Joint prob. distr of the eigenvalues $E_1, \dots, E_N$ of ensembles of random matrices

$$P(E_1, \dots, E_N) = c \prod_{k<j} |E_j - E_k|^\beta e^{-a \sum_j E_j^2}$$

Derivation similar to that of $P(H) = \text{const}\, e^{-a \,\mathrm{tr} H^2}$

Level spacing distr. $P(s)$

$N=2$ $\quad P(E_1, E_2) = c\, |E_2 - E_1|^\beta \exp(-aE_1^2 - aE_2^2)$

$|E_2 - E_1| = \Delta \qquad E = E_1 + E_2$

$$E_1^2 + E_2^2 = \frac{(E_2 - E_1)^2 + (E_1 + E_2)^2}{2} = \frac{\Delta^2 + E^2}{2}$$

$$P(\Delta, E) = c \Delta^\beta \exp\left[-\frac{a}{2}(\Delta^2 + E^2)\right]$$

$$P(\Delta) = c \Delta^\beta e^{-\frac{a\Delta^2}{2}} \underbrace{\int dE\, e^{-\frac{aE^2}{2}}}_{\sqrt{\frac{4\pi}{a}}}$$

$$P(\Delta) = \text{const}\, \Delta^\beta e^{-\frac{a\Delta^2}{2}}$$

· const from $\int P(\Delta)\, d\Delta = 1$

$$\langle \Delta \rangle = \int P(\Delta)\, \Delta\, d\Delta = f(\beta, a)$$

$$s = \frac{\Delta}{\langle \Delta \rangle} \qquad P(s) = \underbrace{\text{const}}_{\text{Known}}\, s^{\beta} e^{-b s^2}$$

$$\Delta = \langle \Delta \rangle s$$

Determine b from $\int P(s)\, s\, ds = 1$

We get "Wigner's surmise" (sometimes loosely called WD distr., but this isn't quite correct)

$$P(s) = \begin{cases} \frac{\pi}{2} s\, e^{-\frac{\pi s^2}{4}} & \text{GOE} \\ \frac{32}{\pi^2} s^2 e^{-\frac{4 s^2}{\pi}} & \text{GUE} \\ \frac{2^{18}}{3^6 \pi^3} s^4 e^{-\frac{64 s^2}{\pi}} & \text{GSE} \end{cases}$$

$P(s) \propto s^{\beta}$ for small s level repulsion: GSE > GUE > GOE

This is level spacing distr. for 2×2 matrices

Generally $P(s) = P(s, N, \beta)$

Especially interesting is $N \to \infty$ limit

But $N=2$ approximates general N surprisingly well. In particular, $P_{N\to\infty}(s) \approx P_{N=2}(s)$ within 1%

General $P(s, N, \beta)$ expressed in terms

of $\det \left[\int_s^\infty \varphi_{2i}(x)\, \varphi_{2j}(x)\, dx \right]$

h.o. wf $\quad i, j = 0, 1, \ldots, m-1$

$N = \begin{cases} 2m \\ 2m-1 \end{cases}$

Rescale energies $E_j = \sqrt{\frac{\beta}{2a}}\, x_j$

$$P = c \prod_{k<j} |x_j - x_k|^\beta \, e^{-\frac{\beta}{2} \sum_j x_j^2} =$$

2D Coulomb

$$= c \exp\left\{ -\beta \underbrace{\left[\sum_j \frac{x_j^2}{2} - \sum_{k<j} \ln |x_j - x_k| \right]}_{W} \right\}$$

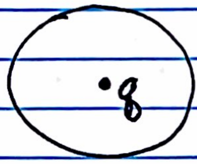
h.o.

$$W = \sum_j \frac{x_j^2}{2} - \sum_{k<j} \ln |x_j - x_k|$$

Boltzman weight for N particles in 1-particle harmonic pot'l w $m\omega^2 = 1 = K$ $\longrightarrow$

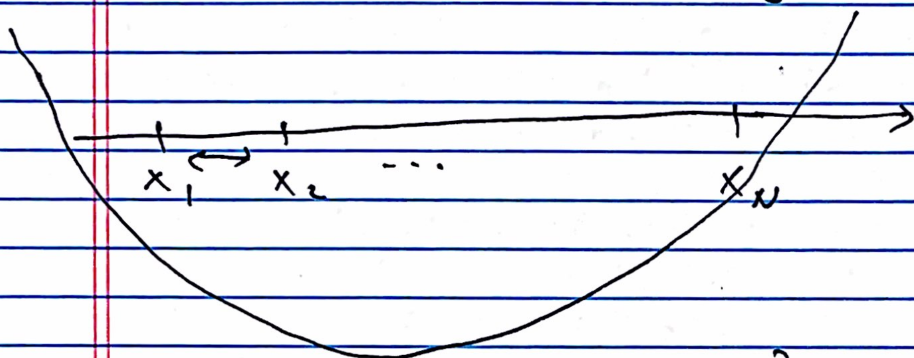
§ 2D Coulomb int

Gauss law



$$E\, 2\pi r = q/\varepsilon_0 \Rightarrow E \propto \frac{1}{r} \Rightarrow W \propto \ln r$$

cf. Pechukas gas



$$H = W + \sum_j \frac{p_j^2}{2m}$$

$$Z(\beta) = \int e^{-\beta H(x_j, p_j)} \prod_j dx_j \prod_j dp_j =$$

$$\Downarrow$$

$$e^{-\beta W} \; e^{-\beta \sum_j \frac{p_j^2}{2m}}$$

$$= \cancel{\text{const}} \int e^{-\beta W(x_j)} \prod_j dx_j$$

General feature of classical physics - momenta integrate out. Not so in QM

$$\beta = \frac{1}{T} \qquad T_{GOE} > T_{GUE} > T_{GSE}$$

Exact DOS $\hat{\rho}(x) \equiv \sum_{i=1}^{N} \delta(x - x_i)$

↑ new variable (aux. variable)

