Applicability of RMT to QDs

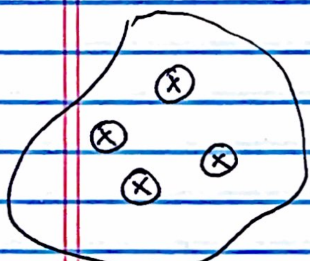
Recall: density – density correlation fn

$$K(\omega) = \langle \rho(\varepsilon)\,\rho(\varepsilon+\omega)\rangle - \langle \rho(\varepsilon)\rangle^2$$

Units $\delta = 1$

RMT $\langle \dots \rangle$: ensemble average

Diffusive QD $V(\vec{r})$ – disorder pot'l



$$\langle \dots \rangle = \frac{1}{\Delta E}\int d\varepsilon$$

$$\Delta E\, t \sim 1 \qquad (\hbar = 1)$$

$$k(t) = \int k(\omega)\, e^{-i\omega t}\, d\omega = t \iint d\varepsilon\, d\omega\, \rho(\varepsilon)\,\rho(\varepsilon+\omega)\, e^{-i\omega t} =$$

$$e^{-i\omega t} = e^{-i(\varepsilon+\omega)t}\, e^{+i\varepsilon t}$$

$$x = \varepsilon + \omega \qquad y = \varepsilon \qquad J = 1$$

$$= t \iint dx\, dy\, \rho(x)\, e^{ixt}\, \rho(y)\, e^{-iyt} = t\,\rho(t)\,\rho(-t)$$

$$\rho(\varepsilon) = \sum_n \delta(\varepsilon - E_n)$$

$$\rho(t) = \int d\varepsilon\, \rho(\varepsilon)\, e^{-i\varepsilon t} = \sum_n e^{-iE_n t}$$

$$Z(-it) = \mathrm{Tr}\left(e^{-i\hat{H}t}\right)$$

spectral form factor $Z Z^*$

$$|\psi\rangle = \sum_n |n\rangle$$

$$\hat{U}|\psi\rangle = e^{-iHt} \sum_n |n\rangle = \sum_n e^{-iE_n t}|n\rangle$$

$$\langle\psi|\hat{U}|\psi\rangle = \sum_n e^{-iE_n t} = \rho(t)$$

$$\rho(t)\rho(-t) = \langle\psi|\hat{U}|\psi\rangle\langle\psi|\hat{U}|\psi\rangle^* =$$

$$= |\langle\psi|\hat{U}|\psi\rangle|^2 = P(t)$$

Return probability (aka Loshmidt echo)

$$K(t) = t\,P(t)$$

$$K(\omega) = \int e^{i\omega t}\, t\, P(t)\, dt =$$

$$= -i\frac{\partial}{\partial\omega}\int e^{i\omega t} P(t)\, dt =$$

$$= -i\frac{\partial P(\omega)}{\partial\omega}$$

Semiclassical determination of $P(t)$

Diffusion:

$$\frac{\partial n}{\partial t} = D\nabla^2 n \qquad n = \sum_{\vec{q}} n_{\vec{q}}\, e^{i\vec{q}\cdot\vec{r}}$$

$$\frac{\partial n_{\vec{q}}}{\partial t} = -Dq^2 n_{\vec{q}}$$

$$n_{\vec{q}} = A_{\vec{q}}\, e^{-Dq^2 t}$$

Set $A_{\vec{q}} = 1$

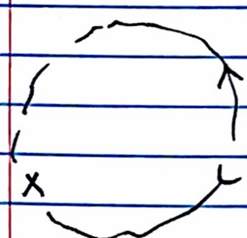
$$n = \sum_{\vec{q}} e^{-Dq^2 t}\, e^{i\vec{q}\cdot\vec{r}}$$

$$P(t) = n(\vec{r}=0, t) = \sum_{\vec{q}} e^{-Dq^2 t}$$

$$P(\omega) = \sum_{\vec{q}} \frac{1}{-i\omega + Dq^2}$$

$$K(\omega) = \frac{1}{\beta\pi^2}\sum_{q} \frac{1}{(-i\omega + Dq^2)^2}$$

$K_{\beta=1} = 2K_{\beta=2}$ due to time reversed pairs



Semiclassical approximation breaks down at the Heisenberg time

$$t_H = \frac{\hbar}{\delta} = \frac{1}{\delta}$$

That's when we see that spectrum is discrete

$$e^{-iE_n t} + e^{-i(E_n+\delta)t}$$

These two terms are no longer close when

$$t\delta \sim 1$$

Consider N equally spaced levels

$E_2 = E_1 + N\delta$

$$Z = \sum_{n=0}^{N} e^{-i(E_1 + n\delta)t}$$

E_1

$$Z(t=0) = N$$

Continuum limit

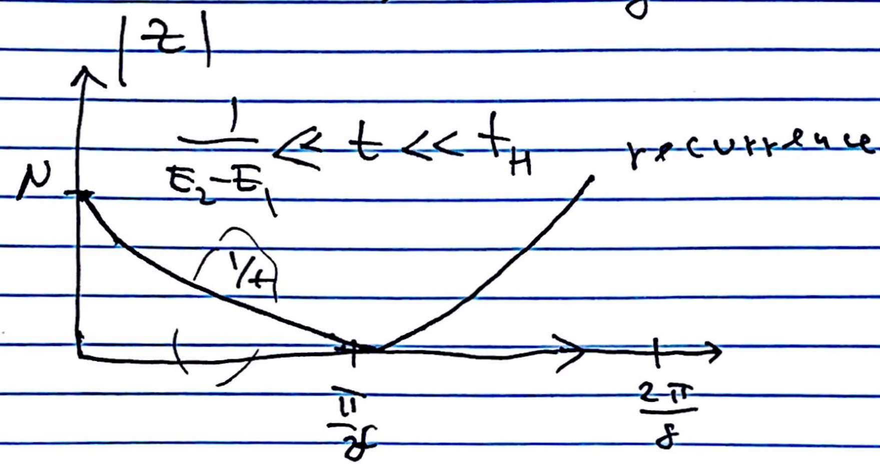
$$Z = \frac{1}{\delta}\int_{E_1}^{E_2} e^{-ixt}\,dx = \frac{1}{\delta}\,\frac{i\left[e^{-iE_2 t} - e^{-iE_1 t}\right]}{t}$$

$$\rightarrow 0 \quad \text{as } t \rightarrow \infty$$

Riemann-Lebesgue lemma

But... Z is periodic with period

$$\delta T = 2\pi \,, \quad T = \frac{2\pi}{\delta}$$

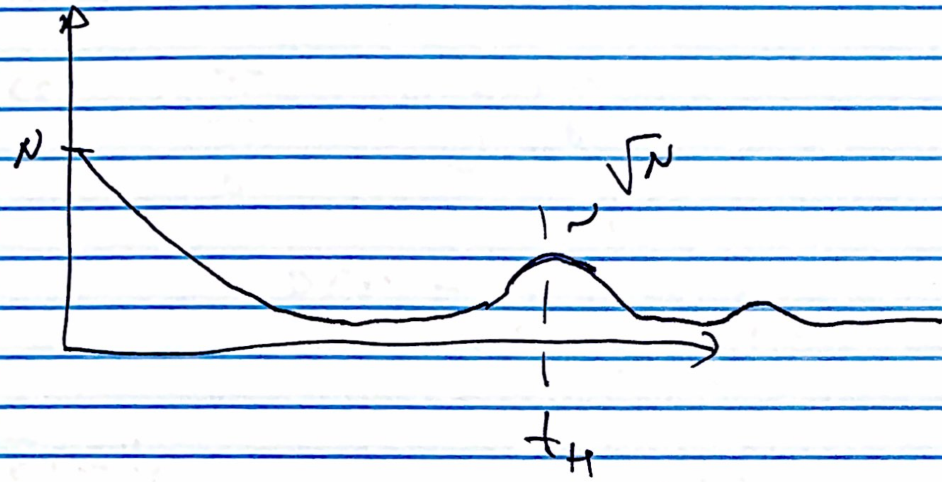


Heisenberg time is where recurrences occur

Dephasing / destructive interference

More realisticly ε_n – not equally spaced.

