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Liquid–Vapor Phase Transitions for Systems with Finite-Range Interactions

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We consider particles in $\mathbb{R}^d$, $d \geqslant 2$, interacting via attractive pair and repulsive four-body potentials of the Kac type. Perturbing about mean-field theory, valid when the interaction range becomes infinite, we prove rigorously the existence of a liquid–gas phase transition when the interaction range is finite but long compared to the interparticle spacing for a range of temperature.

KEY WORDS: Continuum particle system; liquid–gas phase transition; mean-field theory; Pirogov–Sinai theory; cluster expansion; Dobrushin uniqueness.

1. INTRODUCTION

An outstanding problem in equilibrium statistical mechanics is to derive rigorously the existence of a liquid–vapor phase transition in a continuum particle system. While such transitions are observed in all types of macroscopic systems there is at present no proof from first principles of their existence in particles interacting with any kind of reasonable potential, say Lennard–Jones or hard core plus attractive square well. Such potentials are known, by comparison of experiment with low density expansions, to accurately describe the observed behavior of gases. Furthermore the properties of these systems at higher densities, obtained via approximate integral equations, are in good agreement with those of liquids in the ranges of temperatures and pressures where boiling and condensation takes

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place. In fact, computer simulations using classical statistical mechanics of such systems, containing several hundred to several million particles, convincingly show, via extrapolations which take into account finite size effects, that systems with these type of interactions will have true liquid–vapor phase transitions in the thermodynamic (infinite volume) limit. In this paper we go some way toward a proof of such a transition, i.e., we prove for the first time the existence of a liquid–vapor transition in a continuum particle system with finite range interactions and no special symmetry.

Historically the first proof of liquid–vapor type phase transitions was given for lattice systems (which are isomorphic to Ising spins). These systems can be thought of as idealizations of real fluids in which however the natural continuous spatial translation invariance symmetry is replaced by that of the lattice $\mathbb{Z}^d$, $d \geqslant 2$. It was Peierls [P] who first gave a convincing argument (later made fully rigorous by Dobrushin [D1] and Griffiths [G]) of the coexistence of different phases in such systems. The power of Gibbsian statistical mechanics to produce such rigorous results was brought home to the general science community by the dramatic work of Onsager [O] who exhibited the detailed structure of the critical point associated with this transition by explicitly solving the two dimensional Ising model (or lattice gas), with nearest neighbor interactions. Since that time there have been found many other exactly solvable two dimensional lattice systems [Bax]. At the same time the development of various types of inequalities as well as the powerful Pirogov–Sinai formalism [PS] have resulted in a comprehensive rigorous theory of phase transitions in lattice systems, at sufficiently low temperatures.

By contrast, much less is known rigorously about phase transitions in continuum systems. There is basically only the case of the two component Widom–Rowlinson [WR] model, in which the interaction is a hard core repulsion between particles of different species, where Ruelle [R1] was able to generalize the Peierls argument to prove phase coexistence in this system. Ruelle's proof strongly exploits the symmetry between the two components present in the WR model. The same is true, at least to some extent, of various extensions of this model [LL], [GH], [CCK]: see, however, [BKL] where some special multicomponent models without symmetry were treated by an extension of the Pirogov–Sinai (hereafter denoted by P–S) formalism. For general continuum systems without some special symmetry the only proofs of phase transitions so far are for systems with interactions which decay very slowly or not at all. Such one dimensional models, with many particle interactions, were analyzed and proven to exhibit phase transitions by Fisher and Felderhof [FF]. More recently Johansson [J] has considered interactions in one dimension which decay as $r^{-\alpha}$, $\alpha \in (1, 2)$, proving that at low temperatures there is phase transition in the sense that the pressure is not differentiable.

In a genuine continuum particle system with the type of pair interactions discussed above one expects several phase transitions. At high temperatures and small densities we have the gaseous phase, then, lowering the temperature, we first find the liquid and then the solid, crystalline phases, at the appropriate values of the density. We will concentrate here on the vapor–liquid phase transition, which is simpler to investigate than the formation of the periodic structures of crystals. Of course we also want to take advantage of the powerful techniques developed for proving phase transition in lattice systems, but these are generally applicable only at temperatures much smaller than those at which the vapor–liquid transition occurs. We will overcome this problem by performing a coarse graining which may also be considered a one step renormalization group transformation. To do this we divide space into large cubes of side ℓ and introduce variables ϱ_x, which are the particle densities in the cubes labeled by x. After integrating out all the other variables, we will be left with a new system described by the variables ϱ_x, their distribution still Gibbsian with a new Hamiltonian and temperature. The essential point is that the ratio between the new and old temperatures scales as ℓ^{-d}.

In general the verification of such statements is a very hard if not impossible task, but it can be accomplished in a relatively simple fashion if we consider interactions of Kac type in which the range γ^{-1} of the interaction is small. By suitably choosing the side of the cubes we will then enter into the low temperature regime where the Peierls and the P–S methods apply. In such a new perturbative scheme, the unperturbed state is described by mean field (formally $\gamma = 0$) and the small parameter of the expansion is the inverse interaction range γ, instead of the temperature in the traditional approach. By choosing a suitable range of values of chemical potential and temperature we will then be able to put ourselves at the vapor–liquid phase coexistence.

The mean field or van der Waals type of phase transition was first derived rigorously by Kac, Uhlenbeck and Hemmer [KUH] for a class of one dimensional models, hard rods of radius one with an added pair potential

$$\phi_\gamma(q_i, q_j) = -\alpha \tfrac{1}{2} \gamma \exp[-\gamma\, |q_i - q_j|], \qquad \gamma, \alpha > 0 \tag{1.1}$$

in the limit $\Gamma \to 0$, see also van Kampen [vK]. This was later generalized by Lebowitz and Penrose [LP] to d-dimensional systems with suitable short range interactions and Kac potentials of the form

$$\phi_\gamma(q_i, q_j) = -\alpha \gamma^d J(\gamma\, |q_i - q_j|) \tag{1.2}$$

with

$$\int_{\mathbb{R}^d} J(r)\, dr = 1, \qquad J(r) > 0 \tag{1.3}$$

In the thermodynamic limit followed by the limit $\gamma \to 0$ the Helmholtz free energy a takes the form, for a fixed temperature β^{-1},

$$\lim_{\gamma \to 0} a(\varrho, \gamma) = CE\{a_0(\varrho) - \tfrac{1}{2}\alpha\varrho^2\} \tag{1.4}$$

Here ϱ is the particle density, a_0 is the free energy density of the reference system, i.e., the system with $\alpha = 0$ in (1.2). a_0 is convex in ϱ (by general theorems) and $CE\{f(x)\}$ is the largest convex lower bound of f. For α large enough the term in the curly brackets in (1.4) has a double well shape and the CE corresponds to the Gibbs double tangent construction. This is equivalent to Maxwell's equal area rule applied to a van der Waals' type equation of state where it gives the coexistence of liquid and vapor phases [LP]; see also [vK].

As already discussed we prove the coexistence of liquid and gas phases for systems with finite range interactions as small perturbations, at finite $\gamma > 0$, from the mean field behavior at $\gamma = 0$. The same approach was recently applied to the lattice case for Ising models, [CP], [BZ], [BP], where the Peierls argument applies directly, because of the spin flip symmetry of the model. The absence of symmetries in our case requires instead the whole machinery of the Pirogov–Sinai theory. To insure stabilization against collapse, which would be induced by a purely attractive pair potential, the natural choice is to replace the point particles by hard spheres. Our approach however does not work in such a case, as we need a cluster expansion for the unperturbed reference system (i.e., without the Kac interaction) at values of the chemical potential or density for which it is not proved to hold. We avoid the problem by considering point particles and insuring stability by introducing a positive four body potential of the same range as the negative two body potential. In this way we avoid having to control strong short range interactions, something beyond our present day abilities for dense continuum systems. A preliminary announcement of our results is contained in [LMPI] and a fuller description of the proofs (with some variations) can be found in [Pr].

2. DEFINITIONS AND RESULTS

For ease of reference we collect below the main definitions in consecutive subsections.

Particle Configurations and Phase Space

We consider a system of identical point particles in $\mathbb{R}^d$, $d \geqslant 2$ and call *particle configurations* the locally finite subsets of $\mathbb{R}^d$. The collection of all particle configurations in Λ, $\Lambda \subset \mathbb{R}^d$, is the phase space $\mathcal{Q}^{(\Lambda)}$ and we simply write $\mathcal{Q}$ when $\Lambda = \mathbb{R}^d$. The particle configurations are denoted by q and sometimes by $q^{(\Lambda)}$ when we want to underline that they are in $\mathcal{Q}^{(\Lambda)}$. We write $q = (q_1,..., q_n)$ when the configuration consists of $|q| = n$ particles positioned at points $q_1,..., q_n \in \mathbb{R}^d$.

Free measure

For a bounded measurable subset Λ of $\mathbb{R}^d$ the free measure dq on $\mathcal{Q}^{(\Lambda)}$ (also called the Liouville measure) is

$$\int_{\mathcal{Q}^{(\Lambda)}} dq\, f(q) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_\Lambda dq_1 \cdots \int_\Lambda dq_n\, f(q_1,..., q_n) \tag{2.1}$$

where f is any bounded measurable function on $\mathcal{Q}^{(\Lambda)}$.

Hamiltonian

The energy of the configuration $q = (q_i)$ is given by the formal Hamiltonian

$$\begin{aligned} H_{\gamma, \lambda}(q) = {} & -\lambda\, |q| - \frac{1}{2!} \sum_{i_1} \sum_{i_2 \neq i_1} J_\gamma^{(2)}(q_{i_1}, q_{i_2}) \\ & + \frac{1}{4!} \sum_{i_1} \sum_{i_2 \neq i_1} \sum_{i_3 \neq i_1, i_2} \sum_{i_4 \neq i_1, i_2, i_3} J_\gamma^{(4)}(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}) \end{aligned} \tag{2.2}$$

Here λ is the chemical potential while $J_\gamma^{(2)}(\cdot, \cdot)$ and $J_\gamma^{(4)}(\cdot, \cdot, \cdot, \cdot)$ are respectively two and four-body potentials. For notational simplicity we choose $\gamma \in \{2^{-n}, n \in \mathbb{N}\}$ and γ is a scaling factor in the definition of $J_\gamma^{(2)}(\cdot, \cdot)$ and $J_\gamma^{(4)}(\cdot, \cdot, \cdot, \cdot)$. These are chosen of the form

$$J_\gamma^{(2)}(q_{i_1}, q_{i_2}) = \gamma^{2d} \int dr \prod_{j=1}^{2} \mathbb{1}_{|r - q_{i_j}| \leqslant \gamma^{-1} R_d} \tag{2.3}$$

$$J_\gamma^{(4)}(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}) = \gamma^{4d} \int dr \prod_{j=1}^{4} \mathbb{1}_{|r - q_{i_j}| \leqslant \gamma^{-1} R_d} \tag{2.4}$$

where $\mathbb{1}_A$ is the characteristic function of the set A and R_d is the radius of the ball in $\mathbb{R}^d$ having a unit volume. Denoting by $B_\gamma(r)$ a ball of radius $\gamma^{-1}R_d$ centered at $r \in \mathbb{R}^d$ we have

$$J_\gamma^{(2)}(q_{i_1}, q_{i_2}) = \gamma^{2d}\, |B_\gamma(q_{i_1}) \cap B_\gamma(q_{i_2})| \tag{2.5}$$

and

$$J_\gamma^{(4)}(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}) = \gamma^{4d} \left| \bigcap_{j=1}^{4} B_\gamma(q_{i_j}) \right| \tag{2.6}$$

Let $J^{(2)} = J_1^{(2)}$ and $J^{(4)} = J_1^{(4)}$. Then the scaling properties of $J_\gamma^{(2)}$ and $J_\gamma^{(4)}$ can be expressed by

$$J_\gamma^{(2)}(q_{i_1}, q_{i_2}) = \gamma^d J^{(2)}(\gamma q_{i_1}, \gamma q_{i_2}) \tag{2.7}$$

and

$$J_\gamma^{(4)}(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}) = \gamma^{3d} J^{(4)}(\gamma q_{i_1}, ..., \gamma q_{i_4}) \tag{2.8}$$

which is similar to (1.2). It is also clear that

$$\int J_\gamma^{(2)}(0, r)\, dr = \int J_\gamma^{(4)}(0, r_1, r_2, r_3)\, dr_1\, dr_2\, dr_3 = 1 \tag{2.9}$$

as in (1.3).