$$\int_a^b \hat{\rho}(x)\, dx = \#\text{ of levels} \in (a, b)$$

In particular, $\int_{-\infty}^{\infty} \hat{\rho}(x)\, dx = N$

Write W in terms of $\hat{\rho}$

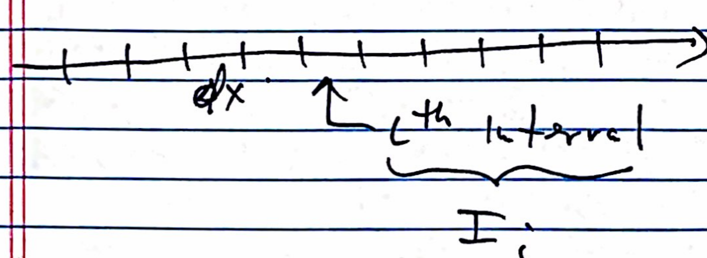
principal value integral

$$W(\hat{\rho}) = \frac{1}{2} \int_{-\infty}^{\infty} dx\, \hat{\rho}(x)\, x^2 - \frac{1}{2} \iint_{-\infty}^{\infty} (dx\, dy)\, \hat{\rho}(x)\, \hat{\rho}(y) \ln|x - y|$$

$\sum_{j<k} + \sum_{j>k}$

included $j = k$, will deal w this later

Convert $\int \prod_j dx_j = \int \mathcal{D}\rho = \prod_i \int_{-\infty}^{\infty} d\rho_i$



coarse graining

$$\sum_{x_j \in I_i} \delta(x - x_i) = \rho_i\, dx$$

$$W(\rho) = \frac{1}{2}\int dx\, \rho(x)\, x^2 - \frac{1}{2}\iint (dx\, dy)\, \rho(x)\rho(y) \ln|x-y|$$

$\hat{\rho} \to \rho$

$$Z(\beta) = \int e^{-\beta W(\rho)} \mathcal{D}\rho$$

$W \propto N$ Use stat. phase method for $N \gg 1$

Constraints: $\int_{-\infty}^{\infty} dx\, \rho(x) = N$, $\rho(x) \geq 0 \;\; \forall x$

$$\widetilde{W}(\rho) = W(\rho) + \mu\left(\int_{-\infty}^{\infty} \rho(x)\, dx - N\right)$$

$$\frac{\delta W}{\delta \rho} = 0: \quad \frac{1}{2}x^2 - \fint_{-\infty}^{\infty} dy\, \rho(y) \ln|x-y| + \mu = 0$$

$$\frac{\delta \rho(x)}{\delta \rho(x')} = \delta(x-x') \qquad \fint_a^b f(x)\, dx = \underset{\text{P.V.}(x_0)}{P}\int_a^b f(x)\, dx =$$

$f(x_0) = \infty$

$$= \lim_{\varepsilon \to 0}\left[\int_a^{x_0-\varepsilon} f(x)\, dx + \int_{x_0+\varepsilon}^b f(x)\, dx\right]$$

Equivalent to $j \neq k$ in $\sum_{j,k}$

$i \neq i_0$

$\hookrightarrow x \quad \hookrightarrow x_0$

Diff. w.r.t. x $\qquad \ln(y-x) = \ln(x-y) + \ln(-1)$

$$(y-x) = (x-y)(-1)$$

$$x = \fint_{-\infty}^{\infty} dy \, \frac{\rho(y)}{x-y} = \pi \rho_H(x)$$

Hilbert transform: $\quad \frac{1}{\pi} \int dy \, \frac{f(y)}{x-y} = f_H(x)$

For which $f(x)$, $f_H(x) \propto x$

cf. $\frac{\partial W}{\partial x_j} = 0$ (force $= 0$ equiv.)

$$x_j = \sum_{j \neq k} \frac{1}{x_j - x_k}$$

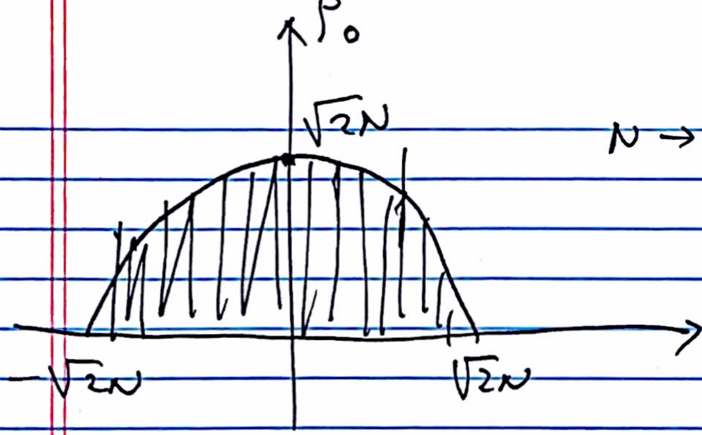
$$\downarrow \qquad \qquad \downarrow$$

$$x = \fint \frac{\hat{\rho}(x')}{x-x'} \, dx' \qquad \hat{\rho}(x') = \sum_j \delta(x'-x_j)$$

$$\int_{-\infty}^{\infty} \rho(x) \, dx = N$$

Can show $\quad \rho_0(x) = \begin{cases} \frac{1}{\pi}\sqrt{2N - x^2} \,, & |x| < 2N \\ 0 & |x| > 2N \end{cases}$

Wigner's semicircle law. Global property



$N \to \infty$

$x = \sqrt{\frac{2a}{\beta}}\, E$

$$\rho_0(x) = \langle \hat{\rho}(x) \rangle = \frac{\int \hat{\rho}(x)\, e^{-\beta W} \prod_i dx_i}{\int e^{-\beta W} \prod_i dx_i} \simeq$$

$$\approx \frac{\rho_0\, e^{-\beta W(\rho_0)} \cdots}{e^{-\beta W(\rho_0)} \cdots} = \rho_0$$

Density - density correlation fn

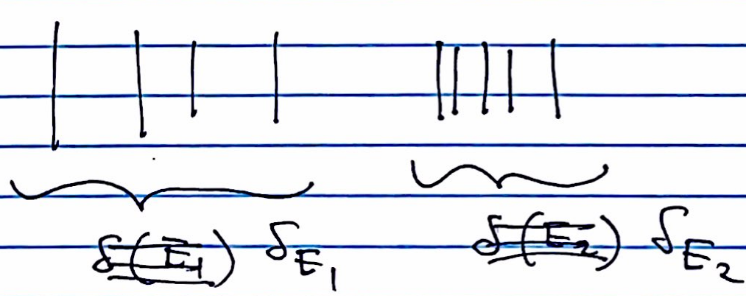
$$K(\omega, \varepsilon) = \langle \hat{\rho}(\varepsilon)\, \hat{\rho}(\varepsilon + \omega) \rangle - \langle \hat{\rho}(\varepsilon) \rangle \langle \hat{\rho}(\varepsilon + \omega) \rangle$$

