Welcome to Phys 682

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What are we going to do?

Only interesting & important stuff!

The rest we can decide together

→ link with tentative outline

1) probably won't be able to cover all of it

2) suggest your own topics

A. Mesoscopic physics

micro < meso < macro

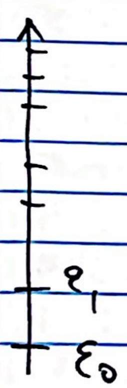
QD, RMT Nanophysics

$L = 10\,nm - 1\,\mu m$

$1 - 10^7$ particles

B. Quantum chaos & integrability

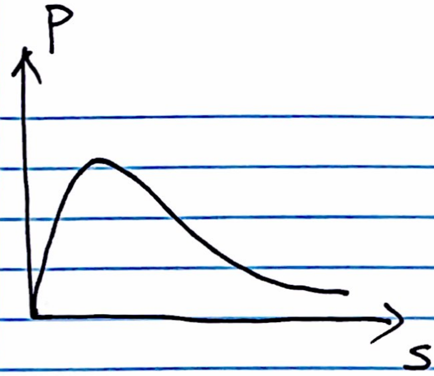
→ Generic many-body $\hat{H}$



Spacings $\varepsilon_{k+1} - \varepsilon_k$

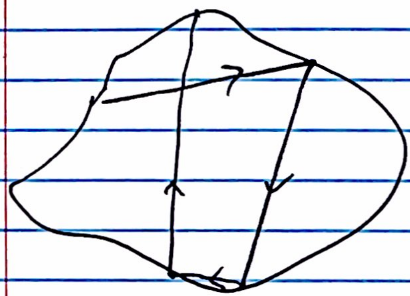
Mean level spacing $\delta = \langle \varepsilon_{k+1} - \varepsilon_k \rangle$

Dimensionless spacing $s_k = \dfrac{\varepsilon_{k+1} - \varepsilon_k}{\delta}$



$P(s) = a s e^{-b s^2}$

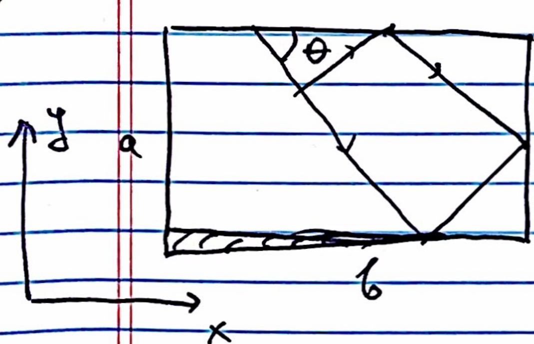
Wigner-Dyson statistics

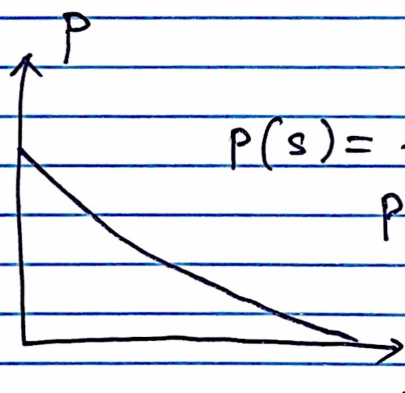


Random shape

WD statistics

chaotic billiard





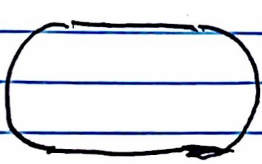
$P(s) = e^{-s}$

Poisson distr.

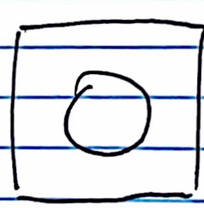
Integrable billiard $\quad |p_x|$ & $|p_y|$ conserved

Same for integrable many-body $\hat{H}$

Q. Determine θ so that comes back to same pt



Bunimovich stadium



Sinai billiard

C. Solid state qubits

- sc qubits
- transmons (Google & IBM quantum computing)

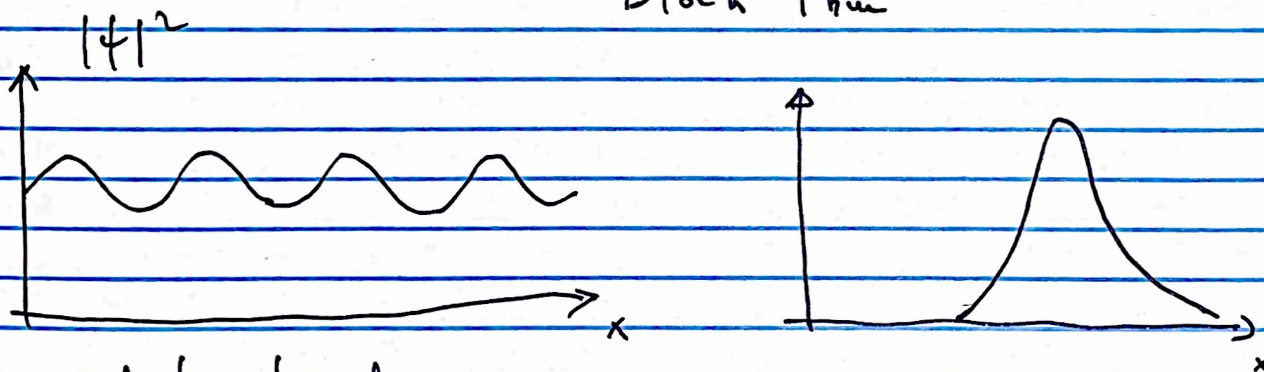
D. Anderson and many-body localization

periodic pot'l

translationally inv ⇒

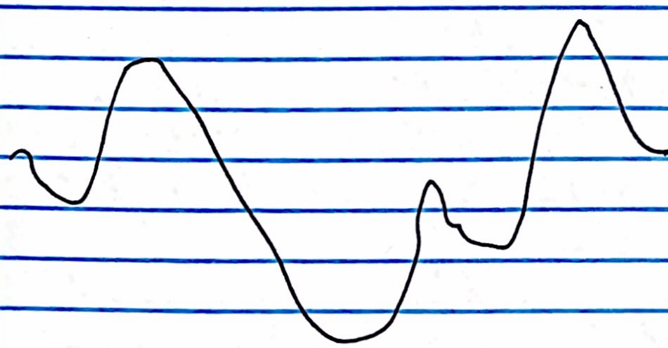
⇒ always delocalized

Bloch thm



delocalized

localized



⇒ Disorder can localize

↓

Anderson localization

+ int = MBL

E. Far from eq. ≠ noneq.

Dynamics with desired $\hat{H}(t)$

$$i \frac{\partial \psi}{\partial t} = \hat{H}(t) \psi$$

no coupl to env (dissipation), unwanted int.

Quantum gases, pump-probe exp.

All subjects are interconnected & modern

⟶ Teaching philosophy

grades, attendance, hw, presentation

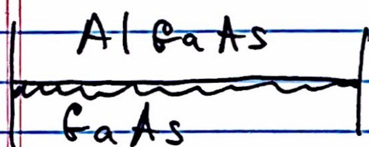
Who's experimentalist?