Behavior depends on statistics of ε_n



Back to $P(\omega)$, Consider first two terms

$$\frac{1}{(-i\omega)^2} + \frac{1}{-i\omega + Dq^2_{min}}$$

$q_{min} \sim \frac{1}{L}$ Suppose $Dq^2_{min} > \omega$

$\Rightarrow$ 1st term dominates

$$\Rightarrow \quad K(\omega) = -\frac{1}{\beta \pi^2 \omega^2}$$

$$\frac{Dq^2_{min}}{\delta} \sim \frac{1}{\delta}\frac{D}{L^2} = \frac{1}{\delta}\frac{1}{t_e} = E_T/\delta$$

(recall everything was normalized by δ)

$\frac{E_T}{\delta} > \omega$, $\omega < g_T$

Also, $t \ll t_H \Rightarrow \omega \sim \frac{1}{t} \gg \delta \Rightarrow \omega \gg 1$

Opposite limit, $L \to \infty$. Diffusion in infinite sys.

$$\sum_{\vec{q}} \longrightarrow \int d\vec{q}$$

$$P(t) = \frac{L^d}{(4\pi D t)^{d/2}}$$

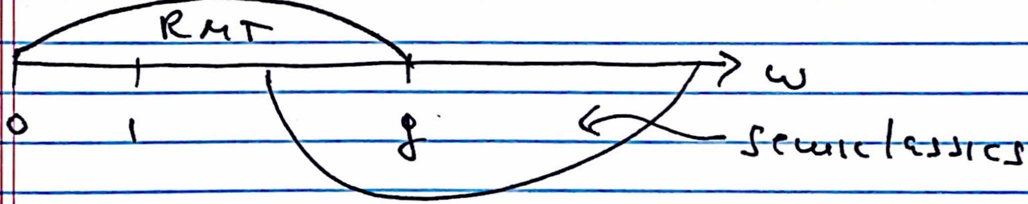
Finally

$$K(\omega) \simeq \begin{cases} \underbrace{-\frac{1}{\beta \pi^2 \omega^2}}_{\text{universal}}, & 1 \ll \omega < g_T \\ -\frac{1}{\beta \omega^2}\left(\frac{\omega}{g_T}\right)^{d/2} \cos\left(\frac{\pi d}{4}\right) & \omega \gg g_T \end{cases}$$

Recall for CUE from "Pechukas gas" construction

$\forall \omega$ $$K(\omega) = -\frac{\sin^2 \pi\omega}{\pi^2 \omega^2} + \delta(\omega) \langle \rho(\varepsilon) \rangle$$

for $\omega \gg 1$ $\sin^2 \pi\omega \approx \frac{1}{2}$ $K(\omega) \approx -\frac{1}{2\pi^2 \omega^2}$



RMT good for $g \gg 1$ – good metal

Weakly disordered (metallic) QD.

Thus, T-inv, no spin-orbit

$$\hat{H}_0 = \sum_\alpha \varepsilon_\alpha \hat{n}_\alpha$$

↑ eigenvalues of an $N \times N$ RM

$$|\varepsilon_\beta - \varepsilon_\alpha| < E_T \qquad N = E_T/\delta$$

Now let's add interactions

2-body, spin-indep. Most general form:

$$\hat{H}_{int} = \sum M^{\alpha\beta}_{\mu\nu}\, c^\dagger_{\alpha\sigma_1} c^\dagger_{\beta\sigma_2} c_{\mu\sigma_2} c_{\nu\sigma_1}$$

$$M^{\alpha\beta}_{\mu\nu} = \int d\vec{r}_1\, d\vec{r}_2\, V(\vec{r}_1 - \vec{r}_2)\, \varphi_\alpha(\vec{r}_1)\, \varphi_\beta(\vec{r}_2)\, \varphi_\mu(\vec{r}_2)\, \varphi_\nu(\vec{r}_1)$$

Two approaches: qualitative & RMT

1) Qualitative.

Suppose int. are short range

$$V(\vec{r}_1 - \vec{r}_2) = \frac{\lambda}{\nu_F}\,\delta(\vec{r}_1 - \vec{r}_2)$$

$$M^{\alpha\beta}_{\mu\nu} = \frac{\lambda}{\nu_F}\int d\vec{r}_1\ \varphi_\alpha \varphi_\beta \varphi_\mu \varphi_\nu$$

~~Arrrr~~ E_F $\quad \lambda_F \sim \frac{1}{a}$ $\quad$ Expect $M = 0$

except $\quad \underbrace{M^{\alpha\alpha}_{\mu\mu}}_{\beta\ \beta} = M^{\alpha\beta}_{\alpha\beta} = M^{\alpha\beta}_{\beta\alpha} = M$

In all 3 cases $\quad \int d\vec{r}_1\ \varphi_\alpha^2 \varphi_\beta^2 = \frac{1}{V}$

$$\int d\vec{r}\ \varphi_\alpha^2 = 1 \qquad \varphi_\alpha^2 \approx \frac{1}{V}$$

$$M = \frac{\lambda}{\nu_F V} = \lambda\,\delta_\xi$$

2) RMT approach

$$\langle M^{\alpha\beta}_{\mu\nu} \rangle \sim \langle \varphi_\alpha(i)\, \varphi_\beta(j)\, \varphi_\mu(j)\, \varphi_\nu(i) \rangle$$

$\langle \varphi_\alpha(i)\, \varphi_\beta(j) \rangle$ – must be rotationally inv

cf. $\vec{a}$ & $\vec{b}$ $\qquad a_x b_y$ – not inv

Must have $\langle \varphi_\alpha(i)\varphi_\beta(j)\rangle \propto \delta_{ij}$

$$\sum_i \varphi_\alpha(i)\varphi_\beta(i) = \delta_{\alpha\beta}$$

$$\Rightarrow \langle \varphi_\alpha(i)\varphi_\beta(i)\rangle = \frac{\delta_{\alpha\beta}}{N}$$

$$\Rightarrow \langle \varphi_\alpha(i)\varphi_\beta(j)\rangle = \frac{1}{N}\delta_{\alpha\beta}\delta_{ij}$$

continuous case $\langle \varphi_\alpha(\vec{r}_1)\varphi_\beta(\vec{r}_2)\rangle = \delta_{\alpha\beta}\,\delta(\vec{r}_1-\vec{r}_2)$

Only bilinears $\langle \varphi_\alpha(i)\varphi_\beta(i)\rangle$ are nonzero

Similarly $\langle M^{\alpha\beta}_{\mu\nu}\rangle \neq 0$ only when indicies are pairwise equal