Can show

$$K(\omega, \varepsilon) = \delta(\omega)\, \rho_0(\varepsilon) - \left(\frac{\sin \pi \omega}{\omega} \right)^2$$

GOE, $N \to \infty$ limit

Unfolding: smooth rescaling of energy so that $\rho(\varepsilon) = 1$



Derived $P = c \prod_{j<k} |E_j - E_k|^\beta \, e^{-a \sum_j E_j^2}$

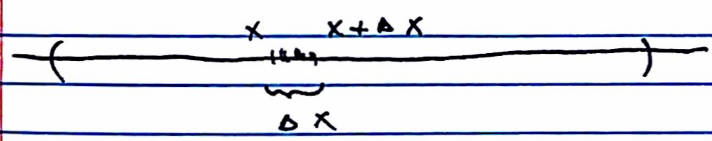
Rescale $E_j = \sqrt{\frac{\beta}{2a}}\, x_j$

$$P = c\, e^{-\beta W}$$

$$W = \sum_j \frac{x_j^2}{2} - \sum_{k<j} \ln |x_j - x_k|$$

1D particles w 2D Coulomb int

Partion fn $\quad Z(\beta) = \int_{-\infty}^{\infty} \cdots \int e^{-\beta W(x_1, \dots, x_N)} \, dx_1 \dots dx_N$



$N \to \infty$

$$\sum_{x_j \in (x, x+\Delta x)} \frac{x_j^2}{2} \approx \rho(x)\, \Delta x \, \frac{x^2}{2}$$

$$\rho(x)\, \Delta x = \# \text{ of } x_j \in (x, x+\Delta x)$$

$$W = \int \rho(x) \frac{x^2}{2} dx - \frac{1}{2} \iint \rho(x) \rho(y) \ln|x-y| \, dx\, dy$$

Exact DOS $\quad \hat{\rho}(x) = \sum_j \delta(x - x_j)$

$$\rho(x)\, \Delta x = \int_x^{x+\Delta x} \hat{\rho}(x)\, dx \qquad \text{coarse graining}$$

mean-field

$$Z(\beta) = \int_0^\infty \cdots \int e^{-\beta W(\rho_1, \dots, \rho_N)} d\rho_1 \cdots d\rho_N = \int e^{-\beta W} \mathcal{D}\rho$$

Constaint $\int_{-\infty}^{\infty} \rho(x)\, dx = N$

$$\tilde{W} = W + \mu \left(\int_{-\infty}^{\infty} \rho(x)\, dx - N \right)$$

$$\frac{\delta \tilde{W}}{\delta \rho(x')} = 0 \qquad \text{Use} \quad \frac{\delta \rho(x)}{\delta \rho(x')} = \delta(x - x')$$

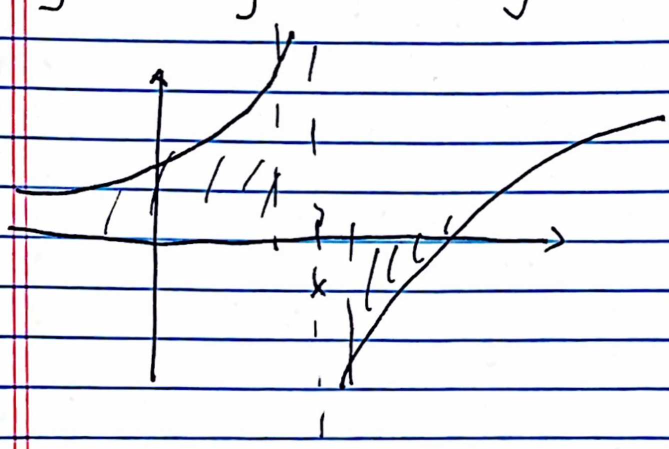
$$\frac{x^2}{2} - \int_{-\infty}^{\infty} dy\, \rho(y) \ln |x - y| + \mu = 0$$

$$\ln |x - y| = \ln \pm (x - y) = \ln (x - y) + \ln \pm 1$$

$$x = ⨍_{-\infty}^{\infty} \frac{dy\, \rho(y)}{x - y} = \lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{x-\varepsilon} \frac{\rho(y)\, dy}{x - y} + \int_{x+\varepsilon}^{\infty} \frac{\rho(y)\, dy}{x - y} \right\}$$

corresponds to $j \neq k$ (general)

$$⨍ \equiv P\int \equiv P.V.\int$$



Ex. $⨍_{-a}^{a} \frac{dx}{x} = 0$

$$x_j = \sum_{k \neq j} \frac{1}{x_j - x_k}$$

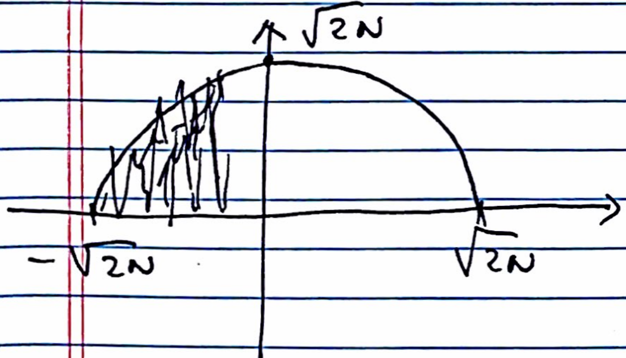
eg. for zeroes of Hermit polys?

$$x = \int_{-\infty}^{\infty} \frac{\rho(y)\,dy}{x-y} \quad \text{- Hilbert transform}$$

$$H(u)(t) = \frac{1}{\pi}\, \text{p.v.} \int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau}\, d\tau$$

$$H(H(u)) = -u \qquad F[H(u)](\omega) = -i \operatorname{sgn} \omega\, F(u)(\omega)$$

$$\rho_0(x) = \begin{cases} \frac{1}{\pi}\sqrt{2N - x^2} & |x| < \sqrt{2N} \\ 0 & |x| > \sqrt{2N} \end{cases}$$