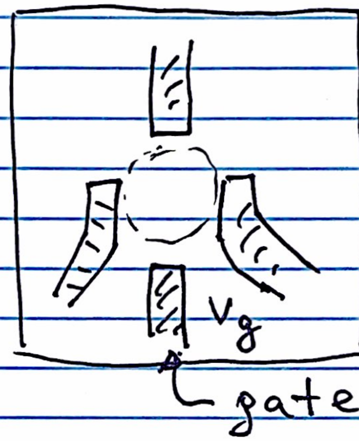
QD

- 2D QD

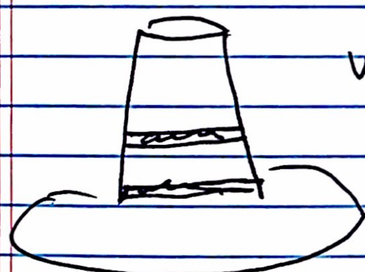
GaAs AlGaAs InGaAs

2DEG





latteral dot

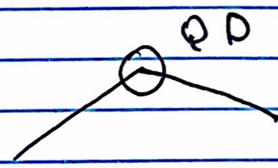


vertical dot (Tarucha)

Can create Sinai & various other billiards this way

- nanoparticles (Al) - sc qubits

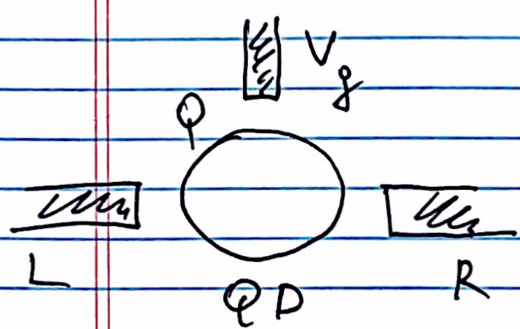
- defects



Transport thru QD

Almost closed dot

closed dot N_e – well-defined (charge)



Electrostatic energy

Ext pot'l $= -V_g Q$

Coulomb repul $= \frac{Q^2}{2C}$

$$U(Q) = -V_g Q + \frac{Q^2}{2C}$$

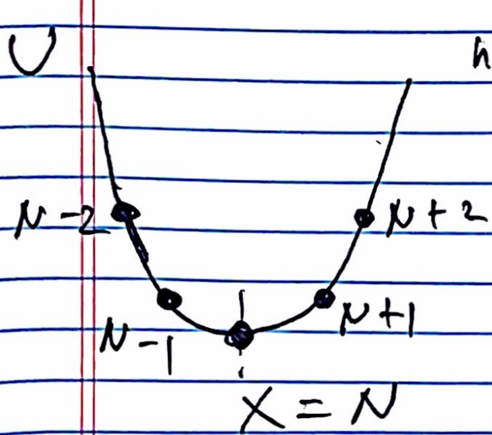
$Q = Xe$ $\quad U(X) = -XeV_g + X^2 \frac{e^2}{2C}$

Minimum of $U(X)$:

$$-eV_g + X\frac{e^2}{C} = 0$$

$$X = \frac{V_g}{eC}$$

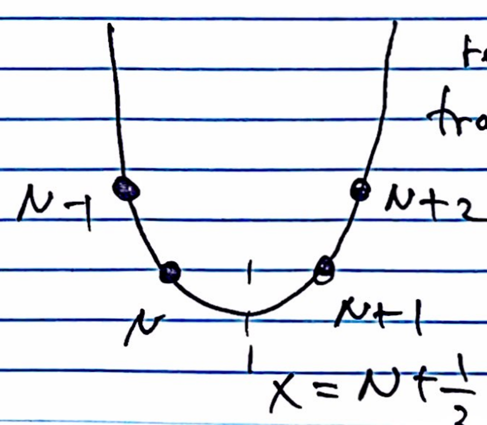
1) X – integer



no transport

Coulomb blockade

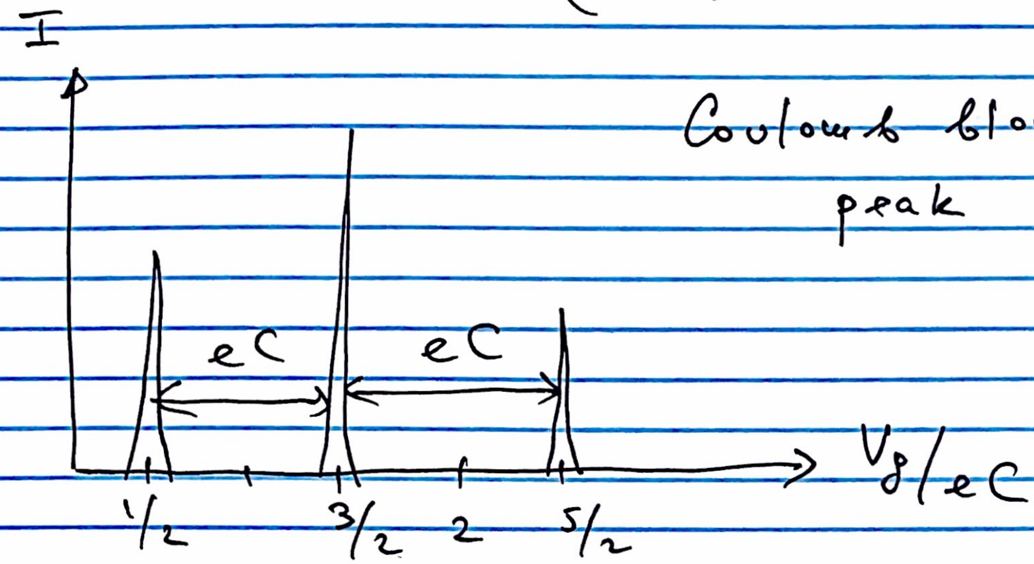
2) X – half-integer



resonant transport

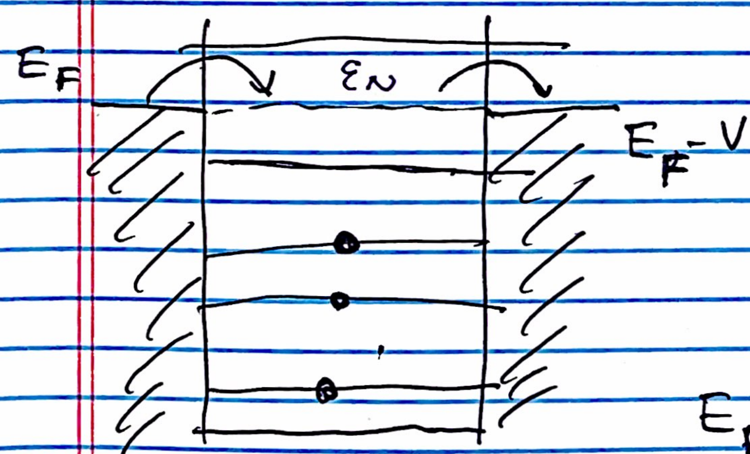
resonance $\qquad \frac{V_g}{eC} = N + \frac{1}{2}$

$$V_g = \left(N + \frac{1}{2}\right) eC$$



+ Confinement energy

resonance condition



$$E_F + U(N-1) = \varepsilon_N + U(N)$$

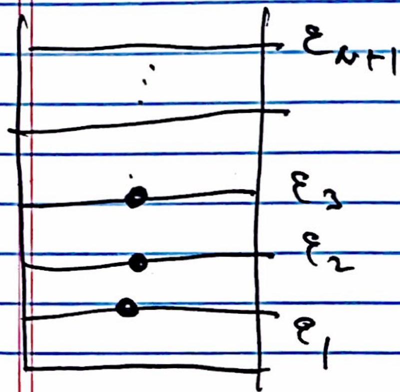
$$E_F = \varepsilon_N - eV_g + \frac{e^2}{C}\left(N - \frac{1}{2}\right)$$