Let's allow $\underset{= A_1}{M^{\alpha\alpha}_{\beta\beta}}$ $\underset{= A_2}{M^{\alpha\beta}_{\alpha\beta}}$ $\underset{= A_3}{M^{\alpha\beta}_{\beta\alpha}}$ to be unequal

1) $M^{\alpha\alpha}_{\beta\beta}$:

$$A_1 C^+_{\alpha\sigma_1} C^+_{\alpha\sigma_2} C_{\beta\sigma_2} C_{\beta\sigma_1}$$

Must have $\sigma_1 \neq \sigma_2$ $\sigma_1 = \uparrow$ $\sigma_2 = \downarrow$

$$A_1 \sum_{\alpha,\beta} C^+_{\alpha\uparrow} C^+_{\alpha\downarrow} C_{\beta\downarrow} C_{\beta\uparrow}$$

Let $\hat{L} = \sum_\beta c_{\beta\downarrow} c_{\beta\uparrow}$

$A_1 \hat{L}^+ \hat{L}$

2) $M^{\alpha\beta}_{\alpha\beta}$: $A_2 \, c^+_{\alpha\sigma_1} c^+_{\beta\sigma_2} c_{\alpha\sigma_2} c_{\beta\sigma_1}$

$M^{\alpha\beta}_{\beta\alpha}$ $-A_2 \, c^+_{\alpha\sigma_1} c_{\alpha\sigma_2} c^+_{\beta\sigma_2} c_{\beta\sigma_1}$

$\sigma_1 = \sigma_2$ $n_{\alpha\downarrow} n_{\beta\downarrow} + n_{\alpha\uparrow} n_{\beta\uparrow}$ $S^+_\alpha S^-_\beta$

$\sigma_1 \neq \sigma_2$ $c^+_{\alpha\uparrow} c_{\alpha\downarrow}$ need also $S^z_\alpha S^z_\beta$

$$\underbrace{2 S^z_\alpha}_{\shortparallel}$$

$(n_{\alpha\uparrow} + n_{\alpha\downarrow})(n_{\beta\uparrow} + n_{\beta\downarrow}) + (n_{\alpha\uparrow} - n_{\alpha\downarrow})(n_{\beta\uparrow} - n_{\beta\downarrow}) =$

$= 2(n_{\alpha\uparrow} n_{\beta\uparrow} + n_{\alpha\downarrow} n_{\beta\downarrow})$

$$n_{\alpha\uparrow} n_{\beta\uparrow} + n_{\alpha\downarrow} n_{\beta\downarrow} = \frac{n_\alpha n_\beta}{2} + 2 S^z_\alpha S^z_\beta$$

Altogether,

$$\underbrace{\text{const}_1}_{E_c} \sum_{\alpha,\beta} n_\alpha n_\beta + \overbrace{\text{const}_2}^{-J} \sum_{\alpha,\beta} \vec{S}_\alpha \cdot \vec{S}_\beta$$

$\hat{N} = \sum_\alpha n_\alpha$ total # of fermions

$\vec{S} = \sum_\alpha \vec{S}_\alpha$ their total spin

Universal Hamiltonian

$$\hat{H}_{univ} = \sum_\alpha \varepsilon_\alpha n_\alpha - g \underbrace{\sum c^+_{\alpha\uparrow} c^+_{\alpha\downarrow} c_{\beta\downarrow} c_{\beta\uparrow}}_{\hat{L}^+ \hat{L}^-} +$$

$$+ E_c \hat{N}^2 - J (\hat{\vec{S}}^2)$$

charging energy cf. $\frac{Q^2}{2C} = \frac{e^2}{2C} \hat{N}^2$

$E_c = \frac{e^2}{2C}$. Coulomb blockade

$-g \hat{L}^+ \hat{L}^-$: sc (pairing) int., BCS pairing

$$H_{BCS} = \sum_\alpha \varepsilon_\alpha n_\alpha - g \hat{L}^+ \hat{L}^-$$ BCS Hamilt.

$J \vec{S}^2$ - spin exchange int

$\hat{L}$, $\vec{S}$ & $\hat{N}$ - the only quadratic opts inv under ortho transforms

Can also add terms linear in $\hat{L}$, $\vec{\hat{S}}$ & $\hat{N}$

$$\hat{H}_{univ} \to \hat{H}_{univ} + V_g \hat{N} + \vec{B}\cdot\vec{\hat{S}} + \underbrace{a\hat{L}^{+} + a^{*}\hat{L}}_{\text{pair tunneling}}$$

gate voltage

$(B\hat{S}_z)$

Zeeman field

Could ~~also~~ also have higher order int

$\hat{N}^3$, $\hat{N}(\vec{B}\cdot\vec{\hat{S}})^2$ etc.

3-body

Occurs in nuclear physics. Higher seniority int

Kurland, Aleiner, Altshuler, PRB (2000).

$$H_{BCS} = \sum_{\alpha} \varepsilon_\alpha n_\alpha - g \sum_{\alpha,\beta} c^{+}_{\alpha\uparrow} c^{+}_{\alpha\downarrow} c_{\beta\downarrow} c_{\beta\uparrow}$$

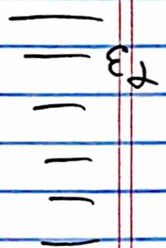
$N \times N$ matrix $\quad |\varepsilon_\alpha - \varepsilon_\beta| < E_T \quad N \approx \frac{E_T}{\delta}$

1st sum $\sim N$ $\qquad$ 2nd sum $\sim N^2$

Must have $g \sim \frac{1}{N}$. Also, $[g] =$ energy

Take $g = \lambda\delta$ dimensionless BCS coupling

Universal Hamiltonian



$$H_{univ} = \underbrace{\sum_{\alpha} \varepsilon_\alpha n_\alpha - g \sum_{\alpha,\beta} c^{+}_{\alpha\uparrow} c^{+}_{\alpha\downarrow} c_{\beta\downarrow} c_{\beta\uparrow}}_{H_{BCS}} +$$

$$+ \cancel{E_c \hat{N}^2} - J\left(\vec{\hat{S}}\right)^2$$

$$g = \lambda \delta = \frac{\lambda}{\nu_F V}$$

λ – dimensionless BCS coupling

$$H_{full} = H_{univ} + O\left(\frac{1}{g_T}\right) \qquad g_T = \frac{E_T}{\delta}$$

In general, small doesn't mean negligible, but terms in H_{univ} capture all main effects/instabilities

Isolated dot, $\hat{N} = \text{const}$, drop $E_c \hat{N}^2$

Original BCS Hamiltonian

Bardeen, Cooper, Schrieffer, "Theory of sc",
Phys. Rev. (1957)

$$\hat{H}_{BCS} = \sum_{\vec{k}} \varepsilon_k n_k - g \sum_{\vec{k},\vec{p}} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} c_{-\vec{p}\downarrow} c_{\vec{p}\uparrow}$$

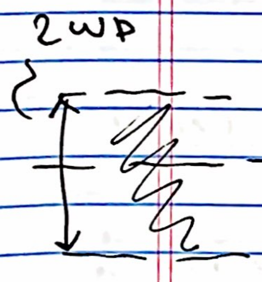
Sometimes also called "reduced BCS Ham."

full (BCS) Hamiltonian:

$$\hat{H}_{full} = \sum_{\vec{k}} \varepsilon_k n_k - \sum_{\vec{k},\vec{p},\vec{q}} g(\vec{q})\, c^\dagger_{\vec{k}+\vec{q}\uparrow} c^\dagger_{-\vec{k}\downarrow} c_{-\vec{p}\downarrow} c_{\vec{p}+\vec{q}\uparrow}$$

$\varepsilon_k = \frac{k^2}{2m}$ $\qquad\qquad$ $\vec{q}$ = C.o.M. momentum

Only $\vec{q} = 0$ terms important for equil. sc.