Wigner's semicircle law

$$x_{k+1} - x_k \sim \frac{1}{\sqrt{N}} \quad \text{OK}$$

global property

$$\rho_0(x) = \langle \hat{\rho}(x) \rangle_{\text{ensemble, st. pt}} = \frac{\int \rho(x)\, e^{-\beta W}\, \mathcal{D}\rho}{\int e^{-\beta W}\, \mathcal{D}\rho} \simeq$$

$$\simeq \frac{\rho_0(x)\, e^{-\beta W(\rho_0)}}{e^{-\beta W(\rho_0)}} = \rho_0(x)$$

Density-density correlation fn.

$$K(\omega, \varepsilon) = \langle \hat{\rho}(\varepsilon)\, \hat{\rho}(\varepsilon+\omega) \rangle - \underbrace{\langle \hat{\rho}(\varepsilon) \rangle \langle \hat{\rho}(\varepsilon+\omega) \rangle}_{\rho_0(\varepsilon)\rho_0(\varepsilon+\omega)}$$

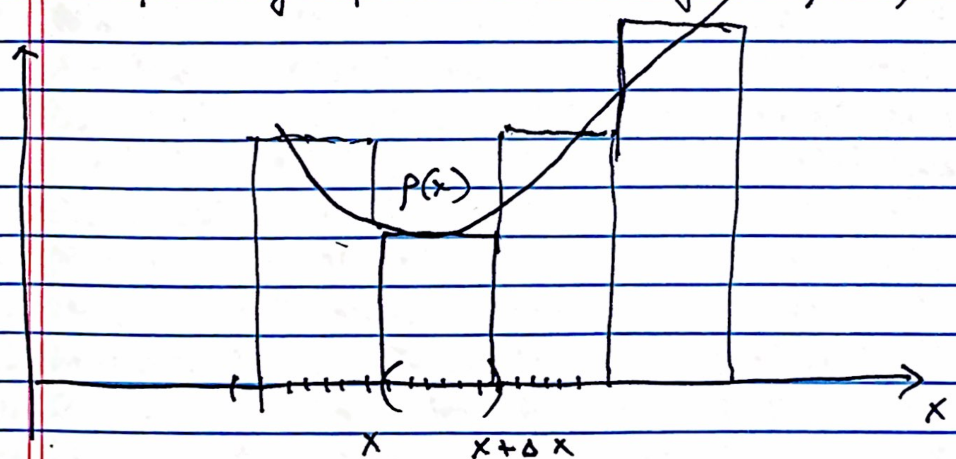
Can show

$$K(\omega, \varepsilon) = \delta(\omega)\, \rho_0(\varepsilon) - \left(\frac{\sin \pi\omega}{\pi\omega} \right)^2 \quad \text{for GUE}$$

For GOE & GSE $K(\omega, \varepsilon)$ is more complicated

With unfolding $\rho_0(\varepsilon) = 1$

Unfolding procedure again $\rho_0(x)$



$$\delta(x) = \frac{1}{\rho_0(x)\, \Delta x}$$

Making level spacing the same $\equiv \rho_0(x) = 1$

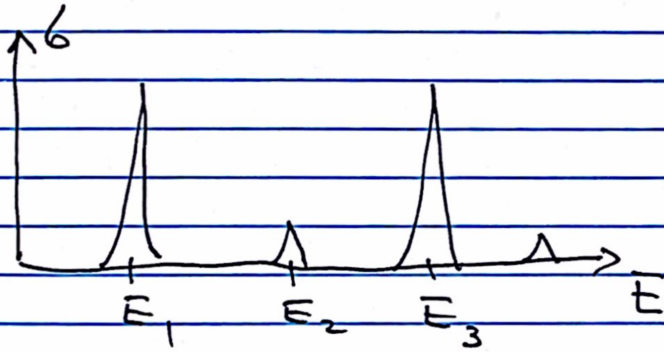
$$\hat{\rho}_{new} = \frac{\hat{\rho}_{old}}{\rho_0(x)} \qquad \langle \hat{\rho}_{new} \rangle_{\text{carse gr.}} = 1$$

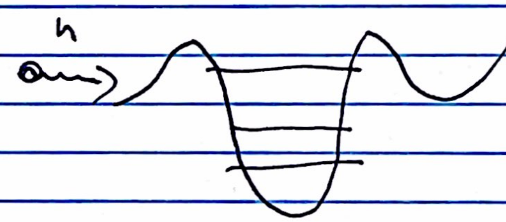
$$E_i \longrightarrow \rho_0(E_i)\, E_i$$

Universality of RMT (where & when RMT works and why)

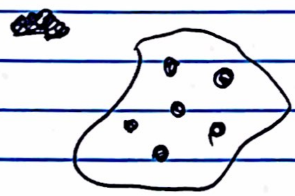
1) Energy levels of heavy nuclei (origin of RMT, Wigner, Dyson) ^{238}U ^{232}Th ← thorium

Measured w slow neutron scattering





See p1 panel (a) $P(s)$ vs GOE & Poisson



complicated (unkown) strong forces ⇒ RMT

$$H_{full} = RM \quad \leftarrow \text{many-body}$$

n-body forces w arbitrary n

$$|t_1 t_2 \cdots t_N\rangle \xrightarrow[H]{} |t_1' t_2' \cdots t_N'\rangle$$

$$\langle t_1 \cdots t_N | H | t_1' \cdots t_N'\rangle - \text{random \# (iid)}$$

cf 2-body int

$$|t_1 t_2 t_3 \cdots t_N\rangle \longrightarrow |t_1' t_2 t_3' \cdots t_N\rangle$$

