + V(r) + \frac{L^2}{2mr^2} = E$$

2D motion

r, φ

H, L^2

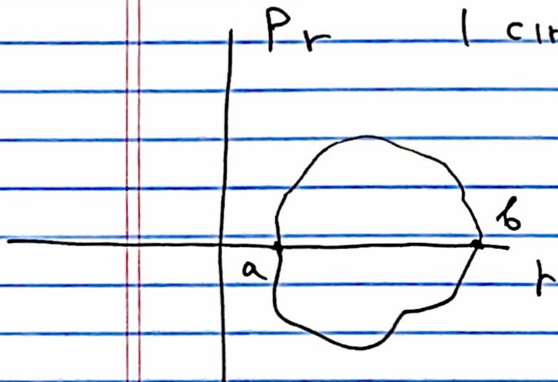
$\parallel$

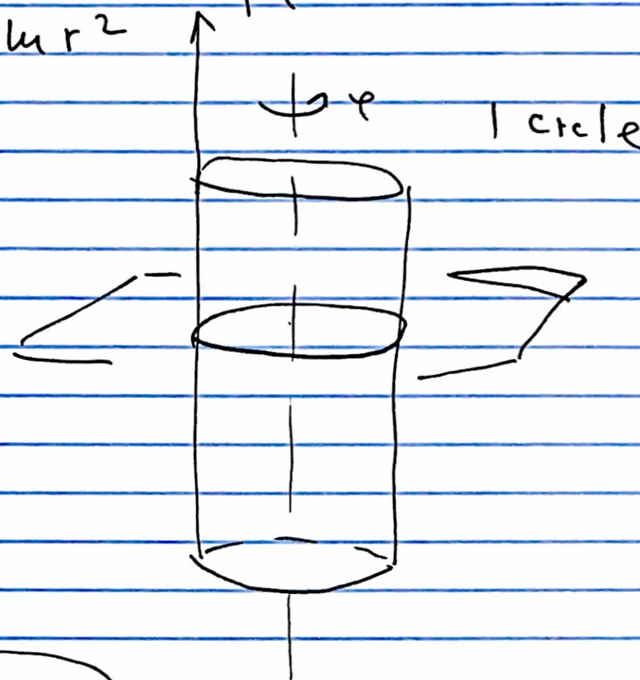
E

$$\int \frac{dr}{\sqrt{\cdots}} = t$$

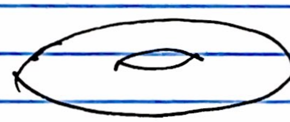
$$mr^2\dot{\varphi} = L$$

$$\varphi = \int \frac{L\,dt}{mr^2} \qquad p_\varphi = L = \text{const}$$





2-torus



Idea of the proof of LA thm

variable change $(q, p) \to (I, \underbrace{\;\;}_{})$ $\hookrightarrow A_1 \dots A_n$

$$I_k(q,p) = I_k \qquad\qquad q = q(I, A)$$
$$A_k(q,p) = A_k \quad \Longrightarrow \quad p = p(I, A)$$

In general, this is not a canonical transformation, i.e., it doesn't preserve Hamiltonian EOM

Canonical transformation $I \to J(I)$

New Integrals $J_1, \dots J_n$ (alternative set)

Now can make a canonical transformation

$$(q, p) \to (J, \varphi)$$

Hamiltonian is a function of $J_1, \dots, J_n$

New Hamiltonian EOM $\qquad H(J)$

$$\dot{J} = \frac{\partial H}{\partial \varphi} = 0 \qquad \dot{\varphi} = \frac{\partial H}{\partial J}$$

$$\dot{\varphi}_i = \frac{\partial H(J_1, \dots J_n)}{\partial J_i} = \omega_i(I_1, \dots, I_n)$$

$$\varphi_i = \omega_i t + \text{const}$$

If motion is bounded φ_i must be compact, i.e., an angle

Integrable motion (bounded) is quasiperiodic with n frequencies

$$q = q(J, \varphi_1, \dots, \varphi_n)$$

$$q(\varphi_i + 2\pi) = q(\varphi_i)$$

$$T_i = \frac{2\pi}{\omega_i}$$

Trajectories don't diverge (confined to n-torus) — no chaos

Many-body integrable systems

General CM integrability (in the Liouville sense)