Can also take $g(\vec{q}) = \text{const} = g$
in this energy window

Then, $\hat{H}_{full} \longrightarrow \hat{H}_{BCS}$

$\hat{H}_{full}$: fermions w δ-fn attraction

$$V(\vec{r}_1 - \vec{r}_2) = \underbrace{\lambda}_{\nu_F V} \delta(\vec{r}_1 - \vec{r}_2)$$

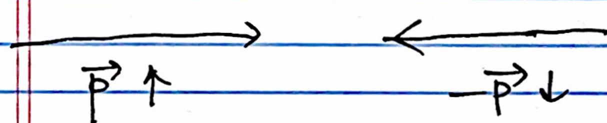
$$\hat{H} = \sum_{\sigma} \int d\vec{r}\ \hat{\psi}^{+}_{\sigma}(\vec{r}) \left[-\frac{\nabla^2}{2m} \right] \hat{\psi}_{\sigma}(\vec{r}) \ -$$

$$- g \int d\vec{r}\ \hat{\psi}^{+}_{\uparrow}(\vec{r})\, \hat{\psi}^{+}_{\downarrow}(\vec{r})\, \hat{\psi}_{\downarrow}(\vec{r})\, \hat{\psi}_{\uparrow}(\vec{r})$$

$$\iint d\vec{r}_1\, d\vec{r}_2\ \hat{\psi}^{+}_{\sigma_1}(\vec{r}_1)\, \psi_{\sigma_2}(\vec{r}_2)\, V(\vec{r}_1 - \vec{r}_2)\, \hat{\psi}_{\sigma_2}(\vec{r}_2)\, \hat{\psi}_{\sigma_1}(\vec{r}_1)$$

Since $\vec{r}_1 = \vec{r}_2$, must have $\sigma_1 \neq \sigma_2$

Back to reduced BCS



T-reversed pairs of fermions interact

$$T\,|\vec{p}\uparrow\rangle = |-\vec{p}\downarrow\rangle$$

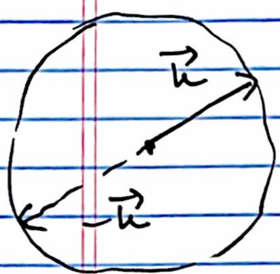
What to do when momentum isn't conserved?

Anderson, "Theory of dirty sc", J. Phys. Chem. Solids (1959)

Always pair up time-reversed pairs

1) momentum conserved + isotropic

$$\varepsilon_{\vec{k}} = \frac{k^2}{2m} \quad \text{– degeneracy in dir of } \vec{k}$$



Many states for given $\vec{k}$, need to select two $\quad T|\varepsilon_k, \vec{k}, \uparrow\rangle = |\varepsilon_k, -\vec{k}, \downarrow\rangle$

$$T|\varepsilon_k, \uparrow\rangle = |\varepsilon_k, \downarrow\rangle$$ need one more quantum #

2) no symmetries except T-inv: only two degenerate states $\forall\ \varepsilon_\alpha$

$\uparrow\downarrow\ \varepsilon_\alpha$

$$T|\alpha\uparrow\rangle = |\alpha\downarrow\rangle$$

energy index

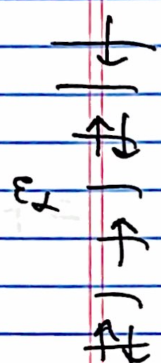
$$T|\varepsilon_\alpha, \uparrow\rangle = |\varepsilon_\alpha \downarrow\rangle$$

$\Rightarrow$ our ~~B~~ version of the BCS Ham.

$$C^{\dagger}_{\beta\uparrow} C^{\dagger}_{\beta\downarrow} C_{\alpha\downarrow} C_{\alpha\uparrow}$$

Blocking effect

$$H = \sum_\alpha \varepsilon_\alpha n_\alpha \underbrace{- g \sum_{\alpha,\beta} c^+_{\alpha\uparrow} c^+_{\alpha\downarrow} c_{\beta\downarrow} c_{\beta\uparrow}}_{H^{int}_{BCS}} - J \vec{S}^2$$



$n_\alpha = 1$ (singly-occupied) – blocked, B

$n_\alpha = 0, 2$ (unoccupied / doubly occupied)
– unblocked, U

H^{int}_{BCS} – hops pars from one level to another.

Singly occupied levels unaffected / spectators

Similarly, $\vec{S}^2$ doesn't affect (see)
unblocked levels

$$\vec{S}^2 = \sum_\alpha \vec{S}_\alpha \cdot \vec{S}_\beta$$

singlet

$$S_\alpha |-\rangle = 0 \qquad S_\alpha |\uparrow\downarrow\rangle = 0$$

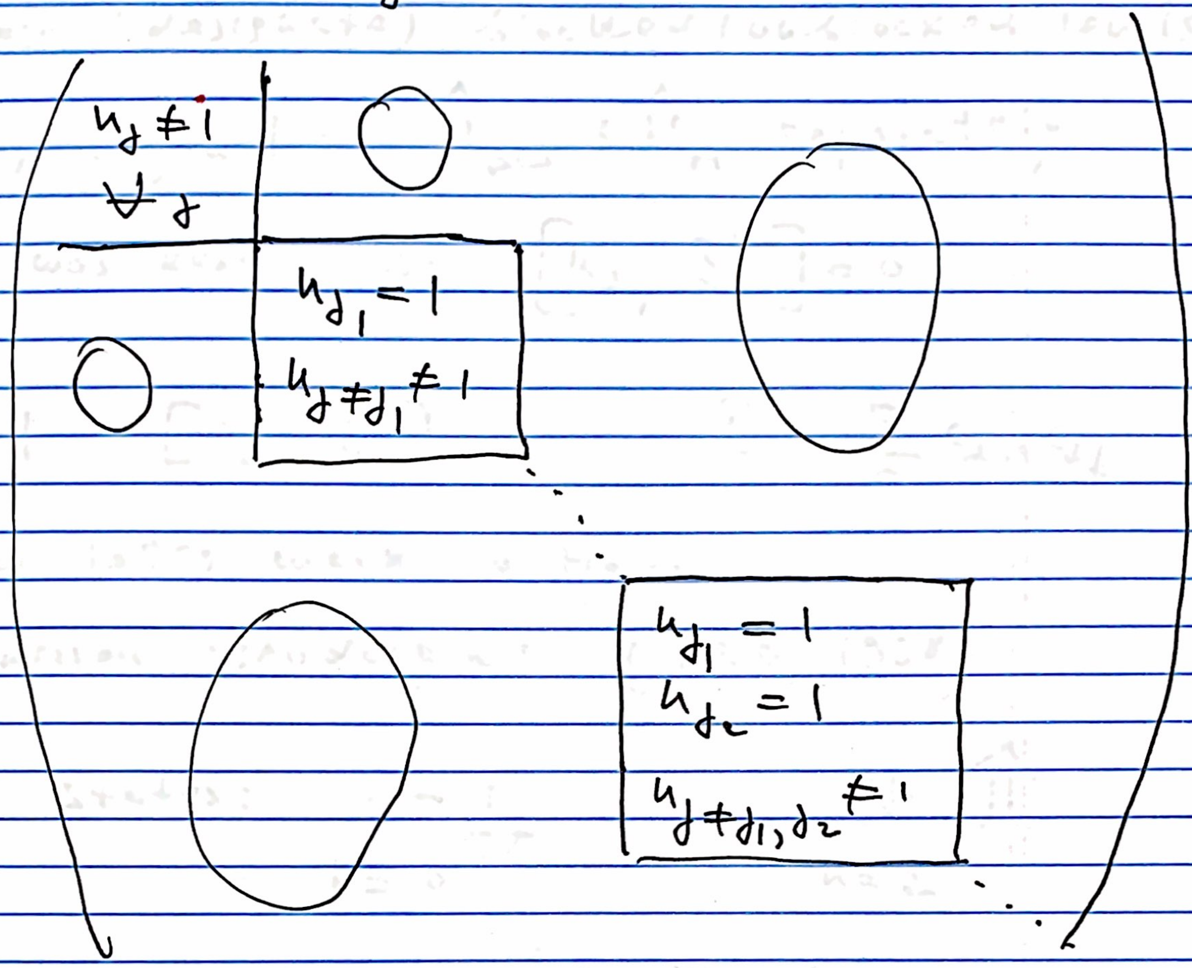
$$S^z_\alpha = \frac{n_{\alpha\uparrow} - n_{\alpha\downarrow}}{2} \qquad S^+_\alpha = c^+_{\alpha\uparrow} c_{\alpha\downarrow}$$