$$eV_g^N = \varepsilon_N + \frac{e^2}{C}N - \frac{e^2}{2C} - E_F$$

$$eV_g^{N+1} - eV_g^N = \varepsilon_{N+1} - \varepsilon_N + \frac{e^2}{C}$$

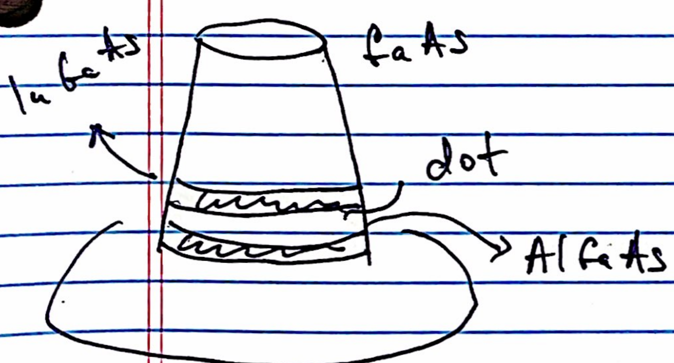
Not quite equally spaced

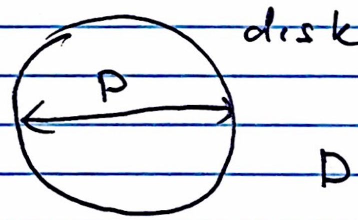
~~Can determine $E_{N+1} - E_N = \varepsilon_{N+1}$~~



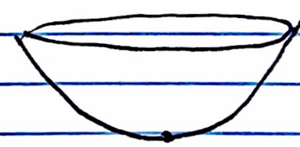
i.e., can determine ε_k

<u>Example:</u> Integrable vertical QD (Tarucha et. al.)





$D = 500\,nm$



$$V(r) = \frac{kr^2}{2} + \cancel{\alpha r^4 + \ldots}$$

2D harmonic osc.

$r^2 = x^2 + y^2$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{kx^2}{2} + \frac{ky^2}{2} =$$

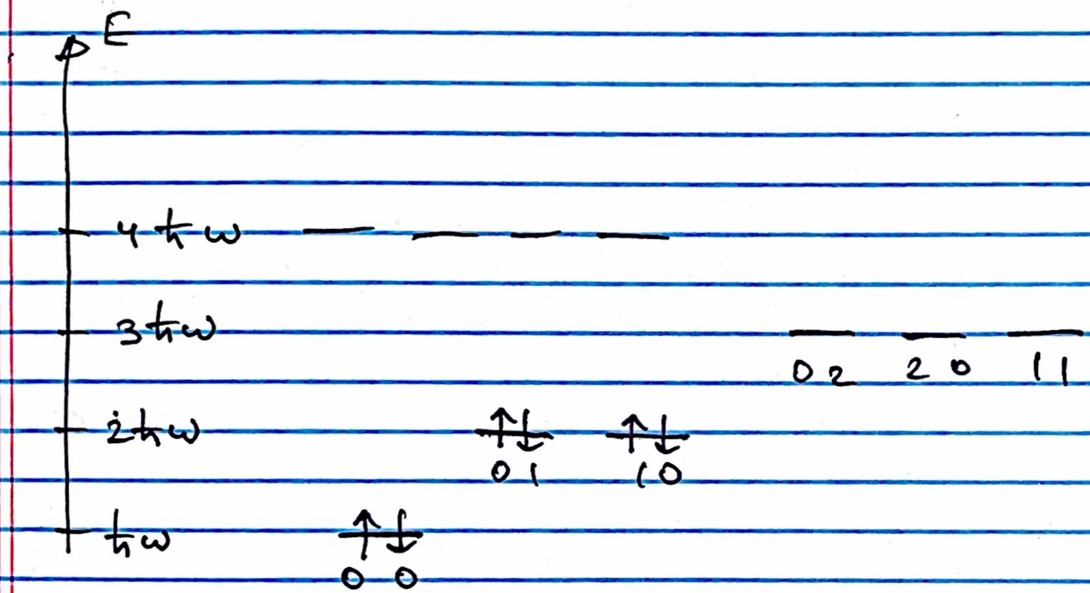
$$= \left(\frac{p_x^2}{2m} + \frac{kx^2}{2}\right) + \left(\frac{p_y^2}{2m} + \frac{ky^2}{2}\right)$$

$\omega = \sqrt{\frac{k}{m}}$ $\qquad E = \hbar\omega\left(n_1 + \frac{1}{2}\right) + \hbar\omega\left(n_2 + \frac{1}{2}\right)$

$$E = \hbar\omega(\underbrace{n_1 + n_2}_{n} + 1) \qquad n_{1,2} = 0, 1, \ldots$$

$$n = 0, 1, 2 \ldots$$

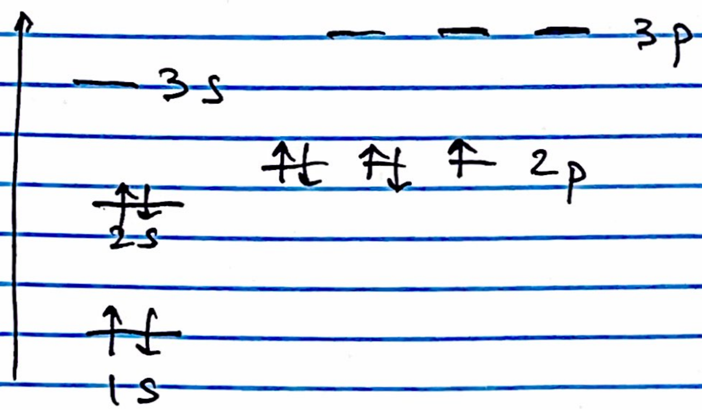
$n_1 + n_2 = n$ $\qquad$ n-fold degeneracy



shell structure

Atom electron configuration ($1/r$ pot-l)

Ex: Fluorine $\quad$ F $\quad$ $1s^2 2s^2 2p^5$



filled shells: $N = 2, 6, 12, 20, \ldots$

$\uparrow$

magic #s – main peaks in addition energy

$\rightarrow$ see bottom panel in Fig. 1

$\rightarrow$ valleys (Coulomb blockade)

Hund's rule favors parallel spin alignment (to minimize spin orbit)

half-filled outer shell

$\uparrow\ \uparrow$

$\uparrow\downarrow$

$N = 4,\quad 6 + 3 = 9,\quad 12 + 4 = 16$

↳ another set of peaks

Can see this in the same Fig.