$\cup$

Lax method (zero curvature)

$||$

Inverse scattering method (ISM)

$\downarrow$

QISM = BA

Nonlinear PDEs, soliton physics

Korteweg - de Vries (KdV) eq. Waves on shallow water

$$u_t + u_{xxx} + u u_x = 0$$

Classical Hamiltonian sys with ∞ degrees of freedom, classical field th (continuum limit)

$$\mathcal{H}(u) = \frac{u^3}{6} + \frac{u u_{xx}}{2}$$

One-soliton solution $\quad u = f(x - ct)$

$$u_t = -c\, u_x = -c f'$$

$$-c f' + f''' + f f' = 0$$

$$-c f + f'' + \frac{f^2}{2} = \text{const}$$

Multiply by f'. Let const $=0$ for simplicity

$$-cff' + \frac{f^2 f'}{2} + f'f'' = 0$$

$$-\frac{cf^2}{2} + \frac{f^3}{6} + \frac{(f')^2}{2} = \text{const}_1$$

$$u = -\frac{c}{2\sinh^2\left[\frac{\sqrt{c}}{2}(x-ct)\right]}$$

How to obtain more general solutions?

Idea: somehow reduce to a linear problem

$$\psi_{xx} + V(x,t)\,\psi = \lambda\psi$$

Similar to FT, but use some more suitable $\psi(x,t)$ instead of plane waves

More suitable $\Rightarrow$ $V(x,t) = V[u(x,t)]$

$\Rightarrow$ ISM – reconstruct the pot'l given the wf. Don't take the name too seriously

Peter David Lax (Hungarian American)

$$L\psi = \lambda\psi \qquad L = -6\frac{d^2}{dx^2} - u$$

Lax operator, Lax matrix

$\psi_t = M\psi$ $\quad\quad$ L, M – Lax pair

$$M = -4\frac{d^3}{dx^3} - u\frac{d}{dx} - \frac{1}{2}u_x$$

λ is time-indep

$$L_t\psi + L\psi_t = \lambda\psi_t$$

$$L_t\psi = \lambda\psi_t - L\psi_t = \lambda M\psi - LM\psi =$$

$$= ML\psi - LM\psi$$

$L_t = [M, L]$ Lax eq.

equivalent to KdV eq.

Now: $\psi(x, t_0) \xrightarrow{L\psi=\lambda\psi} u(x, t_0) \xrightarrow{\psi_t = M\psi} \psi(x, t_0+\Delta t) \rightarrow \ldots$

Systems with $n \gg 1$ particles

$$H = \sum_{j=1}^{n} \frac{p_j^2}{2} + g^2 \sum_{j<k} v(q_j - q_k)$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = p_j$$

Calogero–Moser model: $v(q_j - q_k) = \dfrac{1}{(q_j - q_k)^2}$

$$L_{jk} = p_j \delta_{jk} + \frac{ig(1-\delta_{jk})}{q_j - q_k}$$

$$L = \begin{pmatrix} p_1 & & (jk) \\ & \ddots & \\ & & p_n \end{pmatrix} \qquad (jk) \to \frac{ig}{q_j - q_k}$$

$$M_{jk} = g\,\delta_{jk} \sum_{\ell \neq j} \frac{1}{(q_j - q_\ell)^2} - \frac{g(1-\delta_{jk})}{(q_j - q_k)^2}$$