H_{full} – sparse matrix

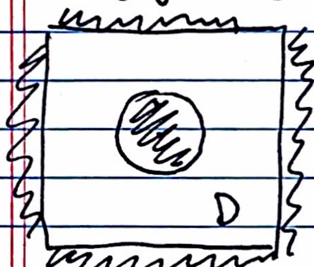
But for QDs we mean $H_{single\text{-}part} = RM$

$$H_{ij} = \langle \varphi_i | \frac{p^2}{2m} + V(r) | \varphi_j \rangle$$

single-part pot'l disorder

2) panel (b) – Sinai billiard (again vs GOE & Poisson)

square



$$\hat{H} = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r})$$

$$V(\vec{r}) = \begin{cases} 0, & \vec{r} \in D \\ \infty, & \vec{r} \notin D \end{cases}$$

$$\hat{H} \Psi_i = E_i \Psi_i \qquad 700\ E_i$$

3) panel (c) – Anderson model (3D)

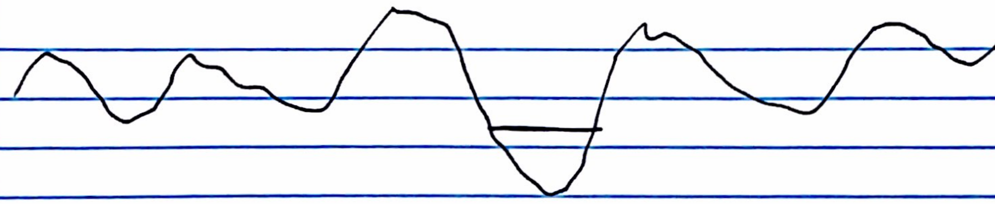
$$H = -t \sum_{\langle ij \rangle} (c_i^\dagger c_j + c_j^\dagger c_i) + \sum_i V_i c_i^\dagger c_i$$

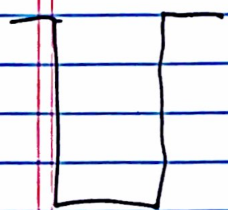
onsite pot'l

$V_i \in \left[-\frac{\omega}{2}, \frac{\omega}{2}\right]$ uniformly distributed

Transition @ $\left(\frac{\omega}{t}\right)_{cr} \approx 16.5$

delocalized – localized (metal-insulator)





∃ critical depth to localize in 2D & 3D
but not in 1D (always a bound state)

⇒ transition in 2D & 3D (Anderson localization)
no transition in 1D
(always localized)

Here $\frac{w}{t} = 2 < 16.5$ – metallic regime, but diffusive metal due to disorder

4) Rabson, Narozhny & Millis PRB (2004)

$$H_{XXZ} = \sum_{i=1}^{N} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + J_1 S_i^z S_{i+1}^z \right)$$

XXZ – "quantum integrable". Exact solution via Bethe ansatz, nontrivial integrals of motion (J_1-dependent) # of such integ $\propto N$

Perturbation: $\delta H = \sum_{i=1}^{N} J_2 S_i^z S_{i+2}^z$

breaks integrability

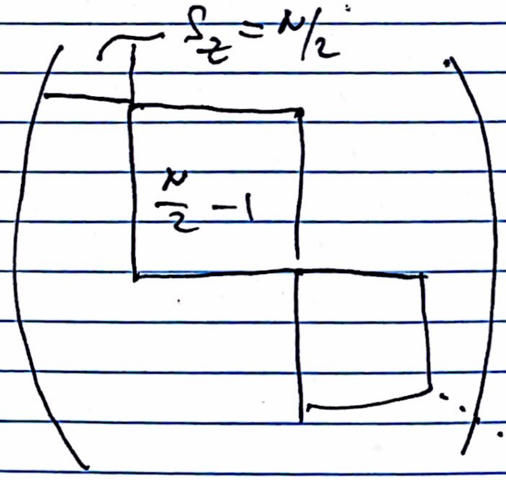
See Fig 1.: crossover from Poisson (e^{-s}, numerics indistinguishable from e^{-s} for $J_2 = 0$) to GOE (right solid curve)

Crossover apparently occurs @ $J_2 \propto \frac{1}{N}$, i.e., immediate in thermodynamic limit: either Poisson (integrable) or GOE (chaotic) [see abstract]

Note that fig. 1 is for $S_z^{tot} = 3$

$[S_z^{tot}, H] = 0$ (w periodic b.c. also $[P_{tot}, H] = 0$)

Here b.c. not periodic, so the only symmetry indep. of J_1 (space symmetry) rotation around z axis

$H =$ 

Need to look @ statistics of individual blocks! w same symm. quantum #s (cf. level crossings)

if these statistic is

General belief:

Poisson : H is integrable

WD : H is chaotic

$\Rightarrow$ Level stats provide an integrability test

Another (closely related) int test: level xings

Levels of same space-time/internal space symm (quantum #s)		
	$\nexists$ xings	chaotic
	$\exists$ xings	integrable

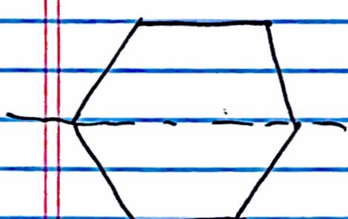
$\uparrow$ violate WvN noncrossing rule

Ex: 1D Hubbard model

$$H = T \sum_{j\sigma} \left(c^{\dagger}_{j\sigma} c_{j+1\sigma} + h.c. \right) + U \sum_{j} n_{j\uparrow} n_{j\downarrow}$$

$\sigma = \uparrow, \downarrow$

model for a benzene molecule



Symmetries:

1) Spatial

- $\hat{C}_6$: rotation by $\frac{2\pi}{6}$

momentum $[\hat{P}, H] = 0$

- reflection σ

2) Internal space

- Spin

$$\hat{S}_z = \frac{N_{\uparrow} - N_{\downarrow}}{2} \qquad \hat{S}_+ = \sum_j c^{\dagger}_{j\uparrow} c_{j\downarrow}$$

$$N_{\sigma} = \sum_j n_{j\sigma} \qquad \hat{S}_- = \left(\hat{S}_+ \right)^{\dagger}$$

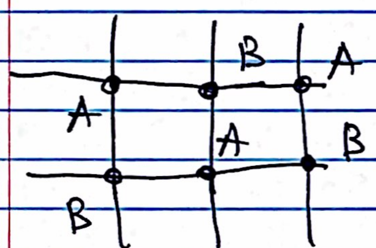
spin $SU(2)$ $\quad [\hat{\vec{S}}, H] = 0$

- η-pairing $SU(2)$

$$\hat{L}_z = \frac{N_\uparrow + N_\downarrow}{2} \qquad L_- = \sum_j (-1)^j c_{j\uparrow} c_{j\downarrow}$$

$$[\hat{\vec{L}}, H] = 0$$

holds $\forall$ bipartite lattice (e.g. on cubic)



only A — B links

Internal space symm.: $\frac{SU(2) \times SU(2)}{Z_2} = SO(4)$

same as in quantum Kepler problem

Symmetry quantum #s

$$\hat{L}_z, \hat{\vec{L}}^2, \hat{S}_z, \hat{\vec{S}}^2, \hat{\sigma}, \hat{P}$$

Can map $T \to -T$ and $U \to -U$ separetelly

Sufficient to consider one sign, e.g.

$$T \leq 0 \quad \text{and} \quad U \geq 0$$

One essential parameter.