Can change $\uparrow \to \downarrow$, but doesn't change level occupancy

$$\langle n_\gamma = 1, \dots | \hat{H} | n_\gamma \neq 1, \dots \rangle = 0$$

$\hat{H}$ is block-diagonal

$H =$



$$\hat{H} = \left(\sum_\alpha \varepsilon_\alpha n_\alpha - g \sum_{\alpha\beta} c^\dagger_{\alpha\uparrow} c^\dagger_{\alpha\downarrow} c_{\beta\downarrow} c_{\beta\uparrow} \right)_{\text{unblocked}} + \underbrace{\left(\sum_\alpha \varepsilon_\alpha n_\alpha - J \vec{S}^2 \right)_{\text{blocked}}}_{\hat{H}_M}$$

$$E = E_U + E_B = E_{BCS} + E_M$$

blocked

$$\left.\begin{matrix} \uparrow \\ \uparrow \\ \uparrow \\ \downarrow \\ \uparrow \end{matrix}\right\} \text{spin } S$$

$$E_B = \sum_{\alpha \in B} \varepsilon_\alpha - J S(S+1)$$

In other words, to obtain eigenstates and energies of $\hat{H}_{univ} = \hat{H}_{BCS} + \hat{H}_M$, first arbitrarily choose (designate) blocked/unblocked levels.

Then, diagonalize $\hat{H}_{BCS}$ & $\hat{H}_M$ separately

$\hat{H}_M$ was easy bc. $[n_\alpha, \vec{S}^{\alpha}] = 0$

But $[n_\alpha, \hat{L}] \neq 0$ $\quad \hat{L} = \sum_\alpha c_{\alpha\uparrow} c_{\alpha\downarrow}$

Now let's work w H_{BCS}

Anderson pseudospins (PRB, 1958)

Two states: $\quad - \; \Downarrow \quad n = 0 \qquad\qquad \uparrow\downarrow \; \Uparrow \quad n = 2$

Define pseudospin-1/2 variable

$$K_z = \frac{n-1}{2} \qquad K_+ = c_\uparrow^+ c_\downarrow^+ \qquad K_- = c_\downarrow c_\uparrow$$

spin-1/2 commutation relations

$$K_\alpha^z = \frac{n_\alpha - 1}{2} \Rightarrow n_\alpha = 2 K_\alpha^z + 1$$

$$\sum_\alpha \varepsilon_\alpha n_\alpha = \sum_\alpha 2 \varepsilon_\alpha K_\alpha^z + \text{const}$$

blocked states: pseudospin-0

$$K_z |\mp\rangle = 0 \,, \quad K_\pm |\mp\rangle = 0$$

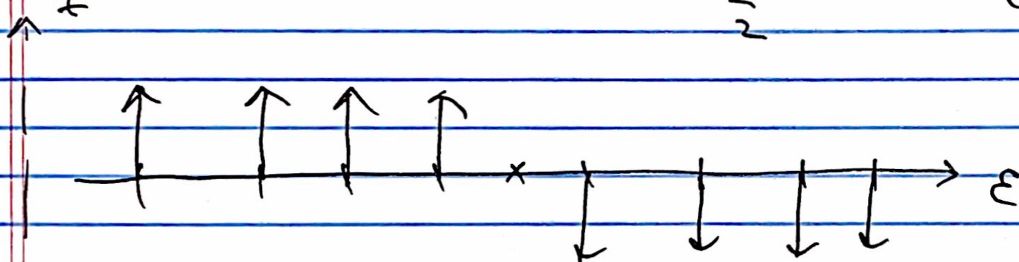
$$H_{BCS} = \sum_\alpha 2\varepsilon_\alpha K_\alpha^z - g \underbrace{\sum_{\alpha,\beta} K_\alpha^+ K_\beta^-}_{K_\alpha^x K_\beta^x + K_\alpha^y K_\beta^y}$$

Inhom. Zeeman magnetic field $\parallel z$

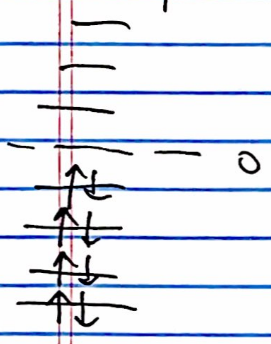
Infinite ferrom. XY int

Zeeman term: $K_\alpha^z = +\frac{1}{2}$ for $\varepsilon_\alpha < 0$

$-\frac{1}{2}$ $\quad \varepsilon_\alpha > 0$

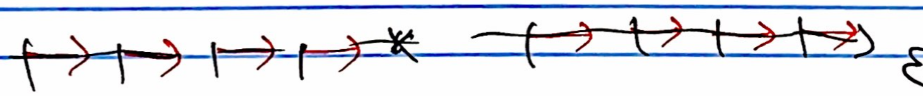


Sharp domain wall



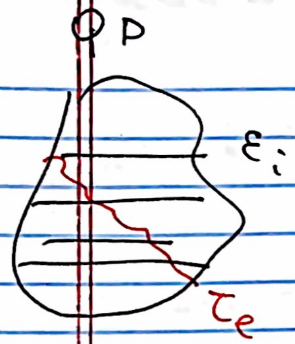
Fermi liquid gr. st.