As both $J_\gamma^{(2)}$ and $J_\gamma^{(4)}$ are positive we have in (2.2) a competition between an attractive pair and a repulsive four-body potential. When the scaling parameter γ is small (but finite), the model has a large but finite interaction radius, $2R_d\gamma^{-1}$, and a small interaction strength between any given two or four particles. Nevertheless, the total strength of the interaction between a given particle and all other particles in a configuration q of bounded nonvanishing density is of order 1. These are characteristic properties of the Kac potentials which, as noted earlier, usually reproduce the van der Waals theory [LP] in the scaling limit $\gamma \to 0$. The specific form (2.3)–(2.4) of the interaction $J^{(2)}$ and $J^{(4)}$ makes the analysis simpler; for more general potentials see (2.14) and the discussion in the beginning of Section 3.

Finite and Infinite Volume Gibbs Measures

For any two locally finite configurations $q = (q_i)$ and $\bar{q} = (\bar{q}_j)$ denote by $q \cup \bar{q} = (q_i, \bar{q}_j)$ the configuration including all particles from both q and $\bar{q}$. The *conditional energy* of $q^{(\Lambda)}$ given a configuration $\bar{q} \in \mathcal{Q}$ is

$$H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}) = H_{\gamma,\lambda}(q^{(\Lambda)} \cup \bar{q}) - H_{\gamma,\lambda}(\bar{q}) \tag{2.10}$$

This conditional energy consist of two parts: the energy, $H_{\gamma,\lambda}(q^{(\Lambda)})$, of the configuration $q^{(\Lambda)}$ itself and the *interaction energy*

$$U_\gamma(q^{(\Lambda)} \mid \bar{q}) = H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}) - H_{\gamma,\lambda}(q^{(\Lambda)}) \tag{2.11}$$

between configurations $q^{(\Lambda)}$ and $\bar{q}$. Both (2.10) and (2.11) are finite sums because of the finite range of the potential.

Let $\Lambda^c = \mathbb{R}^d \setminus \Lambda$ be the complement of Λ. The *Gibbs measure* $\mu^{(\Lambda)}_{\gamma,\beta,\lambda}(dq^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})$ in the bounded measurable set Λ with boundary condition $\bar{q}^{(\Lambda^c)}$ and inverse temperature $\beta > 0$, is the probability measure on $\mathcal{Q}^{(\Lambda)}$ given by

$$\mu^{(\Lambda)}_{\gamma,\beta,\lambda}(dq^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)}) = \frac{e^{-\beta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})}}{\Xi_{\gamma,\beta,\lambda}(\Lambda \mid \bar{q}^{(\Lambda^c)})}\, dq^{(\Lambda)} \tag{2.12}$$

where $\Xi_{\gamma,\beta,\lambda}(\Lambda \mid \bar{q}^{(\Lambda^c)})$ is the *partition function*

$$\Xi_{\gamma,\beta,\lambda}(\Lambda \mid \bar{q}^{(\Lambda^c)}) = \int_{\mathcal{Q}^{(\Lambda)}} dq^{(\Lambda)}\, e^{-\beta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})} \tag{2.13}$$

The infinite volume Gibbs measures $\mu_{\gamma,\beta,\lambda}(dq)$ are probabilities on $\mathcal{Q}$ such that for any bounded measurable set Λ and $\mu_{\gamma,\beta,\lambda}$-almost any $\bar{q}^{(\Lambda^c)} \in \mathcal{Q}^{(\Lambda^c)}$ the conditional measure $\mu_{\gamma,\beta,\lambda}(dq^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})$ is equal to $\mu^{(\Lambda)}_{\gamma,\beta,\lambda}(dq^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})$ given by (2.12).

We say that a translation invariant Gibbs measure has a particle density $\varrho > 0$ if for any bounded set Λ the expectation of $|q \cap \Lambda|$ is equal to $\varrho\, |\Lambda|$.

The Main Result

Our main purpose is to investigate the phase diagram of the model (2.2) and to prove that a phase transition of the liquid–gas type takes place for some values of the parameters. Let $\beta_c = (\frac{3}{2})^{3/2}$ and $\beta_0 > \beta_c$ be a number defined via (3.14) below.

Theorem 2.1. For any $\beta \in (\beta_c, \beta_0)$ there exist functions $\gamma_0(\beta)$ and $\lambda(\gamma, \beta)$ such that for $0 < \gamma < \gamma_0(\beta)$ the model (2.2) has at least two distinct Gibbs measures $\mu^{\pm}_{\gamma,\beta,\lambda(\gamma,\beta)}(dq)$. These measures are translation invariant and ergodic, with an exponential decay of correlations. They have particle densities respectively equal to $\varrho^*_{\gamma,\beta,-} > 0$ and $\varrho^*_{\gamma,\beta,+} > \varrho^*_{\gamma,\beta,-}$.

We will prove Theorem 2.1 by using the P–S theory [PS]. The restriction $\beta \in (\beta_c, \beta_0)$ rather than $\beta > \beta_c$ allows for a simplified proof, we hope however to present in a forthcoming paper a proof for all $\beta > \beta_c$. While technically different and applied to different settings our methods are nevertheless conceptually close to those of [DZ].

The quantities $\lambda(\gamma, \beta)$ and $\varrho^*_{\gamma,\beta,\pm}$ in Theorem 2.1 have limits as $\gamma \to 0$, denoted by $\lambda(\beta)$ and $\varrho_{\beta,\pm}$, which, as already noted, are given exactly by mean field type formulas [LP]. Our proof of Theorem 2.1 is a perturbation theory constructed around the mean field picture.

We could extend Theorem 2.1 to slightly more general interactions where the range of the two and four body potentials are not necessarily the same. In particular we can make the range of the attractive two body forces greater than that of the repulsive four body ones, according to the physical intuition that the force is repulsive at short distances and attractive at longer distances. For brevity we do not give the details of the proofs and only introduce the new class of potentials. The four body interaction is unchanged while $J^{(2)}_\gamma(q_1, q_2)$ in (2.3) is replaced by

$$W^{(2)}_\gamma(q_1, q_2) = \gamma^{2d} \int dr \int dr' \, \mathbb{1}_{|r-q_2| \leqslant \gamma^{-1}R_d} \mathbb{1}_{|r'-q_2| \leqslant \gamma^{-1}R_d} \gamma^d w(\gamma\, |r-r'|) \qquad (2.14)$$

with w a smooth, non negative function having compact support and integral equal to 1. $\gamma^d w(\gamma\, |q_1 - q_2|)$ is the classical choice for a Kac pair potential (with the right scaling properties), what we do in (2.14) is simply to average such a function with the two arguments varied respectively on the balls $B_\gamma(q_1)$ and $B_\gamma(q_2)$. The choice $w = \delta(r - r')$ in (2.14) gives back (2.3).

Outline of the Remaining Sections

In Section 3 we give an outline of the proof formulating a number of statements which are proven in the next sections. In Section 4 we prove Peierls estimates on contours, and in Section 5 we use a cluster expansion to investigate the properties of the effective Hamiltonian obtained after the coarse graining transformation. In Section 6 we study the restricted ensembles proving the basic property of the P–S scheme, namely that it is possible to adjust the value of the chemical potential in such a way that the pressures in the two restricted ensembles are equal. We will conclude from this the proof of Theorem 2.1. Sections 7 and 8 contain more technical material.

3. SCHEME OF PROOF

The starting point of our approach, which as explained in the previous section is of crucial importance, consists of coarse graining or one step renormalization: we partition space into cubes $C_x^{(\ell_2)}$, x the centers of the cubes, of side $\ell_2 := \gamma^{-1+\alpha_2}$, α_2 a small positive number, (the subscript 2 foresees the use of other scales that will be introduced later in the proof). Given a particle configuration q, we call ϱ_x, the particle densities in each cube and we integrate out all the other variables, i.e., we consider the marginal over the variables ϱ_x. The new measure is still Gibbsian and its new effective Hamiltonian (a function of the ϱ_x) can be characterized with remarkable accuracy by cluster expansion techniques, see Section 5. As mentioned earlier, the main point of this procedure is that the new effective inverse temperature becomes $\beta\ell_2^d$. We can thus enter into the very low temperature regime by taking γ small enough and ℓ_2 correspondingly large (in Section 5 we will absorb the temperature into the Hamiltonian). we are then in the right setup for the P–S theory. An analysis à la [LP], which we here omit, would show that the new effective Hamiltonian converges formally, in the limit $\gamma \to 0$, to the mean field free energy functional $\mathcal{F}_{\beta,\lambda}(\varrho)$ given in (3.1) below. We will begin our analysis by characterizing the ground states of $\mathcal{F}_{\beta,\lambda}(\varrho)$ and thereafter use the P–S theory to investigate the perturbations at $\gamma > 0$.

Mean Field Free Energy Functional and Ground States

The mean field Gibbs free energy functional $\mathcal{F}_{\beta,\lambda}(\varrho)$, $\varrho = \varrho(r)$ denoting a non-negative bounded measurable function in $\mathbb{R}^d$, is

$$\mathcal{F}_{\beta,\lambda}(\varrho) = \int dr\, \frac{\varrho(r)}{\beta} (\log \varrho(r) - 1) - \int dr\, \lambda \varrho(r)$$
$$- \frac{1}{2!} \int dr_1\, dr_2\, J^{(2)}(r_1, r_2)\, \varrho(r_1)\, \varrho(r_2)$$
$$+ \frac{1}{4!} \int dr_1 \cdots dr_4\, J^{(4)}(r_1,..., r_4)\, \varrho(r_1) \cdots \varrho(r_4) \tag{3.1}$$

The first integral is the entropy contribution to the free energy (more precisely the product of the temperature times the entropy of the ideal gas with the sign changed). The other three terms arise from the corresponding interaction terms in (2.2). More details concerning the relation between (3.1) and (2.2) can be found in Section 4.