Coulomb blockade peaks in between

Shell structure $\leftarrow$ degeneracy

degeneracy

$\swarrow\!\nearrow \qquad \searrow\!\nwarrow$

integrability $\qquad$ symmetry

$\searrow\!\nwarrow \qquad \swarrow\!\nearrow$

conservation laws

Axial symm. not enough – 1 generator $\hat{L}_z$

$U(1)$

$$H = \hbar\omega\left(a_1^+ a_1 + a_2^+ a_2\right)$$

$$\left[a_i^+ a_j , H\right] = 0 \qquad \hat{n}_1 = a_1^+ a_1 , \quad \hat{n}_2 = a_2^+ a_2$$

$$a_1^+ a_2 \qquad a_2^+ a_1$$

3 generators $\quad SU(2) \quad SU(N) \quad N^2-1$

3D isotropic osc $\quad SU(3)$

$$\left[H, \hat{n}_1\right] = \left[H, \hat{n}_2\right] = \left[H, a_1^+ a_2\right] = 0$$

but $\left[a_1^+ a_2 , n_1\right] \neq 0$

In general, degeneracy when

$$\left[H, \hat{A}\right] = \left[H, \hat{B}\right] = 0 \quad \text{but} \quad \left[\hat{A}, \hat{B}\right] \neq 0$$

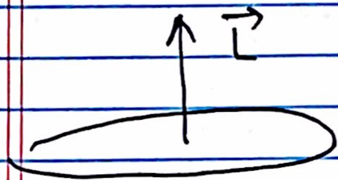
$$\underline{}_{\,01} \xrightarrow{a_1^+ a_2} \underline{}_{\,10}$$

What about atomic shells

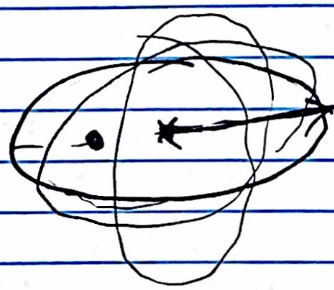
$$H = \frac{p^2}{2m} - \frac{\alpha}{r}$$

Symmetries / conserv. laws

$\vec{L}$ – angular momentum – planar motion



QM: $SU(2)$



$\vec{A} = \text{const}$ – Runge-Lenz vector

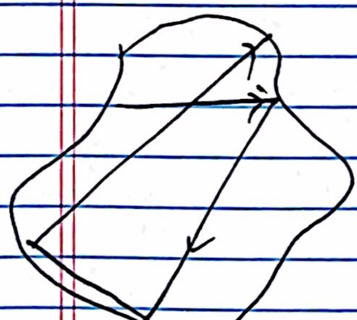
+ another $SU(2)$

$3+3=6$ rotations – 6 planes – 4D rotations

$$SO(4) = SU(2) \times SU(2)$$

QD = artificial atom!

Typically no spatial symm.



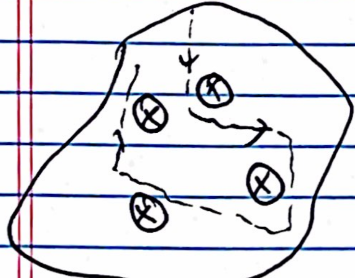
$\rightsquigarrow$ random boundary

ballistic dot

$$\hat{h} = \frac{p^2}{2m} + V(\vec{r})$$

random boundaries and/or impurities

$\varphi_i(r)$ – arbitrary basis



c_i^+, c_i – create / annihilate electron in this state

$$M_{ij} = \langle \varphi_i | \hat{h} | \varphi_j \rangle$$

$$H = \sum_{ij} M_{ij}\, c_i^{\dagger} c_j \qquad M - \text{some matrix}$$

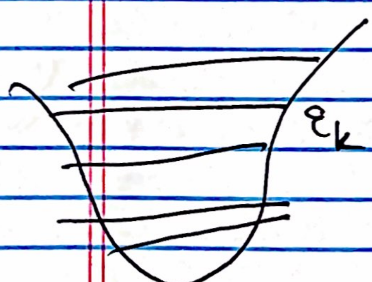
field opt. $\hat{\psi}(\vec{r}) = \sum_i \varphi_i(\vec{r})\, c_i$

$$H = \int \hat{\psi}^{\dagger}(\vec{r}) \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) \right) \hat{\psi}(\vec{r})\, d^3\vec{r}$$

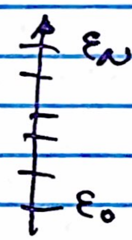
Can change the shape w gates $\Rightarrow$ ensemble of random matrices

Random – means distributed in a certain way

Energies $\quad \varepsilon_k$ – eigenv. of a random matrix



statistics
~~Distribution~~ of $\varepsilon_k = ?$



$$s_k = \frac{\varepsilon_{k+1} - \varepsilon_k}{\delta} \qquad \delta = \langle \varepsilon_{k+1} - \varepsilon_k \rangle = \frac{\varepsilon_N - \varepsilon_0}{N}$$

Distribution of $s_k = ?$

Simplest guess: ε_k – uncorrelated random #s – arrive completely @ random uniform prob. in entire range $(\varepsilon_0, \varepsilon_N)$

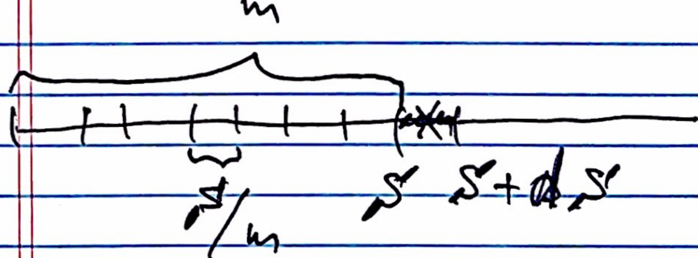
~~$\varepsilon_0 = 0 \quad \varepsilon_N = N\delta$~~

~~Energies in units of δ,~~ $\cancel{\varepsilon_k/\delta}$

prob $\varepsilon_k \in (\varepsilon, \varepsilon + d\varepsilon) = \rho\, d\varepsilon$

indep. of ε

$\rho = \text{DOS} = \dfrac{\#\text{ levels}}{\text{energy}} = \dfrac{1}{\delta}$ $\qquad S = \varepsilon_{k+1} - \varepsilon_k$



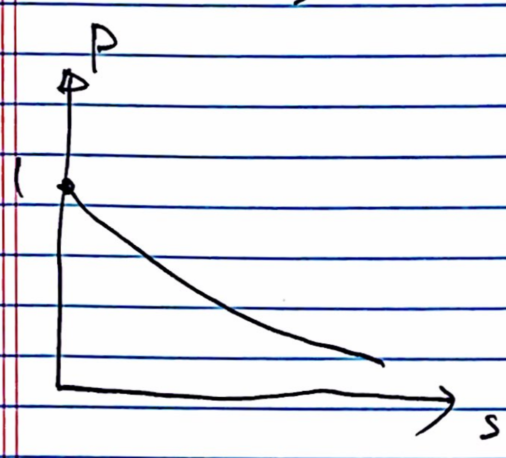
$\rho \dfrac{S}{m} \in$ given int

$1 - \rho \dfrac{S}{m} \notin$ given int

$$P(S)\, dS = \underbrace{\left(1 - \rho \frac{S}{m}\right)^m}_{\text{not before}} \underbrace{\rho\, dS}_{\text{in } (s,\, s+ds) \text{ int}}$$

$s = \dfrac{S}{\delta} = \rho S \qquad \lim_{m \to \infty} \left(1 - \dfrac{s}{m}\right)^m = e^{-s}$

$P(s)\, ds = e^{-s}\, ds$ – Poisson distr.



Note max @ $s = 0$

Not correct for a random shape. Instead, suggestive of degeneracies / symm

2D harmonic osc

$$H = \hbar\omega\left(a_1^+ a_1 + a_2^+ a_2 + 1\right) = \hbar\omega\left(n_1 + n_2 + 1\right)$$

$E_n = \hbar\omega\,(n+1)$ $\quad (n+1)$ - fold degen.