XY term



For $|\varepsilon| \gg \Delta$ Zeeman always wins

(Weakly) disordered metalic QD



$$g_T = \frac{E_T}{\delta} \gg 1 \qquad E_T = \frac{\hbar}{\tau_e}$$

T-inv, no spin-orbit

$\Rightarrow$ ε_i - eigenv. of a GOE RM

Considering 2-body int matrix el and keeping largest int, we derived

$$H_{univ} = \sum_i \varepsilon_i n_i - g \sum_{i,j} c^+_{i\uparrow} c^+_{i\downarrow} c_{j\downarrow} c_{j\uparrow} + E_c \hat{N}_e^2 - J \hat{\vec{S}}^2$$

$$H = H_{univ} + O(g_T^{-1})$$

$g = \lambda\delta$, λ - dimensionless BCS coupling

E_c - charging energy, J - spin-exchange const

$$\hat{N}_e = \sum_i n_i \qquad \hat{\vec{S}}^2 = S_+ S_- + S_z^2 - S_z$$

$$S_z = \sum_i \frac{n_{i\uparrow} - n_{i\downarrow}}{2} \qquad S_+ = \sum_i c^+_{i\uparrow} c_{i\downarrow}$$

$$n_{i\sigma} = c^+_{i\sigma} c_{i\sigma}, \qquad n_i = n_{i\uparrow} + n_{i\downarrow}$$

$\uparrow$ Blocked — Unblocked

$\downarrow$ $n_i = 1$ $\quad$ $\uparrow\downarrow$ $n_i = 0, 2$

Blocked – usual spin-$\frac{1}{2}$

Unblocked – Anderson pseudospin-$\frac{1}{2}$

$$k_{iz} = \frac{n_i - 1}{2}, \qquad k_i^+ = c_{i\uparrow}^+ c_{i\downarrow}^+$$

$$H_{univ} = H_{BCS}\Big|_{unblocked} + E_C N_e^2 - J S(S+1)$$

S – total spin = spin of blocked levels

N_e – total part #

$$H_{BCS} = \sum_i 2\varepsilon_i k_i^z - g \sum_{i,j} k_i^+ k_j^-$$

BCS / Richardson Hamiltonian

Inhom. Zeeman field   all-to-all XY

BCS mean field:

Terms involving $\vec{k}_i$

$$\hat{h}_i = 2\varepsilon_i k_i^z - k_i^+\left(g \sum_j k_j^-\right) - k_i^-\left(g \sum_j k_j^+\right)$$

Total (pseudo) spin $\vec{L} = \sum_{j=1}^{N} \vec{k}_j$

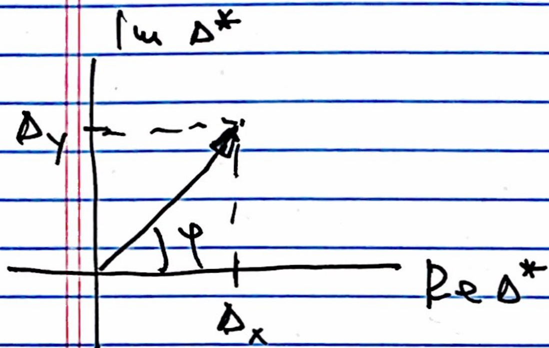
When $N \to \infty$ expect $\vec{L}$ to become classical

$\Rightarrow$ replace $\vec{\hat{L}} \to \langle \vec{\hat{L}} \rangle = \langle \psi | \vec{\hat{L}} | \psi \rangle$

Define $\Delta = g \sum_j \langle K_j^- \rangle = g \langle \hat{L}^- \rangle$

$$h_i = 2\varepsilon_i k_i^z - \underbrace{\left(\Delta k_i^+ + \Delta^* k_i^- \right)}$$

$$\Delta = \Delta_x - i\Delta_y \qquad \Delta^* = \Delta_x + i\Delta_y = |\Delta| e^{i\varphi}$$



2D vector $\vec{\Delta} = (\Delta_x, \Delta_y)$

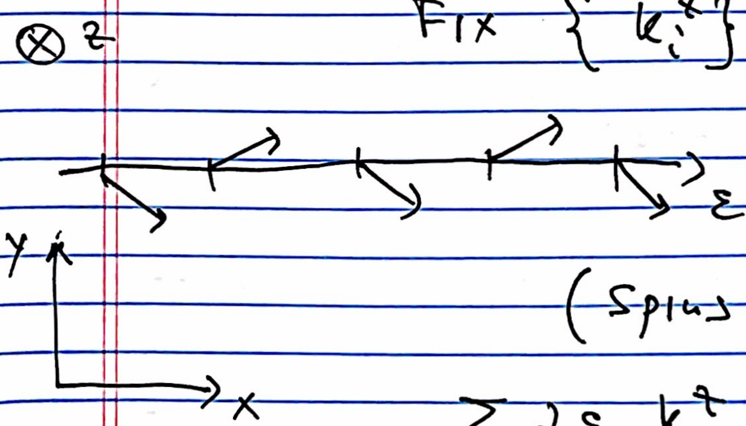
$$\underbrace{\qquad} = (\Delta_x - i\Delta_y)(k_i^x + i k_i^y) + (\Delta_x + i\Delta_y)(k_i^x - i k_i^y) =$$

$$= 2\Delta_x k_i^x + 2\Delta_y k_i^y$$

$$\hat{h}_i = \vec{b}_i \cdot \vec{\hat{k}}_i \qquad \vec{b}_i = (-2\Delta_x, -2\Delta_y, 2\varepsilon_i)$$

In the gr. st. spins are coplanar

Fix $\{ k_i^z \}$



Top view

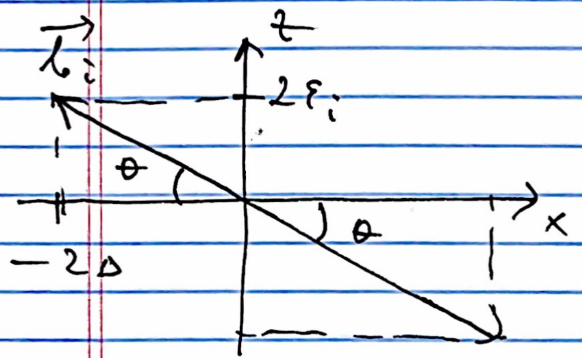
(spins decoupled)

$$\sum_i 2\varepsilon_i k_i^z = \text{fixed}$$

To minimize XY ferro Int, align spins in the same direction in XY plane

Let $\langle k_i^y \rangle = 0 \Rightarrow \Delta = g \sum_i \langle k_i^x \rangle \in \mathbb{R}$

$\vec{b}_i = (-2\Delta, 0, 2\xi_i)$ Let $\Delta > 0$



For gr. st. $\langle \vec{k}_i \rangle \downarrow\uparrow \vec{b}_i$

$$\langle k_i^x \rangle = \frac{\cos\theta}{2}$$

$$\langle k_i^z \rangle = -\frac{\sin\theta}{2}$$

$$\cos\theta = \frac{2\Delta}{|\vec{b}_i|} = \frac{2\Delta}{\sqrt{(2\Delta)^2 + (2\xi_i)^2}} = \frac{\Delta}{\sqrt{\xi_i^2 + \Delta^2}}$$

$$\sin\theta = + \frac{\xi_i}{\sqrt{\xi_i^2 + \Delta^2}}$$

$$\langle k_i^x \rangle = \frac{\Delta}{2\sqrt{\xi_i^2 + \Delta^2}}, \quad \langle k_i^z \rangle = -\frac{\xi_i}{2\sqrt{\xi_i^2 + \Delta^2}}$$

Δ – the mean field produced by other spins

Self-consistency

$$\Delta = g \sum_i \frac{\Delta}{2\sqrt{\xi_i^2 + \Delta^2}}$$

One more piece of info

$$\sum_i \langle k_i^z \rangle = -\sum_i \frac{\varepsilon_i}{2\sqrt{\varepsilon_i^2+\Delta^2}}$$