The *mean field ground states* are, by definition, the minimizers of $\mathscr{F}_{\beta,\lambda}(\varrho)$. To find them we set

$$\mathscr{R}(r,\varrho)=\int_{|r-r_1|\leqslant R_d} dr_1\,\varrho(r_1) \tag{3.2}$$

and

$$\mathscr{I}(r,\varrho)=\int_{|r-r_1|\leqslant R_d} dr_1\,\frac{\varrho(r_1)}{\beta}(\log\varrho(r_1)-1) \tag{3.3}$$

$\mathscr{R}(r,\varrho)$ and $\mathscr{I}(r,\varrho)$ are respectively the mean density and the mean entropy of ϱ in the ball $B(r)$. With this notation we can rewrite (3.1) as

$$\mathscr{F}_{\beta,\lambda}(\varrho)=\int dr\left(\mathscr{I}(r,\varrho)-\lambda\mathscr{R}(r,\varrho)-\frac{1}{2!}\mathscr{R}(r,\varrho)^2+\frac{1}{4!}\mathscr{R}(r,\varrho)^4\right) \tag{3.4}$$

which is true only because of the special form, (2.3) and (2.4), of $J^{(2)}$ and $J^{(4)}$. In fact (3.4) is the main reason for choosing $J^{(2)}$ and $J^{(4)}$ of this form. Observe however that if the potential $J^{(2)}$ is replaced by $W^{(2)}$ as given in (2.14) (with $\gamma=1$) then the r.h.s. of (3.4) simply becomes, after some easy manipulations,

$$\int dr\left(\mathscr{I}(r,\varrho)-\lambda\mathscr{R}(r,\varrho)-\frac{1}{2!}\mathscr{R}(r,\varrho)^2+\frac{1}{4!}\mathscr{R}(r,\varrho)^4\right)$$
$$+\frac{1}{4}\int dr\int dr'\,w(|r-r'|)[\mathscr{R}(r,\varrho)-\mathscr{R}(r',\varrho)]^2 \tag{3.5}$$

By convexity

$$\mathscr{I}(r,\varrho)\geqslant\frac{\mathscr{R}(r,\varrho)}{\beta}(\log\mathscr{R}(r,\varrho)-1) \tag{3.6}$$

and equality is achieved only if $\mathscr{R}(r,\varrho)=\varrho(r)$.

It follows then from (3.4) and (3.6) that if $s\geqslant 0$ is a minimizer of the function

$$F_{\beta,\lambda}(s)=\frac{s}{\beta}(\log s-1)-\lambda s-\frac{s^2}{2!}+\frac{s^4}{4!} \tag{3.7}$$

then $\varrho(r) = \mathscr{R}(r, \varrho) \equiv s$ is a minimizer of $\mathscr{F}_{\beta,\lambda}(\varrho)$, (and also of the functional in (3.5), corresponding to $\mathscr{F}$ with the modified potential (2.14)). The second derivative of $F_{\beta,\lambda}(s)$,

$$F''_{\beta,\lambda}(s) = \frac{1}{\beta s} - 1 + \frac{s^2}{2} \tag{3.8}$$

is positive if $\beta < \beta_c = (\frac{3}{2})^{3/2}$. Hence for $\beta < \beta_c$ and any λ the function $F_{\beta,\lambda}(s)$ is convex and has a unique minimizer which is the root of the equation

$$-s + \frac{1}{3!} s^3 + \frac{1}{\beta} \log s - \lambda = 0 \tag{3.9}$$

On the contrary, for $\beta > \beta_c$ this equation has three positive roots with two of them, $s = \varrho_{\beta,\lambda,-}$ and $s = \varrho_{\beta,\lambda,+}$, local minimizers of $F_{\beta,\lambda}(s)$. Furthermore, there exists a unique $\lambda = \lambda(\beta)$ for which both local minima are the global ones and the function $F_{\beta,\lambda}(s)$ has a "double well" shape. We set $\varrho_{\beta,\pm} = \varrho_{\beta,\lambda(\beta),\pm}$. Clearly for $\beta > \beta_c$ and $\lambda = \lambda(\beta)$ the densities $\varrho(r) \equiv \varrho_{\beta,-}$ and $\varrho(r) \equiv \varrho_{\beta,+}$ are distinct mean field gound states. For later purposes we remark that

$$-1 < \lambda(\beta) < 0 \tag{3.10}$$

which can be checked by direct calculation.

Mean Field Equations and Contraction Property

As the ground states are minimizers of $\mathscr{F}_{\beta,\lambda}(\varrho)$, they satisfy the mean field equation $\delta\mathscr{F}_{\beta,\lambda}(\varrho)/\delta\varrho(r) = 0$. By an explicit calculation we then find that they are fixed points of the transformation

$$\varrho(\cdot) \to \Phi(\varrho(\cdot), \cdot) \tag{3.11}$$

where

$$\Phi(\varrho(\cdot), r) := \exp\left\{\lambda + \int dr_1\, J^{(2)}(r, r_1)\, \varrho(r_1) - \frac{1}{3!} \int dr_1 \cdots \int dr_3\, J^{(4)}(r,..., r_3)\, \varrho(r_1)\, \varrho(r_2)\, \varrho(r_3)\right\} \tag{3.12}$$

Setting $\varrho^* = \varrho_{\beta,-}$ or $\varrho^* = \varrho_{\beta,+}$ the derivative $\psi(r) = \delta\Phi(\varrho(\cdot), r)/\delta\varrho(r)$ computed at $\varrho(r) \equiv \varrho^*$ is equal to

$$\psi(r) = \beta\varrho^* J^{(2)}(0, r)[1 - (\varrho^*)^2/2] \tag{3.13}$$

We define β_0 in Theorem 2.1 so that for all $\beta \in (\beta_c, \beta_0)$

$$\int dr\, |\psi(r)| = \beta\varrho^*\, |1 - \varrho^{*2}/2| < 1 \tag{3.14}$$

If (3.14) holds, the transformation Φ is a contraction (in a neighborhood of the ground states and using sup norms).

The existence of $\beta_0 > \beta_c$ follows by noticing that since $F''_{\beta,\lambda}(\varrho^*) > 0$ then by (3.8) $\beta\varrho < 1$ and the only condition to check for (3.14) to hold is $\beta\varrho^*[1 - \varrho^{*2}/2] > -1$. By (3.8) it can be rewritten as

$$F''_{\beta,\lambda}(\varrho^*) < \frac{2}{\beta\varrho^*} \tag{3.15}$$

and $F''_{\beta,\lambda}(\varrho^*) \to 0$ as $\beta \to \beta_c$ while both $\varrho_{\beta,\pm}$ remain bounded.

We will see in Section 5 that for $\beta \in (\beta_c, \beta_0)$ and $\gamma > 0$ small enough, the Dobrushin uniqueness condition is satisfied by the effective Hamiltonian in the system restricted to the ground state ensemble (defined later in Section 3). That property is the $\gamma > 0$ analogue of the contraction property (3.14). We will in the sequel restrict ourselves to $\beta \in (\beta_c, \beta_0)$ which ensures the following technical conditions.

For $\beta_c < \beta < \beta_0$ there exists a positive number $\zeta(\beta)$ such that

$$\max_{\sigma_1 = \pm 1, \sigma_2 = \pm 1} \beta(\varrho_{\beta,\sigma_1} + 2\zeta(\beta)) \left| 1 - \frac{(\varrho_{\beta,\sigma_1} + \sigma_2 2\zeta(\beta))^2}{2} \right| = a(\beta) < 1 \tag{3.16}$$

Consequently there exist a positive number $\delta(\beta)$ such that for any $\lambda \in (\lambda(\beta) - \delta(\beta), \lambda(\beta) + \delta(\beta))$ one has

$$\max_{\sigma_1 = \pm 1, \sigma_2 = \pm 1} \beta(\varrho_{\beta,\lambda,\sigma_1} + \zeta(\beta)) \left| 1 - \frac{(\varrho_{\beta,\lambda,\sigma_1} + \sigma_2 2\zeta(\beta))^2}{2} \right| \leqslant a(\beta) \tag{3.17}$$

Note $(\varrho_{\beta,\lambda,\sigma_1} + \zeta(\beta))$ in (3.17) instead of $(\varrho_{\beta,\sigma_1} + 2\zeta(\beta))$ in (3.16). This allows for the same $a(\beta)$ in (3.17) and (3.16) provided $\zeta(\beta)$ and $\delta(\beta)$ are small enough.

We expect that for γ small enough the particle configurations will have densities close to the mean field ground states. To investigate the issue we need a spatial scale for computing particle densities and a notion of closeness between densities. We will then have a picture of the particle configurations in terms of spatial regions where there is agreement or disagreement with the ground states.

The Partitions $\mathscr{D}^{(\ell)}$, the Cubes $C_x^{(\ell)}$ and the Densities $\varrho_x^{(\ell)}$

Let $\mathscr{D}^{(\ell)}$, $\ell \in \{2^n, n \in \mathbb{N}\}$, be decreasing partitions of $\mathbb{R}^d$ into cubes $C^{(\ell)}$ of side ℓ, i.e., $\mathscr{D}^{(\ell)}$ is coarser than $\mathscr{D}^{(\ell')}$ if $\ell > \ell'$. For any $r \in \mathbb{R}^d$ we denote by $C^{(\ell)}(r)$ the cube of $\mathscr{D}^{(\ell)}$ that contains r. We suppose that the centers of the cubes $C^{(1)}$ are in $\mathbb{Z}^d$. Consequently the centers of cubes $C^{(\ell)}$ are in $\mathbb{Z}^d_\ell = \ell\mathbb{Z}^d + ((\ell-1)/2,..., (\ell-1)/2)$. For $x \in \mathbb{Z}^d_\ell$ we denote $C_x^{(\ell)}$ the cube of $D^{(\ell)}$ centered at x.

Given a region Λ, $[\Lambda]^{(\ell)}$ is the maximal $D^{(\ell)}$ measurable subset of Λ, i.e., the union of all $C^{(\ell)}$ contained in Λ. We also identify $[\Lambda]^{(\ell)}$ with $\{x \in \mathbb{Z}^d_\ell \mid x \in [\Lambda]^{(\ell)}\}$. The $\mathbb{R}^d$ volume of $[\Lambda]^{(\ell)}$ is denoted by $|[\Lambda]^{(\ell)}|$ while its $\mathbb{Z}^d_\ell$ volume, i.e., the number of lattice points, is denoted by $\|[\Lambda]^{(\ell)}\|$.

The density of a configuration q in a box $C_x^{(\ell)}$ is

$$\varrho_x^{(\ell)}(q) = \ell^{-d}\, |q \cap C_x^{(\ell)}| \tag{3.18}$$

Accordingly $\varrho^{(\ell)}(q) = (\varrho_x^{(\ell)}(q))$ is called a ($\mathscr{D}^{(\ell)}$ measurable) *density configuration* corresponding to a particle configuration q. We denote by $\varrho^{(\ell)}(\Lambda) = (\varrho_x^{(\ell)}(\Lambda))$ a density configuration in $[\Lambda]^{(\ell)}$ not related to any particle configuration.

The notation $[q]^{(\ell)}$ is used for the configuration obtained from $q = (q_i)$ by shifting all q_i to the centers of the corresponding boxes $C^{(\ell)}(q_i)$. For a configuration $q^{(\Lambda)}$ we set

$$H_{\gamma,\lambda}(\varrho^{(\ell)}(q^{(\Lambda)})) = H_{\gamma,\lambda}([q^{(\Lambda)}]^{(\ell)}) \tag{3.19}$$

and

$$H_{\gamma,\lambda}(\varrho^{(\ell)}(q^{(\Lambda)}) \mid \bar{q}^{(\Lambda^c)}) = H_{\gamma,\lambda}([q^{(\Lambda)}]^{(\ell)} \mid \bar{q}^{(\Lambda^c)}) \tag{3.20}$$

where Λ is a region and $\bar{q}^{(\Lambda^c)}$ is a boundary condition.