Lax eq $\quad i\dot{L} = [M, L]$

Similar to quantum Hamiltonian EOM

$$\hat{O}(t) = \hat{U}(t)\,\hat{O}(0)\,\hat{U}^{+}(t), \qquad \hat{H} = i\dot{\hat{U}}\,\hat{U}^{+}$$

In nonunitary case, $\hat{U}^{+} \to \hat{U}^{-1}$

Lax eq – similarity transform, i.e.

$$L(t) = U(t)\,L(0)\,U^{-1}(t) \quad \text{with} \quad M = i\dot{U}U^{-1}$$

$\Rightarrow$ eigenvalues of L are time-indep

$$L\psi = \lambda\psi \qquad U L U^{-1}(U\psi) = \lambda(U\psi)$$

$\Rightarrow$ they are integrals of motion

More convenient: $\quad I_k = \mathrm{tr}(L^k)$

For int sys often in addition to some general approach (ISM here) other parallel and simple approaches arise. But they work for a particular sys or a class of sys only.

For Calogero-Moser, consider $n \times n$ matrix linear in t, $x(t) = at + b$

a, b – $n \times n$ Hermitian matrices

$$\Rightarrow \quad x(t) = U(t)\, Q\, U^{-1}(t), \qquad U^{+} = U^{-1} \qquad (1)$$

$$Q(t) = \begin{pmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{pmatrix}$$

Will interpret $q_j(t)$ as coordinates of the particles

Let $p_j = \dot{q}_j$

Let's derive EOM for q_j & p_j

$$(1) \Rightarrow \quad \dot{U} Q U^{-1} + U \dot{Q} U^{-1} - U Q U^{-1} \dot{U} U^{-1} = a$$

$$\frac{d(U^{-1})}{dt} = -U^{-1}\dot{U}U^{-1} \qquad \text{Let } M = -i U^{-1}\dot{U}$$

$$U \Big[\dot{Q} + \underbrace{U^{-1}\dot{U} Q - Q U^{-1}\dot{U}}_{i[M, Q]} \Big] U^{-1} = a$$

$$L = \dot{\phi} + i[M, \phi], \quad \underline{P} = \dot{\phi}, \quad (*)$$

$$U(t)\, L(t)\, U^{-1}(t) = a$$

Differentiating once more, we obtain the Lax eq

$$\dot{L} + i[M, L] = 0$$

Calogero – M is of the type $(*)$

$\Rightarrow$ particle coordinates are eigenvalues of a Hermitian matrix!

Go to a basis where b is diagonal $\Rightarrow$ $x(t)$ is diagonal @ $t=0$. $x(0) = b = \phi(0)$

$$\Rightarrow U(0) = I \Rightarrow a = L(0)$$

$$\Rightarrow x(t) = \underbrace{L(0)\,t + \phi(0)}_{\text{Fixed by the initial condition}}$$

For this reason, Calogero – M model is often used in RMT

Classical BCS

$$H = \sum_j 2\varepsilon_j S_j^z - g \sum_{j,k} S_j^+ S_k^-$$

$\vec{S}_j$ – vector of fixed length

$$\dot{\vec{S}}_j = \{H, \vec{S}_j\}$$

$$\{S_j^\alpha, S_k^\beta\} = \varepsilon^{\alpha\beta\gamma}\,\delta_{jk}\,S_j^\gamma$$

For $H = \vec{B}_j \cdot \vec{S}_j$, $\quad \dot{\vec{S}}_j = \vec{B}_j \times \vec{S}_j$

Recall: $\quad h_j = \vec{B}_j \cdot \vec{S}_j$

$$\vec{B}(\varepsilon_j) = \vec{B}_j = -2\vec{\Delta} + 2\varepsilon_j\hat{z} = -2g\,\vec{J}_\perp + 2\varepsilon_j\hat{z}$$

$$\Delta = g\sum_k S_k^- = \Delta_x + i\Delta_y \qquad \vec{\Delta} = (\Delta_x, \Delta_y)$$

$$\vec{J} = \sum_k \vec{S}_k \qquad \vec{J}_\perp = (J_x, J_y)$$

$$\dot{\vec{S}}_j = \left(-2g\,\vec{J}_\perp + 2\varepsilon_j\hat{z}\right) \times \vec{S}_j \qquad (2)$$