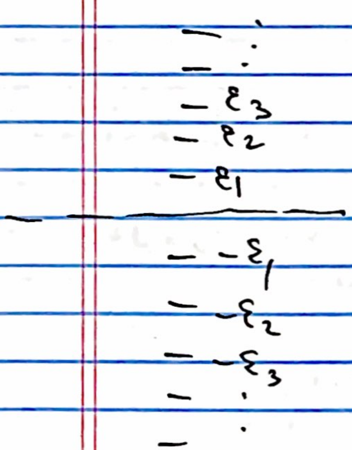
$$k_i^z = \frac{n_i - 1}{2} \qquad \sum_i k_i^z = \frac{1}{2}\sum_i (n_i - 1) =$$

$$= \frac{N_e - N}{2}$$

$$\frac{N_e - N}{2} = -\sum_i \frac{\varepsilon_i}{2\sqrt{\varepsilon_i^2+\Delta^2}}$$

2 eq. on Δ, not always consistent

Often $\exists$ particle-hole symm.



$$H_0 = \sum_i \varepsilon_i n_i$$

particle-hole (fermions)

$$c_{i\sigma} \to c^+_{i\sigma}, \quad c^+_{i\sigma} \to c_{i\sigma}$$

$$n_{i\sigma} \to c_{i\sigma} c^+_{i\sigma} = 1 - n_{i\sigma}$$

$$H_0 \to \sum_i \varepsilon_i (1 - n_i) = \cancel{\sum_i \varepsilon_i} - \sum_i \varepsilon_i n_i =$$

$$= \sum_i (-\varepsilon_i) n_i = H_0$$

$$N_e = 2\frac{N}{2} = N$$

$$0 = -\sum_i \frac{\varepsilon_i}{2\sqrt{\varepsilon_i^2+\Delta^2}} = 0$$

In the absence of p-h symm, recall

$$H_{BCS}^{mf} = \sum_i h_i = \sum_i \left\{ 2\varepsilon_i k_i^z - \Delta k_i^+ - \Delta^* k_i^- \right\} =$$

$$= \sum_i \varepsilon_i n_i - \Delta \sum_i c_{i\uparrow}^+ c_{i\downarrow}^+ - \Delta^* \sum_i c_{i\downarrow} c_{i\uparrow}$$

Particle # not conserved. Use grand canonical formulation

$$Z = \mathrm{tr}\, e^{-\frac{H-\mu N}{T}}$$

Need eigenstates of $H_{BCS} - \mu N$ rather than H_{BCS}

$$\mu N = \mu \sum_i n_i$$

$$\sum_i \varepsilon_i n_i \rightarrow \sum_i (\varepsilon_i - \mu) n_i$$

$$\varepsilon_i \rightarrow \varepsilon_i - \mu$$

$$\Delta = g \sum_i \frac{\Delta}{2\sqrt{(\varepsilon_i - \mu)^2 + \Delta^2}}$$ BCS gap. eq.

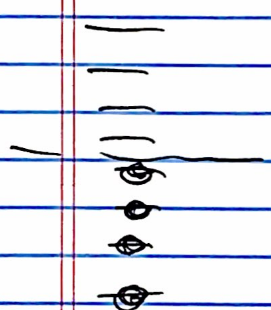
$$\frac{N_e - N}{2} = - \sum_i \frac{\varepsilon_i - \mu}{2\sqrt{(\varepsilon_i - \mu)^2 + \Delta^2}}$$ chem pot'l eq.

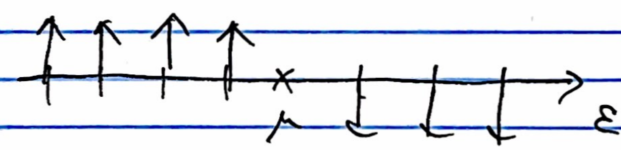
Trivial solution $\Delta = 0$

$$H^{mf}_{BCS} = \sum_i 2(\varepsilon_i - \mu) n_i^{\uparrow} = \sum_i (\varepsilon_i - \mu) n_i$$

need to minimize Zeeman energy

$\Rightarrow$ Fermi gas gr. st.





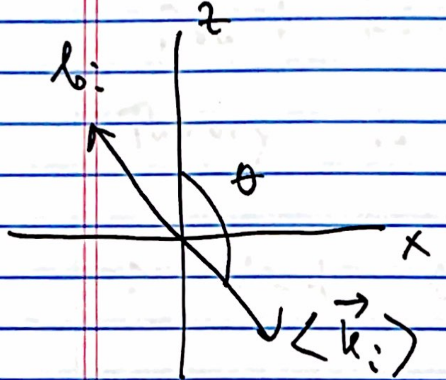
Nontrivial solution $\Delta \neq 0$

$$1 = g \sum_i \frac{1}{\sqrt{(\varepsilon_i - \mu)^2 + \Delta^2}}$$

$$H_{BCS} - \mu \hat{N} = \sum_i 2(\varepsilon_i - \mu) k_i^z - g \sum_{i,j} k_i^+ k_j^-$$

Mean field: $g \sum_j k_j^- \to g \sum_j \langle k_j^- \rangle \equiv \Delta$

$\hat{h}_i = \vec{b}_i \cdot \vec{k}_i \qquad \vec{b}_i = (-2\Delta_x, -2\Delta_y, 2(\varepsilon_i - \mu))$



$\langle k_i^x \rangle = \ldots \qquad \langle k_i^z \rangle = \ldots$

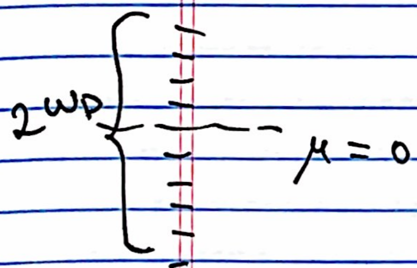
$\langle k_i^y \rangle = 0$

$$\Delta = g \sum_i \frac{\Delta}{\sqrt{(\varepsilon_i - \mu)^2 + |\Delta|^2}}$$

BCS gap eq.

Nontrivial solution: $\Delta \neq 0$

$$1 = \lambda \delta \sum_i \frac{1}{\sqrt{\varepsilon_i^2 + \Delta^2}}$$



$$1 = \lambda \int_{-\Omega}^{\Omega} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}}$$

For $|\varepsilon| \gg \Delta$ $\quad \displaystyle\int_\Delta^\infty \frac{d\varepsilon}{\varepsilon}$ log divergence

$$\Delta = \frac{\Omega}{\sinh(1/\lambda)} \approx \frac{\omega_D}{\sinh(1/\lambda)}$$

Ω – ultraviolet cutoff

$\Omega \sim \omega_D$ for acoustic phonons

$\Omega \sim \omega_0$ optical phonons

Don't know Δ accurately!