$SU(2)$ - symm

$a_1^+ a_1,\ a_2^+ a_2,\ a_1^+ a_2$ - commute w H - right # of gen

$SU(2)$ - spin algebra

S_x, S_y, S_z - generators of rotations

$$[S_x, S_y] = i S_z, \quad [S_y, S_z] = i S_x, \ \ldots$$

$$S_\pm = S_x \pm i S_y$$

$$[S_z, S_\pm] = \pm S_\pm \qquad [S_+, S_-] = 2 S_z$$

$$S^2 = S_x^2 + S_y^2 + S_z^2 = S_+ S_- + S_z^2 - S_z$$

Casimir

$$[S^2, S_i] = 0 \qquad i = x, y, z$$

spin states: $|s, m\rangle$ $\qquad S^2 |s\,m\rangle = s(s+1)\,|s\,m\rangle$

$$S_z |s\,m\rangle = m\,|s\,m\rangle$$

$$S_\pm |s\,m\rangle = \overset{\text{const}}{\checkmark}\,|s, m\pm 1\rangle \qquad m = -s, \ldots, s$$

$(2s+1)$ states w same s

Back to harm. osc. Many different ways to write the spin algebra

Schwinger bosons, Holstein-Primakoff bosons, Anderson pseudospins, diff. opt.

$$S_z = \frac{a_1^+ a_1 - a_2^+ a_2}{2} \qquad S_+ = a_1^+ a_2 , \quad S_- = a_2^+ a_1$$

$$[a_i, a_i^+] = 1 \qquad i = 1, 2$$

$\Rightarrow$ correct $so(2)$ comm. relations

$$S = \frac{a_1^+ a_1 + a_2^+ a_2}{2} = \frac{n}{2}$$

$$E_n = E_s = \hbar\omega (2s+1)$$

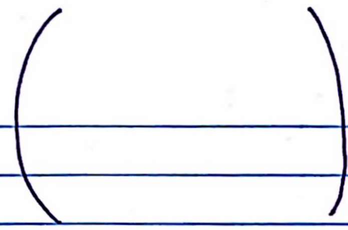
$$|s\, m\rangle \qquad m = -s, \ldots, s$$

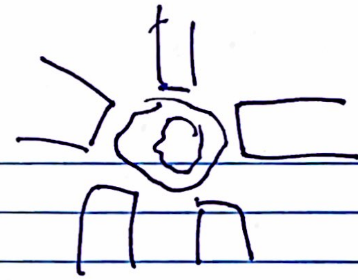
$2s+1 = n+1$ fold deg.

$n = 2$ $\qquad \overline{}_{20} \quad \overline{}_{11} \quad \overline{}_{02} \qquad S = 1$

$E_2 = 3\hbar\omega \qquad S_z = 1 \quad S_z = 0 \quad S_z = -1$

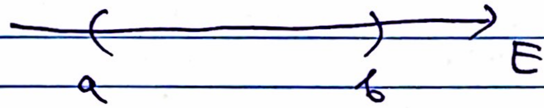
Recall:





$$H = \frac{p^2}{2m} + V(r)$$

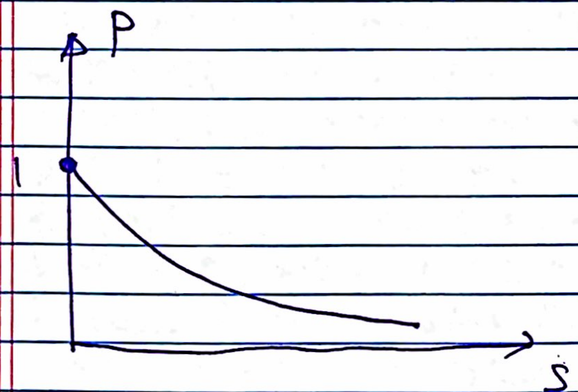
Suppose



$$\text{prob } \varepsilon_k \in (\varepsilon, \varepsilon + d\varepsilon) = \rho \, d\varepsilon$$

$P(s) = e^{-s}$ Poisson distrib.

$$s_k = \frac{\varepsilon_{k+1} - \varepsilon_k}{\delta}$$



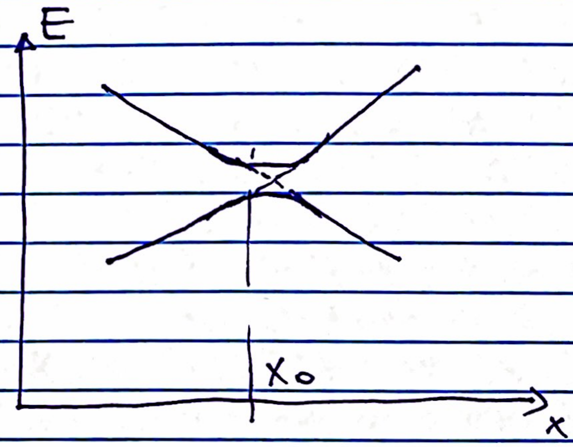
max @ $s = 0$

Incorrect for a random shape. Instead, suggests depen./symm.

Highest probability of $s=0$ prohibited by

Level repulsion

$H(x)$ x – real parameter, e.g., gate voltage



crossing/anticrossing
impossible to determine
numerically

Long history: Hund (1927)

Wigner – von Neumann noncrossing rule (1929):
Energy levels of same symm. (same quantum #s)
typically don't cross

Generic $H(x)$ – no crossings

Landau QM

$$H(x) \approx H_0 + \delta x\, V = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} + \delta x \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^* & V_{22} \end{pmatrix}$$

$$\delta x = x - x_0$$

Quadratic eq.

$$E_1 - E_2 = \sqrt{\mathcal{D}} = \left[\left(\varepsilon_1 - \varepsilon_2 + \delta x \left(V_{11} - V_{22} \right) \right)^2 + \right.$$
$$\left. + 4\, \delta x^2 \left| V_{12} \right|^2 \right]$$

For $E_1 = E_2$ need

$$1) \quad \varepsilon_1 - \varepsilon_2 = \delta x \left(V_{22} - V_{11} \right)$$

$$2) \quad V_{12}(x) = 0$$

Two conditions on a single parameter – generally impossible

Why same symmetry?

What's symm?

Symm: Hermitian S such that $[S, H] = 0$

e.g. central pot'l $\quad [L_z, H] = [\vec{L}^2, H] = 0$

Suppose $[S, H(x)] = 0 \quad \forall x$

$$\shortparallel$$

$$[S, H_0] + \delta x\, [S, V] = 0 \quad \forall\, \delta x$$

$$[S, H_0] = [S, V] = 0$$

$$H_0 |\psi_1\rangle = \varepsilon_1 |\psi_1\rangle \,, \quad H_0 |\psi_2\rangle = \varepsilon_2 |\psi_2\rangle$$

$$\langle t_1 | [S, V] | t_2 \rangle = 0$$

$S|t_1\rangle = s_1|t_1\rangle$

$S|t_2\rangle = s_2|t_2\rangle$

$s_{1,2}$ – quantum #s

$$\langle t_1 | S V | t_2 \rangle -$$

$$- \langle t_1 | V S | t_2 \rangle = 0$$

$\langle t_1 | S = \langle t_1 | s_1$

$$s_1 \langle t_1 | V | t_2 \rangle - s_2 \langle t_1 | V | t_2 \rangle = 0$$

$$V_{12} (s_1 - s_2) = 0$$

If $s_1 \neq s_2$ – levels w diff symm (quantum #s)

$\Rightarrow$ $V_{12} = 0$ – levels allowed to cross

Levels repell

If $s_1 = s_2$ – same symm.