Set $U - 4T = 1$

Let $U = u \Rightarrow T = \frac{u-1}{4}$

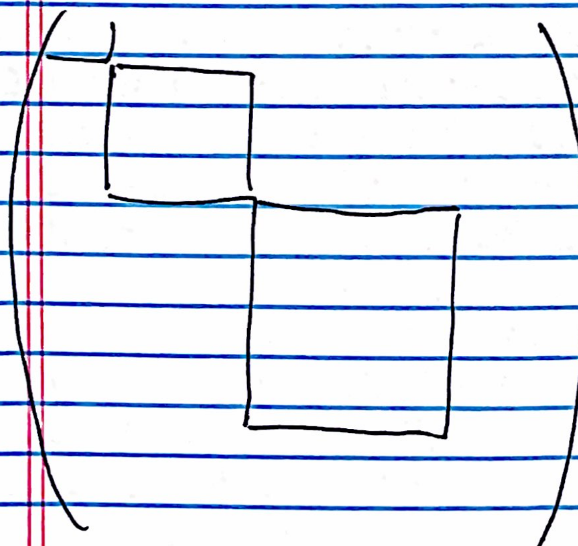
$$0 \le u = \frac{U}{U-4T} \le 1$$

All $T \le 0$, $U \ge 0$ map to $u \in (0,1)$

Let $N_\uparrow = N_\downarrow = 3$ (half-filling)

$L_z = 3$ $\quad S_z = 0$

$$\dim(\mathcal{H}) = \left(\frac{6!}{3!\,3!}\right)^2 = 400$$

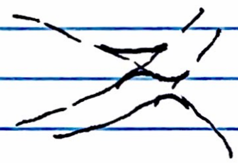


Blocks of sizes from 1×1 to 14×14

See Fig. 1 of Yuzbashyan, Shastry, Altshuler JPA (2002)

$P=3$, $\sigma=1$, $S=0$, $L=1$

Xings @ $u=1$ ($\bar{U}=\infty$) and $u=0$ don't count. Note 2 xings near $u=0.6$



Cannot tell numerically. Here can explicitly determine exact eigenstates ~~x~~ for small blocks and rigorously prove $\exists$ xings

$\Rightarrow$ WvN rule is violated for 1D Hubbard!

But... can it be we missed some of the symm. and in fact these are xings of levels w different symm?

$$H(u) = \frac{u-1}{4} M_1 + u M_2 = A + u B$$

u → real parameter

What do we call symm?

Space-time / internal space symm. as above.

These are u-indep!

$$[S, H(u)] = 0 \quad \forall u$$

$$[S, A] + u[S, B] = 0 \quad \forall u$$

$$\Rightarrow \quad [S, A] = [S, B] = 0$$

Suppose A is nondeg. (In other blocks $\exists\, u_0$ such that $H(u_0)$ is nondeg. Shift to that u_0 $\quad \underbrace{A + u_0 B}_{A_{new}} + \underbrace{(u-u_0)}_{U_{new}} B$)

Go to basis where A is diagonal

$$[S, A]_{ij} = S_{ij} a_j - a_i S_{ij} = 0 \Rightarrow S_{ij} = 0$$

$\Rightarrow S$ is diagonal too. But B isn't diagonal

Suppose $B_{ij} \neq 0 \ \forall i \neq j$

$$(S_i - S_j) B_{ij} = 0$$

$$\Rightarrow S_i = S_j \quad \Rightarrow \quad S = s\,\mathbb{1} \text{ trivial}$$

1D Hubbard has "nontrivial" U-dependent symm. Shastry (1986) Conserved currents

$$\hat{I}_2 = -iT \sum_{j\sigma} \left(c^\dagger_{j+2\sigma} c_{j\sigma} - h.c. \right) - iU \sum_{j\sigma} \left(c^\dagger_{j+1\sigma} c_{j\sigma} - h.c. \right) \times$$

$\sigma = \uparrow \quad -\sigma = \downarrow$

$$\times \left(n_{j+1,-\sigma} + n_{j,-\sigma} - 1 \right)$$

linear in the parameter

$$\left[\hat{H}(u), \hat{I}_2(u)\right] = 0 \quad \forall u$$

Trivial symm. $\hat{H}(u), \hat{H}^2(u), \hat{H}^3(u), \ldots$

Integrals of motion:

$$[\hat{H}, \hat{I}] = 0 \quad \text{and} \quad \frac{\partial \hat{I}}{\partial t} = 0$$

$$\Rightarrow \quad \frac{d\hat{I}}{dt} = i[\hat{H}, \hat{I}] = 0$$

What constitutes a nontrivial int of motion?

$$\hat{H} \longrightarrow \underbrace{\begin{pmatrix} \text{Hermitian} \\ \text{matrix} \end{pmatrix}}_{N \times N} \xrightarrow[\text{eigenbasis}]{} \begin{pmatrix} \times & & & & \\ & \times & & & \\ & & \times\, \varepsilon_k & & \\ & & & \times & \\ & & & & \times \end{pmatrix}$$

Any diag. matrix D commutes w. $\hat{H}$

Projection opts $\quad P_k = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \sim k$

$$\hat{H} = \sum_k \varepsilon_k P_k \qquad [P_k, \hat{H}] = 0 \quad \forall k$$

$$D = \sum_k d_k P_k$$

All quantum Hamiltonians have a full set of integrals of motion?