Spatial Scales, η Functions and Ground-State Configurations

Several scales are of special interest for us

$$\ell_1 = \ell_{1,\gamma} = \gamma^{-1+\alpha_1}, \qquad \ell_2 = \ell_{2,\gamma} = \gamma^{-1+\alpha_2}, \qquad \ell_3 = \ell_{3,\gamma} = \gamma^{-1-\alpha_3} \tag{3.21}$$

where the numbers $0 < \alpha_i < 1$ are rational, $\alpha_1 = 1/2$ and $\alpha_2 + \alpha_3 \ll (2d)^{-1}$.

The scale ℓ_2 is our coarse graining scale. Agreement or disagreement with the ground state is indicated by the η-functions, $\eta = \eta(q) = (\eta_x(q))$, $x \in \mathbb{Z}^d_{\ell_2}$:

$$\eta_x(q) = \begin{cases} -1, & \text{for } \quad |\varrho_x^{(\ell_2)}(q) - \varrho_{\beta,-}| \leqslant \zeta \\ +1, & \text{for } \quad |\varrho_x^{(\ell_2)}(q) - \varrho_{\beta,+}| \leqslant \zeta \\ 0, & \text{otherwise} \end{cases} \tag{3.22}$$

$\zeta = \zeta(\beta)$ being taken from (3.16). Particle configurations q or density configurations $\varrho^{(\ell)}(q)$ are called compatible with η if $\eta = \eta(\varrho^{(\ell)}(q)) = \eta(q)$.

We then say that a particle configuration q belongs to the *ground state ensemble* of the $\pm$ *phase* or equivalently *liquid/vapor phase* if $\eta(q) \equiv \pm 1$. For brevity we will sometimes simply say that q is a $\pm$ ground state configuration.

For a scale ℓ we define the corresponding *standard ground state configurations* $q_-^{(\ell)}$ and $q_+^{(\ell)}$ as those which have $\varrho^{(\ell)}(q_-^{(\ell)}) \equiv \varrho_{\beta,-}$ and $\varrho^{(\ell)}(q_+^{(\ell)}) \equiv \varrho_{\beta,+}$ with all the particles placed at the centers of the corresponding boxes $C^{(\ell)}$. Of course for the irrational $\varrho_{\beta,\pm}$ and even for some rational $\varrho_{\beta,\pm}$ it is impossible to achieve these densities exactly. With an abuse of the notation we set $\varrho^{(\ell)}(q_\pm^{(\ell)}) \equiv \varrho_{\beta,\pm}$ when the densities are the closest possible to $\varrho_{\beta,\pm}$. When $\ell = \ell_2$ we will drop the superscript (ℓ).

The other scales, ℓ_3 and ℓ_1, are used to construct contours (see below) and to do some approximate calculations (see Section 4) respectively.

Our proof of the phase transition or coexistence of phases is based on the following qualitative picture describing typical particle configurations in the two pure phases. In the gas phase a typical configuration q coincides in most of $\mathbb{R}^d$ with some typical configuration in the gas ground state ensemble. Inside this "sea" of the gas ground state there are rare "islands" occupied by the liquid ground state. These two types of ground states are separated from each other by regions in $\mathbb{R}^d$ which are called Peierls contours. The "excess" free energy of q with respect to the free energy of the gas ground state occupying all of $\mathbb{R}^d$ is concentrated in the vicinity of these contours and is proportional to their volume. A similar, inverse, picture describes typical configurations of the liquid phase; the liquid ground state now forms a "sea" with rare "islands" of the gas ground state. Thus the typical configurations of the gas and liquid phases are distinct with the density of particles being an order parameter distinguishing them. A rigorous verification of the above picture is fairly straightforward for simple lattice gases or Ising systems, in which the gas and liquid ground states correspond respectively to "all sites empty" and "all sites occupied". It requires however considerable work for continuum systems and we start with precise definitions.

Correct and Incorrect Sets, Contours

A family of cubes from the partition $\mathscr{D}^{(\ell)}$ is called $*$-*connected* if the closure of the union of the cubes in this family is a connected subset of $\mathbb{R}^d$; we consider from now on $d \geqslant 2$. We then say that r is a $(+)$-*correct* point of a configuration q if $\eta_x(q) = +1$ whenever $C_x^{(\ell_2)}(r) \subset \bar{C}^{(\ell_3)}(r)$ where the later set is the union of all cubes of scale ℓ_3 that are $*$-connected to $C^{(\ell_3)}(r)$. Similarly one defines $(-)$-*correct* points. Finally, if r is not a correct point of q it is an incorrect one. The connected components of the incorrect points of q form the supports of the contours. The pair consisting of the support of the contour and the restriction of $\eta(q)$ to this support is called a contour of q and is denoted $\Gamma(q)$. Observe that any q uniquely defines its contours $\Gamma_i(q)$.

Axiomatically a *contour* $\Gamma = (\mathrm{Supp}(\Gamma), \eta^\Gamma)$ is defined as a pair which consists of a bounded $*$-connected $\mathscr{D}^{(\ell_3)}$ measurable set $\mathrm{Supp}(\Gamma)$ called the spatial *support* of Γ and a spin valued function η^Γ on $\mathrm{Supp}(\Gamma)$, with the condition that there exists at least one configuration q that gives rise to Γ.

To motivate the next definitions and to outline further constructions let us consider first an oversimplified problem.

The Case of a Single Contour

We restrict in this examples the system to have only one contour Γ of the type that we define with reference to Fig. 1 where its typical representative is drawn. To be more precise we consider the ensembles of contours inside the finite domain Λ containing exactly one contour of the special form shown in Fig. 1.

We choose boundary conditions $\bar{q}^{(\Lambda^c)}$ in the $+$ phase and we impose that in the region A of Fig. 1 the configuration is in the $+$ phase, in D it is in the $-$ phase while in $\mathrm{Supp}(\Gamma)$ it agrees with η^Γ. We assume that in Fig. 1 $\mathrm{Supp}(\Gamma)$ is the whole region extending between the bold lines.

The simplified contour ensemble gives rise to the simplified partition function Z (we are using here a notation that will be discontinued in the sequel)

$$Z := \sum_{\Gamma} \int_{\mathscr{Q}^{(\Lambda)}} dq^{(\Lambda)}\, \mathbb{1}_{\eta(q^{(A)}) \equiv 1}\, \mathbb{1}_{\eta(q^{(D)}) \equiv -1}\, \mathbb{1}_{\eta(q^{(\mathrm{Supp}(\Gamma))}) \equiv \eta^\Gamma}\, e^{-\beta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})} \tag{3.23}$$

Here the external sum is taken over all contours Γ of the type described by Fig. 1 and the goal now is to rewrite Z as an integral over the $+$ ground

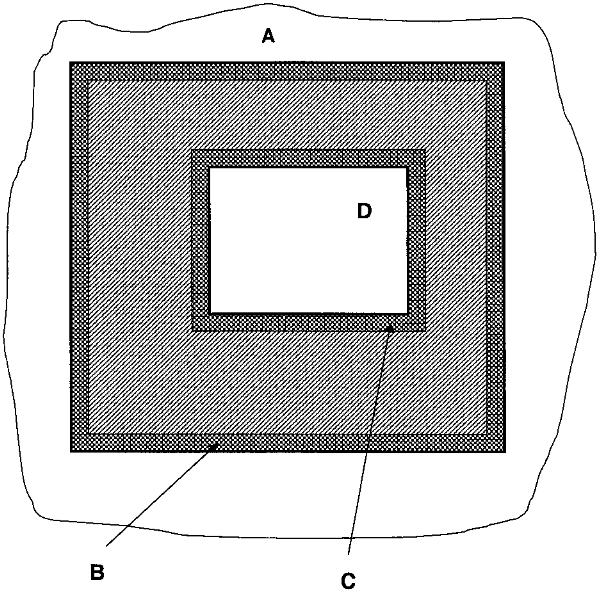

Fig. 1. A typical contour from he simplified contour ensemble.

state ensemble, i.e. as the integral over $q^{(\Lambda)}$ with $\{\eta(q^{(\Lambda)}) \equiv 1\}$. Let B and C be the strips of width γ^{-1} in Supp(Γ), as in Fig. 1, then we write

$$Z = \sum_{\Gamma} \int_{\mathcal{Q}^{(\Lambda)}} dq^{(\Lambda)} \, \mathbb{1}_{\eta(q^{(\Lambda)}) \equiv 1} \, e^{-\beta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})} W(\Gamma \mid q^{(B)}) \tag{3.24}$$

where the statistical weight $W(\Gamma \mid q^{(B)})$ is

$$W(\Gamma \mid q^{(B)}) = \frac{\int_{\mathcal{Q}(\mathrm{Supp}(\Gamma)\setminus B)} dq \, \mathbb{1}_{\eta(q) = \eta^{\Gamma}} \, e^{-\beta H_{\gamma,\lambda}(q \mid q^{(B)})} \int_{\mathcal{Q}^{(D)}} dq^{(D)} \, e^{-\beta H_{\gamma,\lambda}(q^{(D)} \mid q^{(C)})} \, \mathbb{1}_{\eta(q^{(D)}) \equiv -1}}{\int_{\mathcal{Q}(\mathrm{Supp}(\Gamma)\setminus B)} dq \, \mathbb{1}_{\eta(q) \equiv 1} \, e^{-\beta H_{\gamma,\lambda}(q \mid q^{(B)})} \int_{\mathcal{Q}^{(D)}} dq^{(D)} \, e^{-\beta H_{\gamma,\lambda}(q^{(D)} \mid q^{(C)})} \, \mathbb{1}_{\eta(q^{(D)}) \equiv 1}} \tag{3.25}$$

In (3.24) the configuration $q^{(\Lambda)}$ is not related to η^{Γ} and the contours Γ appear only through their statistical weights as extra variables in the partition functions. This is the contour model in the present over-simplified context. The advantage of the contour model is to work in a ground state ensemble, the price is the extra variables Γ. The game, in the general case with many contours, is to get good estimates on their statistical weights (3.25) in order to control their proliferation.

Let us examine this issue in the context of the oversimplified example. We should think of the integral in (3.25) over the region $\text{Supp}(\Gamma)\setminus B$ as a "surface term" and of the integral over D as a "volume term". The second one looks therefore more likely to take large values and we consider it first, with reference to the numerator in (3.25) shorthanded by N. The integral is over all configurations in the region D that are in the ground state ensemble of the $-$ phase. The boundary conditions for this partition function are fixed in the region C and they are also configurations of the $-$ phase. In fact the $-$ phase extends over the whole ℓ_3 cubes $*$-connected to D, recall the definition of contours.