Lax formulation

$$L = \begin{pmatrix} \sum_j \frac{S_j^z}{u-\varepsilon_j} - \cancel{+}\frac{1}{g} & \sum_j \frac{S_j^+}{u-\varepsilon_j} \\ \sum_j \frac{S_j^-}{u-\varepsilon_j} & -\sum_j \frac{S_j^z}{u-\varepsilon_j} \cancel{+}\frac{1}{g} \end{pmatrix} \qquad M = \begin{pmatrix} 2\overset{u}{\cancel{\xi_j}} & 2\Delta^* \\ 2\Delta & -2\overset{u}{\cancel{\xi_j}} \end{pmatrix}$$

(2) is equivalent to $\qquad i\dot{L} = [M, L]$

Lax vector

$$\vec{L}(u) = -\frac{\hat{z}}{g} + \sum_j \frac{\vec{S}_j}{u - \varepsilon_j}$$

$$L = \begin{pmatrix} L_z & L^+ \\ L^- & L_z \end{pmatrix} = L_z \sigma_z + L_x \sigma_x + L_y \sigma_y =$$

$$= \vec{L} \cdot \vec{\sigma}$$

$$M = 2\varepsilon_j \sigma_z + 2\Delta_x \sigma_x + 2\Delta_y \sigma_y = \vec{B}(u) \cdot \vec{\sigma}$$

Lax eq $\Leftrightarrow$ $\dot{\vec{L}}(u) = \vec{B}(u) \times \vec{L}(u)$ (3)

$$\sum_j \frac{\dot{\vec{S}}_j}{u - \varepsilon_j} = \frac{\vec{B}(u) \times \hat{z}}{g} + \sum_j \frac{B(u) \times \vec{S}_j}{u - \varepsilon_j}$$

Comparing res @ $u = \varepsilon_j$ on both sides

$$\dot{\vec{S}}_j = \vec{B}_j \times \vec{S}_j$$

Integrals of motion: $I_k = \mathrm{tr}\, L^k$

$I_1 = 0$ Instead of $I_2 = \mathrm{tr}\, L^2$ take

$\det(L) = \vec{L}^2$. Indeed, (3) $\Rightarrow$ $\vec{L}^2(u)$ is conserved $\forall u$

Let's evaluate $\vec{L}^2(u)$

$$\vec{L}^2(u) = \left(-\frac{\hat{z}}{g} + \sum_j \frac{\vec{S}_j}{u-\varepsilon_j}\right)\left(-\frac{\hat{z}}{g} + \sum_k \frac{\vec{S}_k}{u-\varepsilon_k}\right) =$$

$$= \frac{1}{g^2} + \sum_j \frac{2H_j}{u-\varepsilon_j} + \sum_j \frac{\vec{S}_j^2}{(u-\varepsilon_j)^2}$$

$$2H_j = 2\sum_k \frac{\vec{S}_j \cdot \vec{S}_k}{\varepsilon_j - \varepsilon_k} - 2\frac{S_j^z}{g}$$

$$\frac{A_j f(u)}{u-\varepsilon_j} = \frac{A_j f(\varepsilon_j)}{u-\varepsilon_j}$$

$$H_j = -\frac{S_j^z}{g} + \sum_{k\neq j} \frac{\vec{S}_j \cdot \vec{S}_k}{\varepsilon_j - \varepsilon_k}$$