Normally (as, e.g., in CM) want to say that there some (rare) $\hat{H}$ is a full set of int of motion which we call integrable, and the rest nonint, but all Hermitian opts look the same from this point of view.

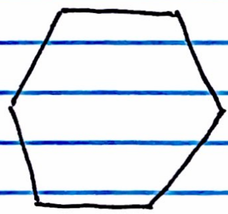
Also, if $\hat{H}$ is nondeg, can show that

$$[I, H] = 0 \iff I = \sum_{k=1}^{N} a_k H^k$$

Q: How to define a nontrivial ~~nondeg~~ integral of motion?

Violation of WvN noncrossing rule in 1D Hubbard

Xings of levels of same space-time/internal space symm



$$\hat{H} = T\underbrace{\ \ \ }\ + U\underbrace{\ \ \ }$$

$T = 1 \qquad u = U/T$

$$H(u) = A + uB \qquad 400 \times 400$$

(int. of motion)

Usual symm (total spin, momentum, etc.)

$$[S, H(u)] = 0 \ \forall u \iff [S, A] = [S, B] = 0$$

Hubbard model has additional u-dependent int of motion ~~symm.~~ $I_2(u) = C + uD$

Do these make Hubbard special, do they explain the xings?

Any Hamiltonian has trivial int of motion

$$[\hat{H}, \hat{I}] = 0 \qquad \hat{I} = \hat{H}, \hat{H}^2, \ldots$$

They cannot explain any crossings or make $\hat{H}$ special. Must have a nontrivial in some sense integral

What constitutes a nontrivial int. of motion?

$$\hat{H} \longrightarrow \begin{pmatrix} \text{Hermitian} \\ \text{matrix} \end{pmatrix} \xrightarrow[\text{eigenbasis}]{} \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_N \end{pmatrix}$$

Any diag matrix commutes w $\hat{H}$, $[\hat{H}, D] = 0$

Projection opts. $\hat{P}_k = |k\rangle\langle k|$

$$\begin{array}{c} \\ P_k = \end{array} \begin{array}{c} k \\ \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \\ & & & & \ddots 0 \end{pmatrix} \end{array} \qquad \hat{H} = \sum_k \varepsilon_k \hat{P}_k$$

$$[\hat{P}_k, \hat{H}] = 0 \quad \forall k \qquad D = \sum_k d_k P_k$$

Moreover, suppose $\hat{H}$ – nondegenerate

Can show $[\hat{I}, \hat{H}] = 0 \iff \hat{I} = \sum_k a_k \hat{P}_k$

Equivalently, $\hat{I} = \sum_{k=1}^{N} c_k \hat{H}^k$,

i.e., cannot have any nontrivial in this sense int. of motion.

Main paradox of quantum integrability

On one hand "nontrivial" additional symm, on the other there cannot be any nontrivial symm. at all

Back to "universality of RMT" (where and why RMT works)

Two more examples

1) Common buzzards (buteo buteo)

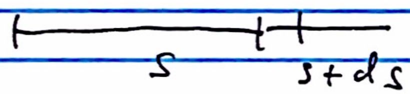
– show picture of the bird from Wikipedia

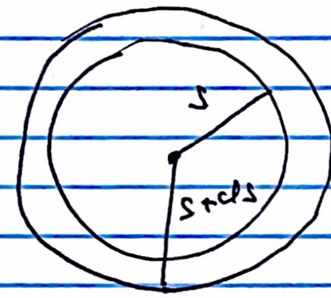
– nest locations Poisson vs WD (Gubre)

Note: 2D

a) Poisson

1D





$P(s) \propto e^{-s}\,ds$ $\qquad\qquad$ $P(s) \propto e^{-as^2}\,s\,ds$

Also some level repulsion, but notice clustering

b) RMT: need complex matrices w complex eigenvalues

Note $\beta = 0.3 - 1$

$$z = x + iy \qquad s = \sqrt{x^2+y^2} = |z|$$

2D particles w 2D Coulomb interactions

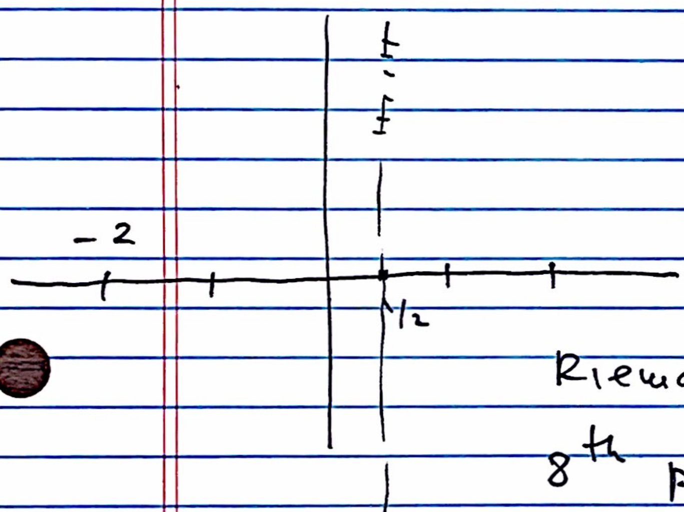
2) Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \mathrm{Re}(s) > 1$$

Otherwise – analytic continuation

Plays key role in number theory, important applications in physics (e.g. zeta function regularization)

$$\sum_{n=1}^{\infty} n = 1+2+3+\ldots = \zeta(-1) = -\frac{1}{12}$$



$\zeta(z)$

1) single pole @ $z=1$

2) trivial zeroes @ $z = -2, -4, -6, \ldots$

Riemann hypothesis! (Hilbert's 8th problem US \$1 million)