Theory is valid at energies $|\varepsilon| \ll \omega_D$

$\Rightarrow$ Need $\Delta \ll \omega_D \Rightarrow \lambda \ll 1$

$$\sinh\frac{1}{\lambda} = \frac{e^{1/\lambda} - e^{-1/\lambda}}{2} \approx \frac{e^{1/\lambda}}{2}$$

$$\Delta \approx 2\omega_D e^{-1/\lambda}$$

BCS theory is an effective weak coupling theory valid the ~~[illegible]~~ limit $\frac{\Delta}{\omega_D} \to 0$

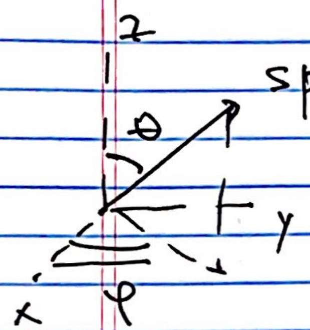
Ex: $$\frac{2\Delta}{T_c} = \sqrt{3.5^2 + \left(\frac{\Delta}{\omega_D}\right)^2}$$

$$\Rightarrow \frac{2\Delta}{T_c} = 3.5$$

Note cutoff cancels from various ratios

$\Rightarrow$ these are legitimate answers

BCS wf



$$\begin{pmatrix}\cos\frac{\theta}{2}\, e^{i\varphi} \\ \cancel{e^{-i\varphi}}\sin\frac{\theta}{2}\end{pmatrix} =$$

$$= \cos\frac{\theta}{2}\,|\uparrow\rangle + e^{-i\varphi}\sin\frac{\theta}{2}\,|\downarrow\rangle =$$

$$= \left(e^{-i\varphi}\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\,k^{+}\right)|\downarrow\rangle$$

$$|BCS\rangle =$$

$$= \prod_j \left(\cancel{e^{-i\varphi}}\,\overbrace{\boxed{\sin\frac{\theta_j}{2}}}^{u_j} + \overbrace{\boxed{\cos\frac{\theta_j}{2}}}^{v_j}\,k_j^{+}\right)|\downarrow\rangle$$

$e^{i\varphi}$

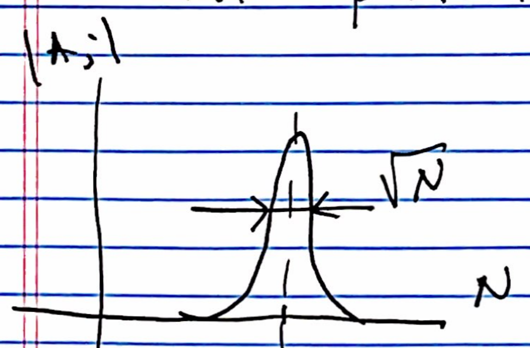
$$k_j^{+} = c_{j\uparrow}^{+}c_{j\downarrow}^{+}$$

u_j, v_j – Bogolioubov amplitudes, coherence factors

$$\sin\theta_j = \frac{|\Delta|}{\sqrt{(\epsilon_j-\mu)^2+|\Delta|^2}}, \quad \cos\theta_j = \ldots$$

$$\Psi(\varphi) = |BCS\rangle = \prod_j \left(\cancel{e^{-i\varphi}}\,u_j + e^{i\varphi}\,v_j\,c_{j\uparrow}^{+}c_{j\downarrow}^{+}\right)|0\rangle$$

Variable particle #



$$\langle BCS|\hat{N}|BCS\rangle = \bar{N}$$

Gr. st. energy indep. of φ

Anderson (1958)

$$H_{BCS}\,\Psi(\varphi) = E_0\,\Psi(\varphi)$$

$$H_{BCS}\int A(\varphi)\,\Psi(\varphi)\,d\varphi = E_0\int A(\varphi)\,\Psi(\varphi)\,d\varphi$$

Project oust states with given $N_e = 2M_0$

M_0 – # of pairs

$$|PBCS\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\varphi M_0}\,\Psi(\varphi)\,d\varphi$$

$$\Psi(\varphi) = \sum_{M=0}^{2N} e^{i\varphi M}\,|M\rangle$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\varphi(M-M_0)}\,d\varphi = \delta_{MM_0}$$

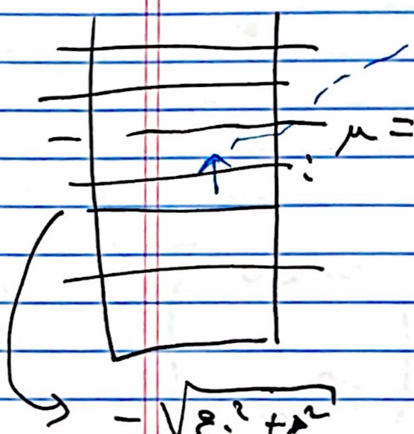
$$|PBCS\rangle = |M_0\rangle$$

Projected BCS

Excitations:

1) single-part (tunneling) 2) pair-breaking

3) excited pairs



$$H_{BCS}^{int} = \sum_i \vec{b}_i \cdot \vec{k}_i$$

$$|\vec{b}_i| = 2\sqrt{\xi_i^2+\Delta^2}$$

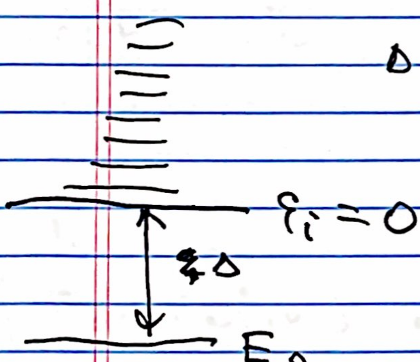
$$\vec{b}_i = (-2\Delta, 0, 2\xi_i)$$

$$h_i = \pm\sqrt{\xi_i^2+\Delta^2}$$

Min energy = all levels unblocked

1) excitation energy

$$\Delta E = 0 - \left(-\sqrt{\xi_i^2+\Delta^2}\right) = \sqrt{\xi_i^2+\Delta^2}$$



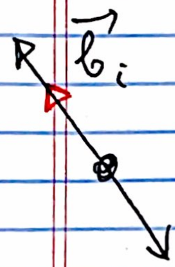
energy gap $= \Delta = (\Delta E)_{min}$

2) pair breaking = 2 single particle

$$\Delta E = \sqrt{\xi_i^2+\Delta^2} + \sqrt{\xi_j^2+\Delta^2}$$

$$(\Delta E)_{min} = 2\Delta$$

3) spin-flips = excited pairs = fermions



$$\Delta E = \sqrt{\xi_i^2 + \Delta^2} - \left(-\sqrt{\xi_i^2+\Delta^2}\right) =$$

$$= 2\sqrt{\xi_i^2 + \Delta^2}$$

$$(\Delta E)_{min} = 2\Delta$$

$$\begin{pmatrix} \cos\frac{\theta_i}{2} \\ \sin\frac{\theta_i}{2} \end{pmatrix} \qquad \begin{pmatrix} -\sin\frac{\theta_i}{2} \\ \cos\frac{\theta_i}{2} \end{pmatrix} \qquad [\theta \to \pi - \theta]$$

$$\begin{pmatrix} v_i \\ u_i \end{pmatrix} \qquad \begin{pmatrix} -u_i \\ v_i \end{pmatrix}$$

pair wf $\quad (u_i + v_i c_{i\uparrow}^{+} c_{i\downarrow}^{+}) |0\rangle$

$$(v_i - u_i c_{i\uparrow}^{+} c_{i\downarrow}^{+}) |0\rangle$$

Excitations of the condensate

When does BCS mf break down as we decrease the size of the system?

When quantum fluctuations of

$$\hat{\vec{J}} = \sum_j \hat{\vec{\tau}}_j \quad \text{are relatively large}$$