The intuition (which is made rigorous in Section 6) is that since C is well inside the $-$ phase, then the configurations in a neighborhood of C are actually not only in the $-$ ground state ensemble but they are very close to the standard ground state configuration. We thus write (recall that N denotes the numerator in (3.25))

$$N = T_1 T_2 T_3 \tag{3.26}$$

where T_1, T_2 and T_3 are defined as follows:

Splitting 1/2-1/2 the interaction between $\text{Supp}(\Gamma)$ and D and recalling the intuition that the configurations involved in such interactions are very close to the $-$ ground state configuration, we write

$$T_1 := \int_{\mathcal{Q}(\text{Supp}(\Gamma)\setminus B)} dq \, \mathbb{1}_{\eta(q) = \eta^\Gamma} \, e^{-\beta H_{\gamma,\lambda}(q \mid q^{(B)}, q_-^{(D)}) + \beta U_\gamma(q_-^{(C)} \mid q_-^{(D)})/2} \tag{3.27}$$

Denoting by $f_{\lambda,-}$ the thermodynamic limit of the pressure associated with the partition function restricted to the $-$ ground state ensemble we set

$$T_2 = e^{\beta f_{\lambda,-} |D|} \tag{3.28}$$

The third term T_3 is implicitly defined so that (3.26) holds. T_3 takes into account the errors made by replacing in (3.27) the interaction

$$\tfrac{1}{2} U_\gamma(q^{(C)} \mid q^{(D)}) \to U_\gamma(q^{(C)} \mid q_-^{(D)}) - \tfrac{1}{2} U_\gamma(q_-^{(C)} \mid q_-^{(D)}) \tag{3.29}$$

and the error which comes from T_2 where the partition function is replaced by its thermodynamic limit in the sense of (3.28).

The error term T_3 appears to be bounded by

$$|\log T_3| \leqslant c\gamma^{1/4} \,|\text{surface of } D| \tag{3.30}$$

where c is a constant. The above inequality is proven in Section 6 in the general case.

After an analogous decomposition in the denominator, we see that the dangerous volume terms simplify between numerator and denominator if we are able to choose the chemical potential in such a way that $f_{\lambda,-} = f_{\lambda,+}$. Once again, the possibility to solve this equation relies on a rather explicit representation of the pressures which one is able to derive.

The ratio of the terms T_1 (from numerator and denumerator) is bounded by

$$\exp(-c\ell_2^d \ell_3^{-d} \zeta^2 \,|\text{Supp}(\Gamma)|) \tag{3.31}$$

This is the famous Peierls estimate which is proven in the general case in Section 4. Its proof is relatively simpler than the proof of (3.30) and relies directly on properties of the mean free field energy functional. This step is similar to the analogous one in Ising models [BP].

Combining (3.30) and (3.31) we conclude that the whole statistical weight $W(\Gamma \mid q^{(B)})$ is bounded as in (3.31).

Let us now go from the oversimplified example to the general case.

More About Contours

Given a (large enough) region V and $\ell > 0$ we call

$$\begin{aligned} \partial^{(\ell)} V &= \{ r \in V^c : \text{dist}(r, V) \leqslant \ell \}; \\ \delta^{(\ell)} V &= \{ r \in V : \text{dist}(r, V^c) \leqslant \ell \} \end{aligned} \tag{3.32}$$

and we set $\partial V = \partial^{(\gamma^{-1})} V$ and $\delta V = \delta^{(\gamma^{-1})} V$.

Denoting by $\text{Ext}(\Gamma)$ and by $\text{Int}_m(\Gamma)$ respectively the unbounded and the bounded connected components of $(\text{Supp}(\Gamma))^c$ we observe that η^Γ takes the same values $\sigma(\Gamma)$ and $\sigma_m(\Gamma)$ on each of the sets $\partial^{(\ell_3)} \text{Ext}(\Gamma)$ and $\partial^{(\ell_3)} \text{Int}_m(\Gamma)$ respectively. The regions $\text{Ext}(\Gamma)$ and $\text{Int}(\Gamma) = \bigcup_m \text{Int}_m(\Gamma)$ are respectively called the *exterior* and the *interior* of Γ, in the previous example they are A and D.

For a contour Γ denote

$$\delta_m(\Gamma) = \partial \,\text{Int}_m(\Gamma) \tag{3.33}$$

(the set C in the example of Fig. 1)

$$\delta^{=}(\Gamma) = \partial\, \mathrm{Ext}(\Gamma) \cup \left(\bigcup_{m:\, \sigma_m(\Gamma) = \sigma(\Gamma)} \delta_m(\Gamma) \right) \tag{3.34}$$

(the set B in Fig. 1, the last term being absent in Fig. 1)

$$\delta^{\neq}(\Gamma) = \bigcup_{m:\, \sigma_m(\Gamma) \neq \sigma(\Gamma)} \delta_m(\Gamma) \tag{3.35}$$

(this is the set C in Fig. 1) and

$$\delta(\Gamma) = \delta^{=}(\Gamma) \cup \delta^{\neq}(\Gamma) = \delta\, \mathrm{Supp}(\Gamma) \tag{3.36}$$

Two contours are called *disjoint* if their supports are not $*$-connected. We say that a contour *belongs* to a region if its support is a subset of that region. In any collection of pairwise disjoint contours there is a uniquely defined subcollection of *external contours*, i.e., contours which do not belong to the interior of any other contour in the collection.

A contour Γ *surrounds a point* r if $r \in \mathrm{Supp}(\Gamma) \cup \mathrm{Int}(\Gamma)$. A contour $\Gamma_3(q)$ *separates* contours $\Gamma_1(q)$ and $\Gamma_2(q)$ if $\mathrm{Supp}(\Gamma_1(q)) \in \mathrm{Int}(\Gamma_3(q))$ while $\mathrm{Supp}(\Gamma_2(q)) \in \mathrm{Ext}(\Gamma_3(q))$.

Consider an arbitrary configuration q which differs from a ground state configuration of a given phase a only inside a bounded region Λ. Then all contours of q belong to Λ and for any external contour $\Gamma^{\mathrm{ext}}(q)$ one has $\sigma(\Gamma^{\mathrm{ext}}) = \sigma$. Note that non-external contours $\Gamma(q)$ may have $\sigma(\Gamma(q)) = -\sigma$. Moreover, the contours of q satisfy the following *matching condition.* If $\mathrm{Supp}(\Gamma_1(q)) \in \mathrm{Int}_m(\Gamma_2(q))$ and there is no $\Gamma_3(q)$ separating $\Gamma_1(q)$ and $\Gamma_2(q)$ then $\sigma(\Gamma_1(q)) = \sigma_m(\Gamma_2(q))$. This matching condition is highly non local and represents the main difficulty in understanding contour statistics.

The idea of the P–S theory is to remove matching condition and to construct an equivalent contour model with no matching rules present. Generally this can be done by modification of the statistical weights of contours and shifting the difficulty to the estimate of these modified statistical weights.

The study of the distribution of contours is important because to prove the existence of distinct translation invariant $+$ and $-$ phases it is enough to show that the probability of the event that an external contour surrounds the origin is sufficiently small. In a translation invariant measure this implies that only finitely many contours surround any given point r. Then one of the ground states occupies the infinite region $\bigcap_i \mathrm{Ext}(\Gamma_i^{\mathrm{ext}}(q))$ which is exactly the "sea" discussed earlier. Furthermore, the probability of the event that $\eta_0(q)=1$ can be taken as the order parameter distinguishing two phases. This probability is less than $\frac{1}{2}$ for the $-$ phase and it is larger than $\frac{1}{2}$ for the $+$ phase as follows from a bound on the probability of a contour of the form (3.31) (which will be shown to be true for all contours). As we will see even in the general case when there is more than just one single contour in the contour model the bound (3.31) implies that the contours are very rare and one can control them with an analysis not too distant from the oversimplified example.

Let us recall now that the bound (3.31) on the statistical weight of a contour was obtained after adjusting the chemical potential so that the pressures in the two ground states ensembles are equal. In the general case this is not a simple task because there are contours inside contours, i.e., inside the regions of type D of the example in Fig. 1 there are other contours and so on. One needs to treat contours recursively as in the one-contour example to obtain at the end of the recursion the contour model with the properly modified statistical weights of contours. Only after that one may try to equalize pressures in these contour models. The problem is that even if the bound (3.31) holds, the pressures in the contour models will depend on the contours and we are then in a sort of loop: to have the good bound (3.31) on the contours we need to make the pressures equal, but to control the pressures we need a good bound on the contours.

As explained by the P–S theory it is possible to overcome such an impasse; we will do it by following the Zahradnik approach [Z]. we will define a new cut-offed statistical weight which by its definition cannot exceed the value (3.31) (with a suitable constant c). In this context it will be easy to iterate the procedure of the example with one contour to derive a contour model representation of the true partition function. Such *auxiliary* partition functions (one for each phase) give rise to the corresponding pressures and we will adjust the chemical potential so that these two pressures (one for each phase) are equal. It will then turn out (see [Z]) that for this particular value of the chemical potential the contours satisfy the bound (3.31) without the need of the cutoff so that for this special value of the chemical potential the auxiliary and true partition functions are equal.

The next subsection contains exact definitions and statements which are necessary for application of the general P–S theorem (see [PS], [Z]).

Auxiliary Partition Functions

Given a phase σ and boundary condition $\bar{q}^{(\Lambda^c)}$ belonging to the ground state ensemble of this phase we define the *auxiliary partition functions* and the *truncated statistical weight* as

$$Z^{A}_{\gamma,\beta,\lambda}(\Lambda \mid \bar{q}^{(\Lambda^c)}) = \int_{\mathscr{Q}(\Lambda)} dq\, \mathbb{1}_{\eta(q)\equiv\sigma}\, e^{-\beta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})} \times \sum_{\{\Gamma_i\}^{\sigma}\in\Lambda} \prod_i W^T(\Gamma_i \mid q^{(\delta^=(\Gamma_i))}) \tag{3.37}$$

and

$$W^T(\Gamma \mid q^{(\delta^=(\Gamma))}) = \min(W^A(\Gamma \mid q^{(\delta^=(\Gamma))}), e^{-(c/3)\ell_2^d \ell_3^{-d} \zeta^2 |\mathrm{Supp}(\Gamma)|}) \tag{3.38}$$

where

$$W^A(\Gamma \mid \bar{q}^{(\delta^=(\Gamma))}) = \frac{\left(\displaystyle\int_{\mathscr{Q}(\mathrm{Supp}(\Gamma)\setminus\delta^=(\Gamma))} dq\, \mathbb{1}_{\eta(q)=\eta^\Gamma}\, e^{-\beta H_{\gamma,\lambda}(q \mid \bar{q}^{(\delta^=(\Gamma))})} \times \prod_{m:\, \sigma_m(\Gamma)\neq\sigma(\Gamma)} Z^{A}_{\gamma,\beta,\lambda}(\mathrm{Int}_m(\Gamma) \mid q^{(\partial\,\mathrm{Int}_m(\Gamma))})\right)}{\left(\displaystyle\int_{\mathscr{Q}(\mathrm{Supp}(\Gamma)\setminus\delta^=(\Gamma))} dq\, \mathbb{1}_{\eta(q)=\sigma(\Gamma)}\, e^{-\beta H_{\gamma,\lambda}(q \mid \bar{q}^{(\delta^=(\Gamma))})} \times \prod_{m:\, \sigma_m(\Gamma)\neq\sigma(\Gamma)} Z^{A}_{\gamma,\beta,\lambda}(\mathrm{Int}_m(\Gamma) \mid q^{(\partial\,\mathrm{Int}_m(\Gamma))})\right)} \tag{3.39}$$

To prove that the definition is well posed, we first say that a statement related to a $\mathscr{D}^{(\ell)}$ measurable region is proved by *induction in volume* if this statement is true for a $\mathscr{D}^{(\ell)}$ measurable region Λ as soon as it is true for all $\mathscr{D}^{(\ell)}$ measurable subsets of Λ. We then observe that (3.37)–(3.39) constitute a single inductive definition. At the initial step of this induction in volume one considers only contours with empty interior and uses (3.38)–(3.39) to define their truncated statistical weights. Then (3.37) allows one to calculate the auxiliary partition functions for sufficiently small regions admitting inside them only contours without interior. Then (3.38)–(3.39) are used again to define the truncated statistical weights of contours having sufficiently small but non empty interiors and so on.