All other zeroes are on $\mathrm{Re}(z) = 1/2$ line

See Odlyzko paper for comparison between spacings and GUE

$$z_n = 1/2 + i\,\gamma_n$$

Applicability of RMT

Works well for local (short-range) correlations and doesn't work for global ones

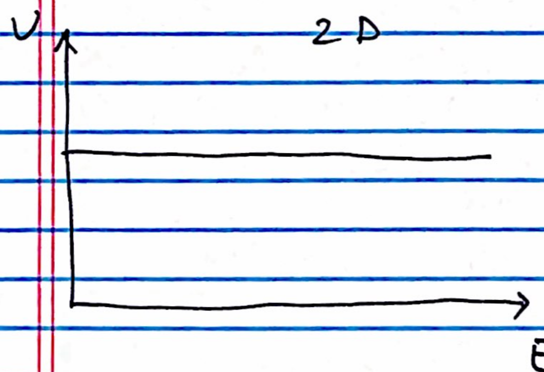
Example of global: DOS

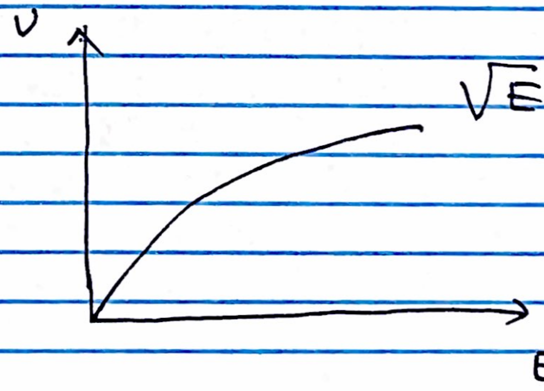
Fermi gas in n-dim

$$\frac{V\, dp_1 \dots dp_n}{(2\pi\hbar)^n} \propto V\, p^{n-1}\, dp \propto V p^{n-2}\, dE$$

$$p^2 = p_1^2 + \dots + p_n^2 \qquad E = \frac{p^2}{2m} \qquad dE \propto p\,dp$$

$$\nu \propto p^{n-2} \propto E^{\frac{n-1}{2}}$$





not in agreement w Wigner semicircle

Short range: captures level repulsion

For anything that repels expect

$$P(s) \propto s^{\beta} \quad \text{for } s \ll 1$$

We understand this power law for Gaussian RM ensembles. Note β is different for birds.

$s \gg 1$ – uncorrelated (repulsion not effective)

By central limit theorem expect normal distribution $P(s) \propto e^{-as^2}$ $\quad s \gg 1$

Together $P(s) = b\, s^{\beta} e^{-as^2}$ simplest thing we can write

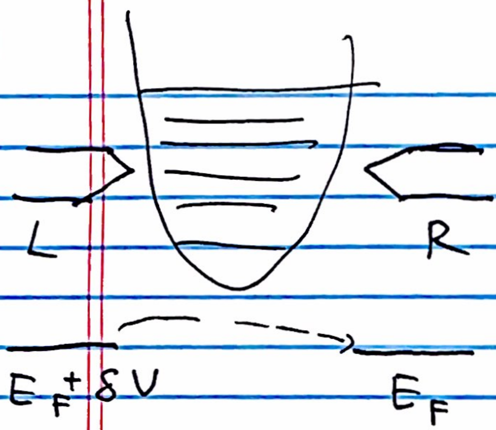
More specifically, let's understand when RMT applies to QDs

<u>Energy scales</u>

1) $\delta_1 = \langle \varepsilon_{k+1} - \varepsilon_k \rangle$ low energy scale (large time)

2) Thouless energy $E_T = \dfrac{\hbar}{\tau_e}$

τ_e – ergodic time: time to fly accross the dot



View tunneling as energy measurement

$\tau_e \sim$ measurement time

$$\Delta E \, \tau_e \sim \hbar$$

$E_T = \Delta E$ – uncertainty in energy

Conductance $\quad G = \frac{1}{R} \propto \frac{\Delta E}{\delta}$ – # of levels within uncertainty window

Dimensionless conductance $\quad g = E_T/\delta$

$$G \propto g$$

Dimensional analysis $\quad IR = V$

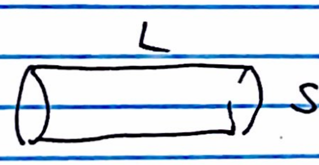
$$G = \frac{1}{R} = \frac{I}{V} = \frac{e/t}{E/e} = \frac{e^2}{Et} = \frac{e^2}{\hbar}$$

Expect $\quad G = \frac{e^2}{\hbar} g$

$\uparrow$ quantum of conductance

~~I. Diffusive regime~~

$$R = \rho \frac{L}{S} \quad \text{(resisters in series)}$$



$$G = \sigma \frac{S}{L} \simeq \sigma \frac{L^{d-1}}{L} = \sigma L^{d-2}$$

I. Diffusive regime

Einstein relation $\quad \sigma = e^2 \nu D$

DOS @ E_F

$$G = e^2 \nu D L^{d-2} = \frac{e^2}{\hbar} \left(\nu L^d\right) \left(\frac{\hbar D}{L^2}\right) = \frac{\hbar}{\tau_e} = E_T$$

$$\ell^2 = D t \qquad L^2 = D \tau_e$$

$$\tau_e = L^2 / D$$

$$\nu L^d = \nu V \text{ — full DOS} = \frac{1}{\delta}$$

ν — DOS/volume

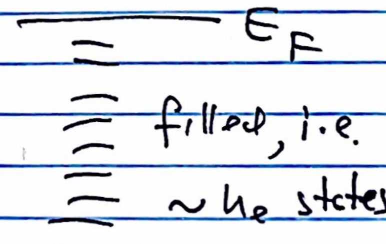
$$G = \frac{e^2}{\hbar} \frac{E_T}{\delta} = \frac{e^2}{\hbar} g$$

II. Ballistic regime

Drude formula $\quad \sigma = \dfrac{e^2 n_e \tau}{m}$ ← time between collisions $\simeq \tau_e$ in ballistic reg.

~~Can show~~ $\quad \nu \propto \dfrac{n_e}{E_F}$

Exact $\quad \nu = \dfrac{d n_e}{E_F}$



$$G \propto \underbrace{\frac{e^2 \nu E_F \tau_e}{m}}_{\sigma} \frac{L^d}{L^2} = \frac{e^2 (\nu L^d) \cancel{E_F} \cancel{\tau_e}}{\cancel{m} v_F^2 \tau_e^2}.$$

$$L = v_F \tau_e$$

$$G \propto \frac{e^2 \, 1/\delta}{\tau_e} \frac{\hbar}{\hbar} = \frac{e^2}{\hbar} \frac{E_T}{\delta}.$$

$$G = \frac{e^2}{\hbar} g$$