$\Rightarrow$ $V_{12} \neq 0$ in general

$$H(x) = H_0 + \cancel{\delta} \overset{d}{} x \, V$$

$$d \varepsilon_k = dx \langle t_k | V | t_k \rangle = dx \, V_{kk}$$

$$\frac{d\varepsilon_k}{dx} = V_{kk}$$ 1st order

$$\frac{dV_{kk}}{dx} = 2 \sum_{\ell \neq k} \frac{|V_{k\ell}|^2}{\varepsilon_k - \varepsilon_\ell} \qquad \frac{dV_{k\ell}}{dx} = \ldots$$

$$\frac{d^2(\varepsilon_1-\varepsilon_2)}{dx^2} = \frac{4|V_{12}|^2}{\varepsilon_1 - \varepsilon_2} + F$$

contributions from all other levels

$$\Delta = \varepsilon_1 - \varepsilon_2$$

$$\frac{d^2\Delta}{dx^2} = \frac{4|V_{12}|^2}{\Delta}$$

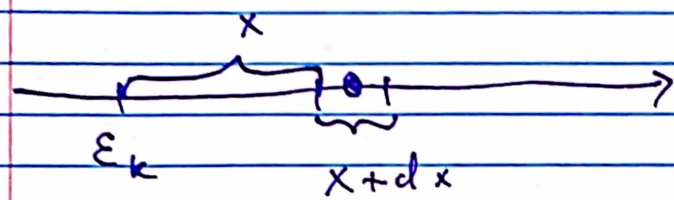
Particles moving in 1D w coordinates

$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$

$1/r$ (3D coulomb) repulsion between them

See also Pechukas PRL (1983)

Levels are correlated!



$\rho(x)\,dx$

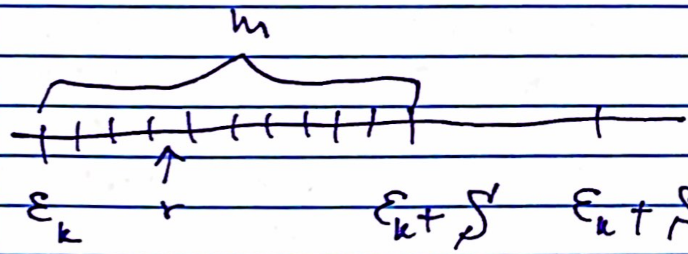
Before: $\rho(x) = \rho = \text{const} = \dfrac{\#\,\text{levels}}{\text{energy}} = \dfrac{1}{\delta}$

Now: Wigner (1951), Landau & Smorodinski (1955)

$$\rho(x) \approx \cancel{\rho(0)} + a\,x + \dots$$

level repulsion

$\rightarrow$ Extrapolate to all x, i.e., $\rho(x) = a\,x$



Before: $$P(S)\,dS = \left(1 - \rho\frac{S}{m}\right)^m \rho\,dS \rightarrow e^{-\rho S}\rho\,dS$$

$$s = \frac{S}{\delta} = \rho S \implies P(s)\,ds = e^{-s}\,ds$$

Now: $$P(S)\,dS = \prod_{r=0}^{m-1}\left(1 - a\frac{rS}{m}\frac{S}{m}\right) a S\,dS =$$

$$= a S\, e^{-aS^2/2}\,dS$$

$$\langle s \rangle = \left\langle \frac{S'}{\delta} \right\rangle = \frac{\langle S' \rangle}{\delta} = 1$$

$$P(s) = a' s \, e^{-a's^2/2} \, ds$$

a' found from $\displaystyle\int_0^\infty s \, P(s) \, ds = 1$

$$a' = \frac{\pi}{2}$$

$N \times N$

H – random Hermitian matrix

Eigenvalues E_i are correlated

$$P(E_1, E_2, \ldots, E_N) \neq P_1(E_1) P_2(E_2) \ldots P_N(E_N)$$

For $P_i(E_i) = P(E_i)$ will get Poisson $\forall P(E)$

In reality, whole matrix

$$\begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

N diagonal elem. (real)

$\frac{N^2 - N}{2}$ off-diagonal

$H^\dagger = H$ general Hermitian: $N + 2\frac{N^2 - N}{2} = N^2$ parameters

$H^T = H$ real symm: $N + \frac{N^2 - N}{2} = \frac{N(N+1)}{2}$

$$P(\{H_{ij}\}_{i \le j}) = P(H_{11}, \ldots H_{1N}, H_{22}, H_{23}, \ldots, H_{2N}, \ldots)$$

What $P(\{H_{ij}\})$ to choose?

$P(\{E_i\})$ will follow

Dyson (1972)

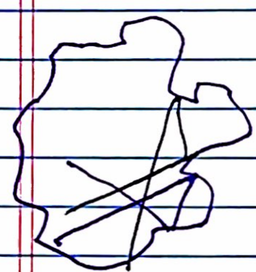
1) physically reasonable (plausible)
2) mathematically treatable

Simplest choice: indep. identically distr. (iid) random #s

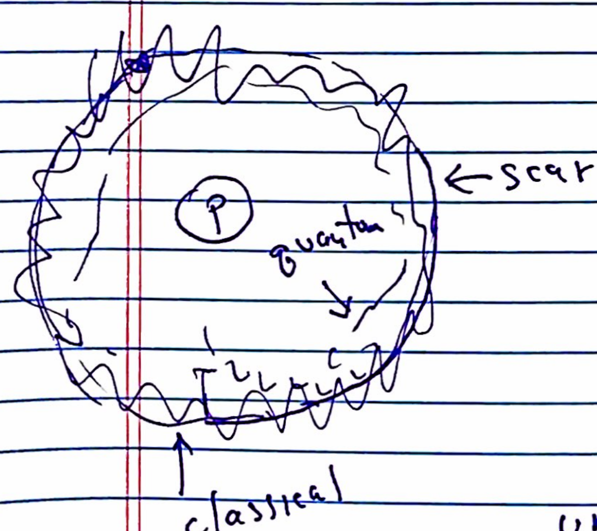
$$P\left(\{H_{ij}\}_{i \le j}\right) = \prod_{i \le j} P(H_{ij})$$

Is this physically reasonable?

Eigenstates are uncorrelated
(cf. eigenvalues)



$$H_{ij} = \int d^3r\, \psi_i^*(\vec{r}) \underbrace{\left[\frac{p^2}{2m} + V(\vec{r})\right]\psi_j}_{\varphi_j(\vec{r})}$$



$$H_{ij} = \int d^3r\, \psi_i^*(\vec{r})\,\varphi_j(\vec{r})$$

H_{ij} H_{ik} H_{re}
unrelated / uncorrelated

But there can still be some symm. No spatial symm, but can have other symm. (time, spin)

Time reversal

$$i\frac{\partial \psi}{\partial t} = H\psi \qquad i\frac{\partial \psi^*}{\partial(-t)} = H^*\psi^*$$

If $H^* = H$, $\psi^*(-t)$ is also a solution

$$H_{ij} = \varphi_i^T H \varphi_j \qquad H_{ij}^* = \varphi_i^T H^* \varphi_j = H_{ij}$$

All matrix el. real

$$H^\dagger = H \qquad H^T = H \text{ — real-symmetric}$$

$$\parallel$$

$$(H^*)^T = H^T$$

Basis change $\qquad O^T H O$ — real symm

$\qquad U^\dagger H U$ — Hermitian

Electron spin

Conserved $\quad [\vec{s}, H] = 0 \quad$ H – spin-indep

$$H = \frac{p^2}{2m} + V(r) \quad \text{vs} \quad H = \frac{p^2}{2m} + V(r) + A\,\vec{L}\cdot\vec{s}$$