We hope that a notational similarity of the numerator and the denominator of (3.39) does not hide from the reader the fact that these expressions are very different. For example, in the numerator of (3.39) configurations $q^{(\partial\,\mathrm{Int}_m(\Gamma))}$ belong to the ground state ensemble of the phase $\sigma = \sigma(\Gamma)$ while in the denominator of (3.39) these configurations belong to the ground state ensemble of the opposite phase, $-\sigma$.

Statement 3.2. If for any contour Γ the truncated statistical weight $W^T(\Gamma \mid q^{(\delta^=(\Gamma))})$ is smaller than the second term in the argument of the min in (3.88) then the auxiliary partition function coincides with the true one.

The proof of the statement is standard in the P–S theory and we omit it (see [Z]).

Another standard observation is that for any region Λ with the boundary condition $\bar{q}^{(\partial\Lambda)}$ belonging to the ground state ensemble of the phase σ the statistics of external contours in the contour model coincides with that of the true particle model. Note that the statistics of non-external contours are different in the contour and particle models. For example, the contours of the opposite phase, $-\sigma$, never appear in the contour model. As we explained before the absence of the matching rules makes the analysis of the contour model much easier than the analysis of the initial particle model. The price paid for this simplification is a rather involved expression for $W(\Gamma \mid \bar{q}^{(\delta^=(\Gamma))})$.

Statement 3.3. For all γ small enough and all chemical potentials $\lambda \in (\lambda(\beta) - \gamma^\alpha, \lambda(\beta) + \gamma^\alpha)$, $\alpha \geqslant \frac{1}{2}$ there are $f^A_{+, \lambda, \gamma}$, resp. $f^A_{-, \lambda, \gamma}$, such that for any sequence of cubes $\Lambda \to \mathbb{R}^d$ and any sequence of boundary conditions $\bar{q}^{(\Lambda^c)}$ belonging to the $+$ (resp. $-$) ground state ensemble

$$\lim_{\Lambda \to \mathbb{R}^d} \frac{1}{\beta\,|\Lambda|} \log Z^A_{\gamma, \beta, \lambda}(\Lambda \mid \bar{q}^{(\Lambda^c)}) = f^A_{\pm, \lambda, \gamma} \tag{3.40}$$

Moreover there exists $\lambda(\gamma, \beta)$ such that

$$\lim_{\gamma \to 0} \lambda(\gamma, \beta) = \lambda(\beta) \tag{3.41}$$

and

$$f^A_{+, \lambda(\gamma, \beta), \gamma} = f^A_{-, \lambda(\gamma, \beta), \gamma} \tag{3.42}$$

The statement is proved in Section 6 using cluster expansion techniques, together with a rather explicit representation of the auxiliary pressures $f^A_{\pm, \lambda, \gamma}$.

In analogue to the first term T_1 in the decomposition (3.26) of the statistical weight we introduce the function

$$w(\Gamma \mid \bar{q}^{(\delta^{=}(\Gamma))})$$

$$= \frac{\left(\begin{array}{l} \displaystyle\int_{\mathcal{Q}(\mathrm{Supp}(\Gamma)\setminus\delta(\Gamma))} dq\, \mathbb{1}_{\eta(q)=\eta^{\Gamma}}\, e^{-\beta H_{\gamma,\lambda}(q \mid \bar{q}^{\delta^{=}(\Gamma)}),\, q_{-\sigma(\Gamma)}^{(\delta^{\neq}(\Gamma))}} \\ \displaystyle\times \prod_{m:\, \sigma_m(\Gamma) \neq \sigma(\Gamma)} e^{(\beta/2) U_{\gamma}(q_{-\sigma(\Gamma)}^{(\Gamma)} \mid q_{-\sigma(\Gamma)}^{(\delta^{\neq}(\Gamma))})} \end{array} \right)}{\left(\begin{array}{l} \displaystyle\int_{\mathcal{Q}(\mathrm{Supp}(\Gamma)\setminus\delta(\Gamma))} dq\, \mathbb{1}_{\eta(q)=\sigma(\Gamma)}\, e^{-\beta H_{\gamma,\lambda}(q \mid \bar{q}^{\delta^{=}(\Gamma)}),\, q_{\sigma(\Gamma)}^{(\delta^{\neq}(\Gamma))}} \\ \displaystyle\times \prod_{m:\, \sigma_m(\Gamma) \neq \sigma(\Gamma)} e^{(\beta/2) U_{\gamma}(q_{\sigma(\Gamma)}^{(\Gamma)} \mid q_{\sigma(\Gamma)}^{(\delta^{\neq}(\Gamma))})} \end{array} \right)} \tag{3.43}$$

In Section 4 we will prove the Peierls bound (which is the analogue of (3.31)):

Statement 3.4. For any contour Γ there is $c > 0$ such that for all γ and $\lambda \in (\lambda(\beta) - \gamma^{\alpha}, \lambda(\beta) + \gamma^{\alpha})$, $\alpha \geqslant \frac{1}{2}$

$$w(\Gamma \mid \bar{q}^{(\delta^{=}(\Gamma))}) \leqslant \exp(-c \ell_2^d \ell_3^{-d} \zeta^2 \, |\mathrm{Supp}(\Gamma)|) \tag{3.44}$$

Finally in Section 6 we will prove

Statement 3.5. For any contour Γ there is $c > 0$ such that for all sufficiently small γ and $\lambda \in (\lambda(\beta) - \gamma^{\alpha}, \lambda(\beta) + \gamma^{\alpha})$, $\alpha \geqslant \frac{1}{2}$

$$\Bigg| \log W^{A}(\Gamma \mid \bar{q}^{(\delta^{=}(\Gamma))}) - \log w(\Gamma \mid \bar{q}^{\delta^{=}(\Gamma))}) - \sum_{m:\, \sigma_m(\Gamma) \neq \sigma(\Gamma)} (\beta \, |\mathrm{Int}_m(\Gamma)| [f^{A}_{-\sigma(\Gamma),\lambda,\gamma} - f^{A}_{\sigma(\Gamma),\lambda,\gamma}]) \Bigg| \leqslant c \, |\mathrm{Supp}(\Gamma)| \, \gamma^{1/4} \tag{3.45}$$

Note that the factor $\gamma^{1/4}$ is not optimal but it is enough for our purposes.

By combining the above statements and choosing $\lambda = \lambda(\beta, \gamma)$ we then obtain that the auxiliary and the true partition functions are equal and consequently that the probability of a contour in the original system is bounded as in (3.44). This proves the existence of two different phases and together with the exponential decay established in Sections 5 and 6 concludes the proof of Theorem 2.1.

4. PEIERLS ESTIMATE

In this section we prove Statement 3.4, i.e., we compare the partition functions in the numerator and denominator of (3.43).

The partition function in the denominator of (3.43) is taken over ground state configurations of the phase $\sigma = \sigma(\Gamma)$ placed in the region $\mathrm{Supp}(\Gamma)\backslash\delta(\Gamma)$ with the boundary condition $\bar{q}^{(\delta(\Gamma))}$ specified on $\delta(\Gamma)$. This boundary condition, $\bar{q}^{(\delta(\Gamma))}$, is different in $\delta^{=}(\Gamma)$ and in $\delta^{\neq}(\Gamma)$: it is a configuration of the ground state ensemble of the phase σ in the region $\delta^{=}(\Gamma)$ and it is the standard configuration of the phase σ in $\delta^{\neq}(\Gamma)$.

The partition function from the numerator of (3.43) is taken over contour configurations, i.e., ones compatible with η^{Γ}, in the same region $\mathrm{Supp}(\Gamma)\backslash\delta(\Gamma)$ with the same boundary condition $\bar{q}^{(\delta^{=}(\Gamma))}$ in $\delta^{=}(\Gamma)$ but with the different boundary condition given on $\delta^{\neq}(\Gamma)$. Contrary to the denominator this boundary condition is the standard configuration of the opposite phase, $-\sigma$.

Scheme of Estimate

The estimate of the ratio in (3.43) is performed in several steps. At each step we achieve a certain simplification at the price of a non essential error which we define as a positive factor not exceeding $\exp(\gamma^{\alpha}\ell_2^d\ell_3^{-d}\zeta^2\,|\mathrm{Supp}(\Gamma)|)$ with $\alpha > 0$. For γ small enough the product of the finite number of such factors is dominated by $\exp(-c\ell_2^d\ell_3^{-d}\zeta^2\,|\mathrm{Supp}(\Gamma)|)$. The last factor appears at the last step of the estimate and the preliminary simplifications make this main step of the Peierls estimate easier.

First we exploit a superstability of the potential and in Lemmas 4.1–4.4 we show that only particle configurations of bounded density need to be taken into account.

As soon as all considerations, are reduced to the case of bounded density it becomes clear that one needs to check (3.44) only for $\lambda = \lambda(\beta)$. Indeed, varying λ in the interval $\lambda \in (\lambda(\beta) - \gamma^{\alpha}, \lambda(\beta) + \gamma^{\alpha})$ with $\alpha \geqslant 1/2$ produces an error in $H_{\gamma,\lambda}(q \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_{\pm\sigma(\Gamma)}^{(\delta^{\neq}(\Gamma))})$ which does not exceed $\gamma^{1/2}\,|\mathrm{Supp}(\Gamma)|$ in absolute value. Hence the total error in (3.43) is negligible with respect to the right hand side of (3.44).

Then in Lemma 4.5 we replace every configuration $q = (q_i)$ contributing to the numerator or denominator of (3.43) by $[q]^{(\ell_1)}$. After shifting particles into the centers of the corresponding boxes integrals over the particle configurations q become sums over density configurations $\varrho^{(\ell_1)}(q) = (\varrho_x^{(\ell_1)}(q))$. Moreover, the entropy factor coming from the summation over different density configurations $\varrho^{(\ell_1)}$ can be neglected and one can consider only a contribution of two density configurations providing the minimum of the energy in the numerator and denominator of (3.43) respectively. These arguments are presented in (4.19)–(4.23) below.

The next observation is that the energy of density configuration is very close to a discrete version, (4.26), of the mean field free energy functional (3.1) with function $\varrho(r)$ of a continuous argument $r \in \mathbb{R}^d$ replaced by its lattice version $\varrho_x^{(\ell_1)}$, $x \in \mathbb{Z}_{\ell_1}^d$. The corresponding errors are estimated in Lemma 4.6.

Thus, we arrive to the variational problem of finding minima of the discrete mean field functional under the constraints corresponding to the numerator and denominator of (3.43). The simplest of the constraints is the discreteness of our density variables. Lemma 4.7 shows that one can treat the density as a continuous variable for the price of non essential error. A more serious constraint is the presence of boundary conditions. (Our density configurations vary only inside the finite region, the support of the contour, being fixed outside this region.) Moreover, these boundary conditions are different for the numerator and denominator of (3.43). To handle this problem in Lemma 4.8 we show that the dependence of the minimizer on the boundary conditions decays exponentially with the distance from the boundary. In particular, at the distance of order $\gamma^{-1-\alpha_3}$ the boundary condition is practically not felt. Our proof of Lemma 4.8 is based on estimate (3.17).