$\vec{L}^2(u)$ - conserved $\overset{\forall u}{\Rightarrow}$ H_j are conserved

$j = 1, \ldots, n$

n spins $\Rightarrow$ n degrees of freedom n integrals

$\Rightarrow$ H_{BCS} is classically integrable

(Can also show H_j are functionally indep

and $\{H_j, H_k\} = 0, \ \forall j, k$)

Let's confirm these are int. of motion of H_{BCS}

$$\sum_j H_j = -\frac{J_z}{g} \qquad \vec{J} = \sum_j \vec{S}_j$$

$$\sum_j 2\varepsilon_j H_j = -\frac{1}{g}\sum_j 2\varepsilon_j S_j^z + \sum_{\text{pairs}}\left[\frac{2\varepsilon_j \vec{S}_j\cdot\vec{S}_k}{\varepsilon_j-\varepsilon_k} + \frac{2\varepsilon_k \vec{S}_k\cdot\vec{S}_j}{\varepsilon_k-\varepsilon_j}\right] =$$

$$= -\frac{1}{g}\sum_j 2\varepsilon_j S_j^z + \underbrace{\sum_{j,k}\vec{S}_j\cdot\vec{S}_k}_{\left(\sum \vec{S}_j\right)^2 = J^2 = J_+J_- + J_z^2} - \sum_j \vec{S}_j^{\,2}$$

$$J_\pm = J_x \pm i J_y$$

$$\sum_j 2\varepsilon_j H_j = -\frac{1}{g} H_{BCS} + \text{const}$$

$$H_{BCS} = -g\sum_j 2\varepsilon_j H_j + \text{const}$$

H_{BCS} is a linear comb. of H_j. It's not a new (funct. indep) integral

$$\Rightarrow \{H_{BCS}, H_j\} = 0$$

Is this classical BCS relevant to any sc physics? We saw that BCS mf is exact for bulk sc. But mf $\equiv$ classical!

Let's see this. Recall:

$$\hat{h}_j = \hat{\vec{B}}_j \cdot \hat{\vec{S}}_j \qquad \text{Now } \hat{\vec{S}}_j - \text{spin-}1/2$$

$$\hat{\vec{B}}_j = -2\hat{\vec{\Delta}} + 2\varepsilon_j \hat{z}$$

$$\hat{\Delta} = g\sum_k \hat{S}_k^- = \hat{\Delta}_x + i\hat{\Delta}_y \qquad \hat{\Delta}_x = g\sum_k \hat{S}_k^x$$

Our Δ from before: $\Delta = \langle \hat{\Delta} \rangle$

mf: $\qquad \hat{\Delta} \to \Delta$

$$\hat{B}_j \to \langle \hat{B}_j \rangle = -2\vec{\Delta} + 2\varepsilon_j \hat{z}$$

$$\hat{h}_j = \vec{B}_j \cdot \hat{\vec{S}}_j$$

Consider spin (quantum) in magnetic field

$$\hat{h} = \vec{B} \cdot \hat{\vec{S}}$$

EOM $\qquad i\dfrac{d\vec{S}}{dt} = [\hat{h}, \vec{S}]$

$$[\vec{B}\cdot\vec{S}, \vec{S}] = i\vec{B}\times\vec{S}$$

$$\frac{d\hat{\vec{S}}}{dt} = \vec{B} \times \hat{\vec{S}} \quad \text{Bloch eqs} \quad (1)$$

Now for classical spin

$$h = \vec{B} \cdot \vec{S}, \quad \frac{d\vec{S}}{dt} = \{h, \vec{S}\} = \vec{B} \times \vec{S}$$

Bloch eqs have the same form in CM & QM

bc Poisson bracket & commutator for spin

is the same algebra

Also $\frac{d\hat{\vec{S}}}{dt} = \hat{\vec{B}} \times \hat{\vec{S}}$ ~~as long as~~ $\vec{B}, \vec{S}$