Time reversal $\xrightarrow{\text{no}}$ unitary ensemble

$H = H^\dagger \qquad H \to U^\dagger H U$

GUE

$\downarrow$ yes

Electron spin conserved $\xrightarrow{\text{no}}$ symplectic ensemble

GSE

$\downarrow$ yes

Orthogonal ensemble, $H^T = H \qquad O^T H O \leftarrow H$

GOE

Wigner–Dyson symm. classification (3-fold way)

cf. similar classification for topological sc/insulators – 10-fold way, 8-fold way

	T	$\hat{\vec{S}}$	QD	β
unitary	no	N/A	$B \neq 0$	2
orthogonal	yes	yes	$B = 0$, spin-orbit $= 0$	1
symplectic	yes	no	$B = 0$, spin-orbit $\neq 0$	4

Def 1 GOE - ensemble of real symm. H:

1) no preferred basis (invariance)

$$H \to H' = O^T H O$$

$$P(H')\, dH' = P(H)\, dH$$

prob of $\in$ volume element dH

$$dH = \prod_{k \le j} dH_{kj}$$

2) H_{kj} $(k \le j)$ statistically indep.

$$P(H) = \prod_{k \le j} f_{kj}(H_{kj})$$

Def 2 GUE same as GOE except H - Hermitian

$$H_{kj} = H'_{kj} + i H''_{kj}$$

1) Invariance $\quad P(\tilde{H}^*)\, d\tilde{H}^* = P(H)\, dH$

$$\tilde{H}^* = U^+ H U \qquad dH = \prod_{k \le j} dH'_{kj} \prod_{k<j} dH''_{kj}$$

2) stat. indep.

$$P(H) = \prod_{k \leq j} f'_{kj}(H_{kj}) \prod_{k<j} f''_{kj}(H_{kj})$$

Amazing fact: these definitions essentially completely determine the form of $P(H)$

$$P(H')\,dH' = P(H)\,dH$$

GOE: $H' = O^T H O$

$$P(O^T H O)\left|\frac{\partial H'}{\partial H}\right| dH = P(H)\,dH$$

Jacobian $\left|\frac{\partial H'}{\partial H}\right| = \mathrm{Det}\underbrace{\left(\frac{\partial H'_{ij}}{\partial H_{ke}}\right)}_{M} = 1$

$\Downarrow$

$$H'_{ij} = O^T_{ik} H_{ke} O_{ej} \qquad\qquad P(O^T H O) = P(H)$$

$$\frac{\partial H'_{ij}}{\partial H_{ke}} = O^T_{ik} O_{ej} = O^T_{ik} O^T_{je}$$

$$M = O^T \otimes O^T$$

$$\det(\underset{n\times n}{A}\otimes\underset{m\times m}{B}) = (\underset{\lambda_i}{\det A})^m(\underset{\mu_i}{\det B})^n$$

$\lambda_i\mu_j$

$\det A = \lambda_1\cdots\lambda_n \qquad \det B = \mu_1\cdots\mu_m$

$$\det(A\otimes B) = \overbrace{(\lambda_1\mu_1\cdots\lambda_1\mu_m)(\lambda_2\mu_1\cdots\lambda_2\mu_m)\cdots(\lambda_n\mu_1\cdots\lambda_n\mu_m)}^{n\text{ terms}} =$$

$$A\otimes B\,|\lambda_i\rangle\otimes|\mu_j\rangle = \lambda_i\mu_j\,|\lambda_i\rangle\otimes|\mu_j\rangle$$

$$= (\lambda_1\cdots\lambda_n)^m(\mu_1\cdots\mu_m)^n$$

$$\det M = (\det O^T)^N(\det O^T)^N = 1$$

$$P(O^THO) = P(H) \quad \forall\, O$$

$\Rightarrow$ $P(H)$ must depend on invariants of H only

Inv of H — eigenvalues $E_1,\dots,E_N$

Symmetric combinations $\qquad N$ #s vs $\frac{N(N+1)}{2}$

$S_1 = E_1+\dots+E_N \qquad \mathrm{Tr}\,H$

$S_2 = E_1E_2+E_1E_3\cdots$

$\vdots$

$S_N = E_1\cdots E_N \qquad \mathrm{Det}\,H$

$p_1 = s_1 = \sum_i E_i$ $\qquad$ $\mathrm{Tr}\, H$

$p_2 = \sum_i E_i^2$ $\qquad$ $\mathrm{Tr}\, H^2$

$\vdots$

$p_N = \sum_i E_i^N$ $\qquad$ $\mathrm{Tr}\, H^N$

$$P(H) = F(\mathrm{tr}\, H, \mathrm{tr}\, H^2, \dots, \mathrm{tr}\, H^N)$$

N parameters (variables)

Turns out (due to stat. indep)

$P(H) = F(\mathrm{tr}\, H, \mathrm{tr}\, H^2)$, moreover

Thm (Porter & Rosenzweig)

$$P(H) = e^{-a\,\mathrm{tr}\, H^2 + b\,\mathrm{tr}\, H + c}$$

a, b, c – real, $a > 0$ for all 3 ensembles (GOE, GUE, GSE)

Proof: $H' = O^T H O \quad (1)$

$$O = \left(\begin{array}{cc|c} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & O \\ \hline & O & \mathbb{I} \end{array}\right)$$

unitary, orthogonal & symplectic

Invariance $\Rightarrow$ $P(H') = P(H)$

$\Rightarrow \dfrac{dP(H')}{d\theta} = 0 \qquad P(H') = \prod_{k \le j} f_{kj}(H'_{kj})$

stat. indep.

$$\ln P = \sum_{k \le j} \ln f_{kj}(H'_{kj})$$

$$0 = \left.\frac{d \ln P(H')}{d\theta}\right|_{\theta=0} = \left.\sum_{k \le j} \frac{1}{f_{kj}} \frac{d f_{kj}}{d H'_{kj}} \frac{d H'_{kj}}{d\theta}\right|_{\theta=0}$$

Determine $\dfrac{dH'_{kj}}{d\theta}$ from (1), substitute and set $\theta = 0$

$$H'_{kj} \xrightarrow[\theta \to 0]{} H_{kj}$$

$$[f_{11}, f_{12}, f_{22}, H_{11}, H_{12}, H_{22}] +$$

$$+ \sum_{k=3}^{N} \left(-\frac{1}{f_{1k}} \frac{d f_{1k}}{d H_{1k}} H_{2k} + \frac{1}{f_{2k}} \frac{d f_{2k}}{d H_{2k}} H_{1k} \right) = 0$$

Only $f_{1k}, f_{2k}, H_{1k}, H_{2k}$ bc only U_{12} & $U_{21} \neq 0$

Recall: H_{ij} – in dep. variables $(i \leq j)$

$N=4$

$$\underset{\substack{\uparrow \quad \uparrow \\ H_{13} \; H_{23}}}{F(x, y)} + \underset{\substack{\uparrow \quad \uparrow \\ H_{14} \; H_{24}}}{G(z, w)} + \underset{H_{11}, H_{12}, H_{22}}{Q(u, v, t)} = 0$$

Must have $F(x,y) = C_1$, $G(z,w) = C_2$, ...

$$-\frac{1}{f_{1k}} \frac{d f_{1k}}{d H_{1k}} H_{2k} + \frac{1}{f_{2k}} \frac{d f_{2k}}{d H_{2k}} H_{1k} = \text{const}$$