Using Lemma 4.8 we perform the last of our simplifications. In (4.57)–(4.58) we show that modulo non essential errors one can replace the initial variational problem by a similar one in a smaller region but with *standard* boundary conditions only.

Finally we come to the key Lemma 4.9 expressing the essence of the Peierls estimate. It is obvious that for the denominator of (3.43) the minimizing density is the constant density configuration coinciding with the boundary condition $\varrho_{\beta,\sigma}$. The minimizing density configuration for the numerator of (3.43) is not a constant and may have a complicated structure. While not trying to calculate it exactly we estimate (from below) the corresponding value of the discrete free energy functional. This is a rather straightforward calculation which takes into account the fact that inside $\mathrm{Supp}(\Gamma)\setminus\delta(\Gamma)$ there are sufficiently many "wrong boxes" $C^{(\ell_2)}$ where the indicator function $\mathbb{1}_{\eta(q)=\eta^\Gamma}$ forces the minimizing configuration to be different from $\varrho_{\beta,\pm}$. Each "wrong box" produces an excess of energy of order ζ^2 and "wrong boxes" are distributed inside $\mathrm{Supp}(\Gamma)$ with the density $c\ell_2^d\ell_3^{-d}$. This finally gives us a factor $\exp(-c\ell_2^d\ell_3^{-d}\zeta^2\,|\mathrm{Supp}(\Gamma)|)$ dominating all error estimates discussed earlier.

The details are given in the rest of this section.

Stability Estimates

First we check that the accumulation of an infinite number of particles in a bounded region is impossible. This is a consequence of

Lemma 4.1. For any particle q_0 and any configuration $q=(q_1, q_2,...)$ the interaction energy

$$U_\gamma(q_0 \mid q) = -\sum_{q_{i_1}\in q} J^{(2)}_\gamma(q_0, q_{i_1}) + \sum_{q_{i_1}, q_{i_2}, q_{i_3}\in q} J^{(4)}_\gamma(q_0, q_{i_1}, q_{i_2}, q_{i_3}) \tag{4.1}$$

is bounded from below

$$U_\gamma(q_0 \mid q) \geqslant H_{\min} \tag{4.2}$$

by an absolute constant

$$H_{\min} = \min_{\mathcal{N}>0,\, 1>\gamma>0} \frac{1}{3!}(\mathcal{N}^3 - 3\gamma^d\mathcal{N}^2 + 2\gamma^{2d}\mathcal{N}) - \mathcal{N} \tag{4.3}$$

Proof. Given q let

$$\mathcal{N}(r, q) = \gamma^d \sum_{q_i\in q} \mathbb{1}_{|r-q_i|\leqslant\gamma^{-1}R_d} \tag{4.4}$$

be the mean number of particles of q situated inside $B_\gamma(r)$, $r\in\mathbb{R}^d$. Then

$$\begin{aligned} -\sum_{q_{i_1}\in q} J^{(2)}_\gamma(q_0, q_{i_1}) &= -\gamma^{2d}\sum_{q_i\in q}\int dr\, \mathbb{1}_{|r-q_0|\leqslant\gamma^{-1}R_d}\mathbb{1}_{|r-q_i|\leqslant\gamma^{-1}R_d} \\ &= -\gamma^d\int dr\, \mathbb{1}_{|r-q_0|\leqslant\gamma^{-1}R_d}\mathcal{N}(r, q) \end{aligned} \tag{4.5}$$

Similarly

$$\begin{aligned} &\sum_{q_{i_1}, q_{i_2}, q_{i_3}\in q} J^{(4)}_\gamma(q_0, q_{i_1}, q_{i_2}, q_{i_3}) \\ &\quad = \gamma^d\int dr\, \mathbb{1}_{|r-q_0|\leqslant\gamma^{-1}R_d}\frac{1}{3!}(\mathcal{N}^3(r, q) - 3\gamma^d\mathcal{N}^2(r, q) + 2\gamma^{2d}\mathcal{N}(r, q)) \end{aligned} \tag{4.6}$$

Hence

$$U_\gamma(q_0 \mid q) \geqslant \gamma^d \int dr\, \mathbb{1}_{|r-q_0| \leqslant \gamma^{-1} R_d} H_{\min} = H_{\min} \tag{4.7}$$

which proves the lemma. ∎

Lemma 4.1 provides the lower bound

$$H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)}) \geqslant H_{\min}\, |q^{(\Lambda)}| = H_{\min}\, |\Lambda|\, \varrho \tag{4.8}$$

where $\varrho = |q^{(\Lambda)}|/|\Lambda|$ is the corresponding density. To obtain an upper bound take some positive $\varrho_{\max} > \varrho_{\beta,+}$.

Lemma 4.2. Consider configurations $q^{(\Lambda)}$ and $\bar{q}^{(\Lambda^c)}$ such that $q^{(\Lambda)} \cup \bar{q}^{(\Lambda^c)}$ has at most $\varrho_{\max}(2\gamma^{-1})^d$ particles in any intersecting Λ cube with the side $2\gamma^{-1}$. Then

$$H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)}) \leqslant H_{\max}(\varrho_{\max})\, |\Lambda| \tag{4.9}$$

where

$$H_{\max}(\varrho_{\max}) = |\lambda(\beta)|\, \varrho_{\max} + 2^{3d} \varrho_{\max}^4 \tag{4.10}$$

Proof. It is clear that $|q^{(\Lambda)}| \leqslant |\Lambda|\, \varrho_{\max}$. The strength of the four-body interaction between any four particles is less than γ^{3d}. The number of interacting quadruples of particles such that one of the elements of the quadruple is a given particle is less than $(\varrho_{\max} 2^d \gamma^{-d})^3$. Hence the total four-body interaction contributing to $H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})$ is less than $|\Lambda|\, \varrho_{\max}^4 2^{3d}$. The two-body interaction is negative and does not contribute to the estimate. ∎

Bad Boxes

In this subsection we treat boxes containing too many particles.

Lemma 4.3. Consider a box $C^{(\ell)}$, $\ell < \frac{1}{2}\gamma^{-1}$, a configuration $\bar{q} \in \mathcal{Q}$ and an integer

$$N \geqslant |C^{(\ell)}|\, e^{c\beta} = \ell^d e^{c\beta} \tag{4.11}$$

then

$$\int_{\mathcal{Q}(C^{(\ell)})} dq\, \mathbb{1}_{|q^{(C^{(\ell)})}| = N}\, e^{-\beta H_{\gamma,\lambda}(q^{(C^{(\ell)})} \mid \bar{q})} \leqslant e^{-N} \tag{4.12}$$

and

$$\int_{\mathcal{Q}(C^{(\ell)})} dq\, \mathbb{1}_{|q^{(C^{(\ell)})}| \geqslant |C^{(\ell)}|\, e^{c\beta}}\, e^{-\beta H_{\gamma,\lambda}(q^{(C^{(\ell)})} \mid \bar{q})} \leqslant 2e^{-|C^{(\ell)}|\, e^{c\beta}} \tag{4.13}$$

Proof. According to Lemma 4.1 the interaction $U_\gamma(q^{(C^{(\ell)})} \mid \bar{q})$ between particles of $q^{(C^{(\ell)})}$ and $\bar{q}$ satisfies the estimate

$$U_\gamma(q^{(C^{(\ell)})} \mid \bar{q}) \geqslant H_{\min} N \tag{4.14}$$

It is clear that

$$H_{\gamma,\lambda}(q^{(C^{(\ell)})}) \geqslant -\gamma^d\, \frac{N(N-1)}{2!} + \frac{1}{2d}\, \gamma^{3d}\, \frac{N(N-1)(N-2)(N-3)}{4!} - |\lambda|\, N \tag{4.15}$$

as the maximal volume of the intersection of two balls of radius $\gamma^{-1} R_d$ is γ^{-d} and the minimal volume of the intersection of four such balls with the centers in $C^{(\ell)}$ is larger than $(1/2^d)\, \gamma^{-d}$, Thus the logarithm of the left hand side of (4.12) does not exceed

$$\begin{aligned} &-N \log N + N + N \log |C^{(\ell)}| - \beta H_{\min} N + \beta\, |\lambda|\, N \\ &\quad + \beta \gamma^d\, \frac{N(N-1)}{2!} - \beta\, \frac{1}{2^d}\, \gamma^{3d}\, \frac{N(N-1)(N-2)(N-3)}{4!} \end{aligned} \tag{4.16}$$

It is not hard to check that for sufficiently small γ and sufficiently large absolute constant $c_{(4.11)}$ the last expression is smaller than $-N$ as soon as $N \geqslant |C^{(\ell)}|\, e^{c_{(4.11)}\beta}$. Note that the most dangerous term in (4.16) is $\beta\gamma^d(N(N-1)/2!)$. It is dominated by $\beta(1/2^d)\, \gamma^{3d}(N(N-1)(N-2)(N-3)/4!)$ for $N > c\gamma^{-d}$ and by $N \log N - N \log |C^{(\ell)}|$ for $|C^{(\ell)}|\, e^{c_{(4.11)}\beta} \leqslant N \leqslant c\gamma^{-d}$. This implies (4.12) and hence (4.13). ∎

Set $\varrho_{\max} = 2e^{c_{(4.11)}\beta}$. An easy consequence of Lemma 4.3 is

Lemma 4.4. For any contour Γ and any $\ell \leqslant \ell_2$ the partition function in the numerator of (3.43) does not exceed

$$\begin{aligned} &e^{\ell^{-d}\, |\mathrm{Supp}(\Gamma)|} \int_{\mathcal{Q}(\mathrm{Supp}(\Gamma) \setminus \delta(\Gamma))} dq\, \mathbb{1}_{\eta(q) = \eta^\Gamma,\, \varrho^{(\ell)}(q) \leqslant \varrho_{\max}}\, e^{-\beta H_{\gamma,\lambda}(q \mid \bar{q}^{(\delta^{=}(\Gamma))},\, q_{-\sigma}^{(\delta^{\neq}(\Gamma))})} \\ &\quad \times \prod_{m:\, \sigma_m(\Gamma) \neq \sigma(\Gamma)} e^{(\beta/2)\, U_\gamma(q^{(\Gamma)}_{-\sigma(\Gamma)} \mid q^{(\delta^{\neq}(\Gamma))}_{-\sigma(\Gamma)})} \end{aligned} \tag{4.17}$$

Proof. To obtain (4.17) we perform a partial integration over the configurations in the bad boxes. Suppose that q is fixed outside the cube $C^{(\ell)} \in \mathrm{Supp}(\Gamma)$ where q has more than $\varrho_{\max} |C^{(\ell)}|$ particles. Now applying Lemma 4.3 we integrate over all such q's leaving exactly $\frac{1}{2}\varrho_{\max} |C^{(\ell)}|$ particles in $C^{(\ell)}$. That means that we integrate over the particles in excess of $\frac{1}{2}\varrho_{\max} |C^{(\ell)}|$, assuming that the total number of particles in $C^{(\ell)}$ is larger than $\varrho_{\max} |C^{(\ell)}|$. This produces an extra factor $1+2e^{-|C^{(\ell)}|\, e^{c\beta}} \leqslant e$. We continue this procedure box by box and the total number of boxes in $\mathrm{Supp}(\Gamma)$ is $\ell^{-d}\, |\mathrm{Supp}(\Gamma)|$. ∎

Reduction to Variational Problem

It was shown in the previous subsection that we may restrict our considerations to configurations with bounded density. Now we estimate the error which is produced by shifting particles in such a configuration into the centers of the corresponding boxes $C^{(\ell)}$.