Take QM average of eq. (1)

$$\langle \hat{\vec{S}} \rangle = \langle \psi(t) | \hat{\vec{S}} | \psi(t) \rangle = \langle \psi(0) | \underbrace{\hat{U}^\dagger(t)\, \hat{\vec{S}}\, \hat{U}(t)}_{\hat{\vec{S}}(t)} | \psi(0) \rangle$$

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

$$\frac{d\langle \hat{\vec{S}} \rangle}{dt} = \langle \psi(0) | \frac{d\hat{\vec{S}}}{dt} | \psi(0) \rangle = \left\langle \frac{d\hat{\vec{S}}}{dt} \right\rangle$$

$$\frac{d\langle \hat{\vec{S}} \rangle}{dt} = \langle \vec{B} \times \hat{\vec{S}} \rangle = \vec{B} \times \langle \hat{\vec{S}} \rangle$$

$\langle \hat{\vec{S}} \rangle$ – classical spin!

$$\hat{H}_{BCS} = \sum_j 2\varepsilon_j \hat{S}_j^z - g \sum_{j,k} \hat{S}_j^+ \hat{S}_k^-$$

$\vec{S}_j$ – spin – 1/2

Terms involving $\vec{S}_j$: $\hat{h}_j = \hat{\vec{B}}_j \cdot \hat{\vec{S}}_j$

mf: $\hat{\vec{B}}_j \to \langle \hat{\vec{B}}_j \rangle = \vec{B}_j = -2\vec{\Delta} + 2\varepsilon_j \hat{z}$

$$\Delta = g \sum_k \langle S_k^- \rangle$$

EOM for $\hat{\vec{S}}_j(t)$:
$$\frac{d\hat{\vec{S}}_j}{dt} = \hat{\vec{B}}_j \times \hat{\vec{S}}_j$$
$$\downarrow \text{mf}$$
$$\vec{B}_j$$

$$\frac{d\langle \hat{\vec{S}}_j \rangle}{dt} = \vec{B}_j \times \langle \hat{\vec{S}}_j \rangle$$

$\langle \hat{\vec{S}}_j \rangle$ behave exactly as classical spins governed by classical BCS Hamiltonian

$$H_{BCS} = \sum_j 2\varepsilon_j S_j^z - g \sum_{j,k} S_j^+ S_k^-$$

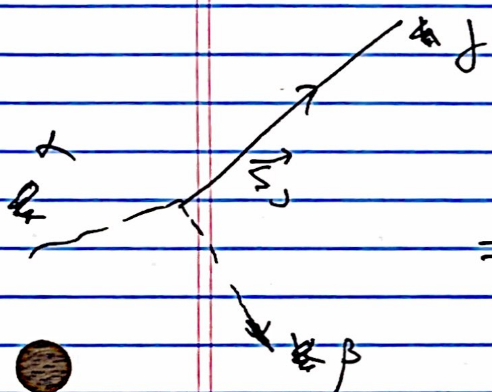
$$\vec{S}_j = \langle \hat{\vec{S}}_j \rangle$$

Consider a product wf: $\Psi(t) = \prod_j \big(u_j(t) |\uparrow\rangle + v_j(t) |\downarrow\rangle \big)$

Time-dep mf approx (not to be confused with DMFT)

Time-evolution starting from a non-stationary state, e.g., quantum quench

$\langle \vec{S}_j \rangle$ in a pure state $= \dfrac{\hat{n}}{2}$



$$\langle S_j^\gamma \rangle = \frac{1}{2} \qquad \langle S_j^\alpha \rangle = \langle S_j^\beta \rangle = 0$$

$$\Rightarrow \quad |\vec{S}_j| = \frac{1}{2}$$

$$\frac{d\vec{S}_j}{dt} = \vec{B}_j \times \vec{S}_j \qquad \vec{B}_j = -2\vec{\Delta} + 2\epsilon_j \hat{z}$$

$$\Delta_{x,y} = g \sum_k S_k^{x,y}$$

Nonlinear, integrable, Hamiltonian system