$f_{1k} \to f_1$, $f_{2k} \to f_2$, $H_{1k} \to x_1$, $H_{2k} \to x_2$

$$-\frac{d \ln f_1(x_1)}{d x_1} x_2 + \frac{d \ln f_2(x_2)}{d x_2} x_1 = \text{const}$$

$2 x_1 \, d x_1 = d(x_1^2)$ $\qquad$ $2 x_2 \, d x_2 = d(x_2^2)$

$$\underbrace{-\frac{d \ln f_1}{d x_1^2}}_{f\left(\frac{1}{x_1}\right)} + \underbrace{\frac{d \ln f_2}{d x_2^2}}_{g\left(\frac{1}{x_2}\right)} = \frac{c}{x_1 x_2} = c \frac{1}{x_1} \frac{1}{x_2}$$

$$x \equiv \frac{1}{x_1}, \quad y \equiv \frac{1}{x_2}$$

$$-f(x) + g(y) = c\, x\, y$$

Must have $c = 0$. If $c \neq 0$, set $y = 0$

$f(x) = +g(0) = \text{const}$. Similarly, $g(y) = \text{const}$

$$\Rightarrow \quad C_1 + C_2 = c\, x\, y \quad \times$$

$\Rightarrow c = 0$

$$-f(x) + g(y) = 0$$

$$f(x) = a = g(y)$$

(2) $\dfrac{d \ln f_1}{d x_1^2} = -a = \dfrac{d \ln f_2}{d x_2^2} \Rightarrow f_1(x_1) = e^{-a x_1^2}$

$$f_{1k}(H_{1k}) = e^{-a H_{1k}^2}$$

Can show a is k-indep bc (2) holds for any two pairs, i.e., $(1k) = (2k) = (ij) \ldots = (12)$

$$\Rightarrow \quad f_{ij}(H_{ij}) = e^{-a H_{ij}^2}$$

$P(H)$ depends on squares of off-diag. elements only

$$\Rightarrow P(H) = F(\operatorname{tr} H^2, \operatorname{tr} H)$$

$\mathrm{tr}\, H^2 = (HH)_{ii} = H_{ik} H_{ki} = \sum_{i,k} H_{ik}^2 =$

Let $P(H) = e^{-a\,\mathrm{tr} H^2 + \cancel{b\,\mathrm{tr} H + c}}$ $\quad f(\mathrm{tr}\, H)$ $\quad = 2\sum_{i<k} H_{ik}^2 + \sum_i H_{ii}^2$

$\uparrow$ cannot have $\mathrm{tr}\, H^2$

$$f\left(\sum_i H_{ii}\right) = \prod_i f_{ii}(H_{ii})$$

Only $f(x) = \text{const}\; e^{bx}$ has this property

$\Rightarrow \quad P(H) = e^{-a\,\mathrm{tr} H^2 + b\,\mathrm{tr} H + c}$

$a, b, c \in \mathbb{R} \qquad a > 0$ so that $P(H)$ normalizable

Let $H_{old} = H_{new} + \frac{b}{2a}\mathbb{1}$

$$P(H) = \text{const}\; e^{-a\,\mathrm{tr} H^2} = \text{const} \prod_{i<k} e^{-2aH_{ik}^2} \prod_i e^{-aH_{ii}^2}$$

$\langle H_{ik} \rangle = 0 \qquad \langle H_{ik} H_{je} \rangle = 0$ if $je \neq ik$ or ki

$$\langle H_{ik}^2 \rangle = \frac{1}{4a} \qquad i \neq k$$

Alternative approach based on information theory

Balian (1968)

Entropy: (stat. mech) logarithmic measure of the # of system states w significant probability of being occupied

$$S = -\sum_i p_i \ln p_i \qquad (k_B = 1)$$

$$= -\langle \ln p \rangle$$

In general basis $\quad S = -\mathrm{Tr}(\hat{\rho} \ln \hat{\rho})$

$\hat{\rho}$ - density matrix $\quad \hat{\rho} = \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_N \end{pmatrix}$ in eigenbasis

Fundamental postulate of stat mech: all microstates w same energy populated w equal probability

$p_i = \frac{1}{\Omega} \qquad \Omega = \#$ of microstates

$$S = \ln \Omega$$

$p_{i_0} = 1 \Rightarrow S = 0$. S - measure of uncertainty (lack of knowledge about the sys. state

Missing information:

$$S[P(H)] = -\int dH\, P(H) \ln P(H) = -\langle \ln P(H) \rangle$$

Most random (least biased) distribution (know least about it) — maximize $S[P(H)]$

Constraints:

1) $\int dH\, P(H) = 1$ normalization

2) $\langle f(H) \rangle = \int dH\, P(H)\, f(H) = \text{const}$

(e.g. $f(H) = \operatorname{tr} H^4$ to ensure eigenvalues of H are bounded)

$$\text{Action} = -\int dH \left[P \ln P + \lambda_1 P + \lambda_2 P f \right] = 0$$

$$\delta P \left[\ln P + 1 + \lambda_1 + \lambda_2 f \right] = 0$$

$$P = \text{const}\; e^{-\lambda_2 f(H)}$$

$f(H) = \operatorname{tr} H^2$ — GOE, GUE, GSE

$H\psi^i = E_i \psi^i$, E_i – eigenvalues of H

$P_J(E_1, \dots, E_N) = ?$ (joint prob. density fn for eigenv.)

Idea

$$\begin{array}{ccc} \underline{\text{old}} & & \underline{\text{new}} \\ H_{jk}\ (j \le k) & \longrightarrow & E_i,\ p_\mu \\ \uparrow & & \uparrow \\ \frac{N(N+1)}{2} & = & N + \frac{N(N-1)}{2} \end{array}$$

$$H = O^T \underset{N}{\underset{\uparrow}{\Sigma}}\ \underset{\frac{N(N-1)}{2}}{\underset{\uparrow}{O}} \qquad \Sigma = \begin{pmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_N \end{pmatrix}$$

$$\begin{array}{ccccc} P(H)\,dH & = & P(E_i) & \left|\dfrac{\partial H}{\partial(E_i, p_\mu)}\right| & \prod_i dE_i \prod_\mu dp_\mu \\ \parallel & & \parallel & \equiv & \\ e^{-a\,\mathrm{tr}\,H^2} & & e^{-a\sum_i E_i^2} & J(E_i, p_\mu) & \end{array}$$

$$P_J(E_1, \dots, E_N) = e^{-a\sum_i E_i^2} \underbrace{\int J(E_i, p_\mu) \prod_\mu dp_\mu}_{I(E_1, \dots, E_N)}$$

When is $H_{jk} \to E_i, p_\mu$ not unique?

When $\exists$ degeneracy (eigenvectors $\Rightarrow p_\mu$ aren't uniquely defined)

$\Rightarrow I(E_1, \dots, E_N) = 0$ when $E_j = E_k \;\forall\, j = k$

Expect: $I \propto \prod_{j<k} |E_j - E_k|$

Real answer: $I = \prod_{j<k} |E_j - E_k|^\beta$

$\beta = 1 \quad 2 \quad 4$

GOE GUE GSE

for $\beta = 2, 4$ H_{jk} & P_μ have more component

$J(E_i, P_\mu)$ is large matrix $\Rightarrow$ large power of $|E_j - E_k|$

Thus $P(E_1, \dots, E_N) = c \prod_{k<j} |E_j - E_k|^\beta e^{-a \sum_j E_j^2}$