Lemma 4.5. Take $\ell < \gamma^{-1}$ and configurations $q^{(\Lambda)}$ and $\bar{q}^{(\Lambda^c)}$ with $\varrho^{(\ell)}(q^{(\Lambda)} \cup \bar{q}^{(\partial\Lambda)})) \leqslant \varrho_{\max}$. Then

$$|H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)}) - H_{\gamma,\lambda}(\varrho^{(\ell)}(q^{(\Lambda)}) \mid \bar{q}^{(\Lambda^c)})| \leqslant \ell\gamma\, |\Lambda|\, (2^d \varrho^2_{\max} + 2^{3d} \varrho^4_{\max}) \tag{4.18}$$

Proof. It is clear that $|q^{(\Lambda)}| \leqslant |\Lambda|\, \varrho_{\max}$. Given two interacting particles the absolute value of the error in their interaction due to shifting these particles to the centers of the corresponding boxes is less than $\gamma^d \ell\gamma$. Given four interacting particles the absolute value of the error in their interaction due to shifting these particles to the centers of the corresponding boxes is less than $\gamma^{3d}\ell\gamma$. As in Lemma 4.2 the number of interacting quadruples of particles such that one of the elements of the quadruple is a given particle is less than $(\varrho_{\max} 2^d \gamma^{-d})^3$. Similarly the number of pairs of interacting particles such that one of the elements of the pair is a given particle is less than $(\varrho_{\max} 2^d \gamma^{-d})$. Hence the total error is less than $\ell\gamma\, |\Lambda|\, (2^d \varrho^2_{\max} + 2^{3d} \varrho^4_{\max})$. ∎

This lemma allows us to replace the integrals over dq in the numerator and denominator of (3.43) by the sums over density configurations $\varrho^{(\ell_1)}$. Namely, consider the partition function on the right hand side of (4.17). The integral over $q \in \mathcal{Q}^{(\mathrm{Supp}(\Gamma)\setminus\delta(\Gamma))}$ with $\eta(q) = \eta^\Gamma$ and $\varrho^{(\ell_1)}(q) \leqslant \varrho_{\max}$ can be calculated in two steps.

First one can fix a density configuration $\varrho^{(\ell_1)} = (\varrho_x^{(\ell_1)})$, $x \in [\mathrm{Supp}(\Gamma)\setminus(\Gamma)]^{(\ell_1)}$ and integrate over configurations $q^{(\mathrm{Supp}(\Gamma)\setminus\delta(\Gamma))}$ with $\varrho^{(\ell_1)}(q) = \varrho^{(\ell_1)}$.

Afterward one can sum over all $\varrho^{(\ell_1)} \leqslant \varrho_{\max}$ compatible with η^Γ. The obvious upper estimate for this sum is the total number of density configurations $\varrho_x^{(\ell_1)} \leqslant \varrho_{\max}$, $x \in [\mathrm{Supp}(\Gamma) \backslash \delta(\Gamma)]^{(\ell_1)}$ times the maximal contribution given by a single density configuration.

The number of density configurations $\varrho^{(\ell_1)} \leqslant \varrho_{\max}$ compatible with η^Γ is less than

$$(\varrho_{\max} \, |C^{(\ell_1)}|)^{\,|\mathrm{Supp}(\Gamma) \backslash \delta(\Gamma)| / |C^{(\ell_1)}|} \tag{4.19}$$

Given $\varrho^{(\ell_1)}$ it follows from Lemma 4.5 that the integral over configurations q with $\varrho^{(\ell_1)}(q) = \varrho^{(\ell_1)}$ does not exceed

$$\begin{aligned} &\exp\big(-\tilde{\mathscr{F}}_{\gamma, \beta, \lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^=(\Gamma))}, q_{-\sigma}^{(\delta^{\neq}(\Gamma))}, *) \\ &\quad + \beta \ell_1 \gamma \, |\mathrm{Supp}(\Gamma) \backslash \delta(\Gamma)| \, (2^d \varrho_{\max}^2 + 2^{3d} \varrho_{\max}^4)\big) \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} &\tilde{\mathscr{F}}_{\gamma, \beta, \lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^=(\Gamma))}, q_{-\sigma}^{(\delta^{\neq}(\Gamma))}, *) \\ &\quad = \beta H_{\gamma, \lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^=(\Gamma))}, q_{-\sigma}^{(\delta^{\neq}(\Gamma))}) - \frac{\beta}{2} \sum_{m:\, \sigma_m(\Gamma) \neq \sigma(\Gamma)} U_\gamma(q_{-\sigma(\Gamma)}^{(\Gamma)} \mid q_{-\sigma(\Gamma)}^{(\delta^{\neq}(\Gamma))}) \\ &\quad\quad - |C^{(\ell_1)}| \log |C^{(\ell_1)}| \sum_{x \in [\mathrm{Supp}(\Gamma) \backslash \delta(\Gamma)]^{(\ell_1)}} \varrho_x^{(\ell_1)} \\ &\quad\quad + \sum_{x \in [\mathrm{Supp}(\Gamma) \backslash \delta(\Gamma)]^{(\ell_1)}} \log((\varrho_x^{(\ell_1)} \, |C^{(\ell_1)}|)!) \end{aligned} \tag{4.21}$$

The notations with $\tilde{}$ and $*$ foresee forthcoming simplifications and variations. In particular $\tilde{\mathscr{F}}_{\gamma, \beta, \lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^=(\Gamma))}, q_\sigma^{(\delta^{\neq}(\Gamma))})$ is defined as $\tilde{\mathscr{F}}_{\gamma, \beta, \lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^=(\Gamma))}, q_\sigma^{(\delta^{\neq}(\Gamma))}, *)$ without the term $(\beta/2) \sum_{m:\, \sigma_m(\Gamma) \neq \sigma(\Gamma)} U_\gamma(q_{-\sigma(\Gamma)}^{(\Gamma)} \mid q_{-\sigma(\Gamma)}^{(\delta^{\neq}(\Gamma))})$ in (4.20).

Suppose that $\bar{\varrho}^{(\ell_1)}$ gives the minimum of $\tilde{\mathscr{F}}_{\gamma, \beta, \lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^=(\Gamma))}, q_{-\sigma}^{(\delta^{\neq}(\Gamma))}, *)$ among all $\varrho^{(\ell_1)} \leqslant \varrho_{\max}$ compatible with η^Γ. Then

$$\begin{aligned} &-\tilde{\mathscr{F}}_{\gamma, \beta, \lambda(\beta)}(\hat{\varrho}^{(\ell_1)} \mid \bar{q}^{(\delta^=(\Gamma))}, q_{-\sigma}^{(\delta^{\neq}(\Gamma))}, *) + |\mathrm{Supp}(\Gamma) \backslash \delta(\Gamma)| \\ &\quad \times (\gamma^{1/2} + \ell_1^{-d} + \ell_1^{-d} \log(\varrho_{\max} \, |C^{(\ell_1)}|) + \beta \ell_1 \gamma (2^d \varrho_{\max}^2 + 2^{3d} \varrho_{\max}^4)) \end{aligned} \tag{4.22}$$

is the upper bound for the log of the numerator of (3.43). Note that $\lambda \in (\lambda(\beta) - \gamma^\alpha, \lambda(\beta) + \gamma^\alpha)$, $\alpha \geqslant 1/2$ in (4.21) while $\lambda = \lambda(\beta)$ in (4.22). The corresponding error is covered by the $\gamma^{1/2}$ term in (4.22).

Similarly if $\hat{\varrho}^{(\ell_1)}$ gives the minimum of $\tilde{\mathscr{F}}_{\gamma,\beta,\lambda(\beta)}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^=(\Gamma))}, q_\sigma^{(\delta^{\neq}(\Gamma))}, *)$ among all density configurations $\varrho^{(\ell_1)}$ from the ground state ensemble of the phase σ then

$$-\tilde{\mathscr{F}}_{\gamma,\beta,\lambda(\beta)}(\hat{\varrho}^{(\ell_1)} \mid \bar{q}^{(\delta^=(\Gamma))}, q_\sigma^{(\delta^{\neq}(\Gamma))}, *) \\ - |\mathrm{Supp}(\Gamma)\backslash\delta(\Gamma)|\, (\gamma^{1/2} + \beta\ell_1\gamma(2^d\varrho_{\max}^2 + 2^{3d}\varrho_{\max}^4)) \tag{4.23}$$

is the lower bound for the log of the denominator of (3.43).

Now denoting $b_{\gamma,\ell} = \| [B_\gamma(\cdot)]^{(\ell)} \|$ let

$$I_\gamma^{(2)}(x_1, x_2) = b_{\gamma,\ell}^{-2} \, \| [B_\gamma(x_1)]^{(\ell)} \cap [B_\gamma(x_2)]^{(\ell)} \| \tag{4.24}$$

and

$$I_\gamma^{(4)}(x_1,..., x_4) = b_{\gamma,\ell}^{-4} \left\| \bigcap_{j-1}^{4} [B_\gamma(x_j)]^{(\ell)} \right\| \tag{4.25}$$

be discrete versions of $J_\gamma^{(2)}(x_1, x_2)$ and $J_\gamma^{(4)}(x_1,..., x_4)$. For any region $[\Lambda]^{(\ell)}$ (we do note exclude the case $[\Lambda]^{(\ell)} = \mathbb{Z}_\ell^d$) and any density configuration $\varrho^{(\ell)}(\Lambda) = (\varrho_x^{(\ell)})$, $x \in [\Lambda]^{(\ell)}$ define a functional

$$\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda)) = \ell^d \Bigg(\sum_{x\in[\Lambda]^{(\ell)}} \frac{\varrho_x^{(\ell)}}{\beta} (\log \varrho_x^{(\ell)} - 1) - \sum_{x\in[\Lambda]^{(\ell)}} \lambda \varrho_x^{(\ell)} \\ - \frac{1}{2!} \sum_{x_2, x_1 \in [\Lambda]^{(\ell)}} I_\gamma^{(2)}(x_1, x_2)\, \varrho_{x_1}^{(\ell)} \varrho_{x_2}^{(\ell)} \\ + \frac{1}{4!} \sum_{x_1, x_2, x_3, x_4 \in [\Lambda]^{(\ell)}} I_\gamma^{(4)}(x_1,..., x_4)\, \varrho_{x_1}^{(\ell)} \cdots \varrho_{x_4}^{(\ell)} \Bigg) \tag{4.26}$$

which is a discrete analogue of the mean field free energy functional (3.1). We also define a conditional functional

$$\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(\Lambda^c)) = \mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) + \bar{\varrho}^{(\ell)}(\Lambda^c)) - \mathscr{F}_{\gamma,\beta,\lambda}(\bar{\varrho}^{(\ell)}(\Lambda^c)) \tag{4.27}$$

with the boundary condition $\bar{\varrho}^{(\ell)} = (\bar{\varrho}_x^{(\ell)})$, $x \in [\Lambda^c]^{(\ell)}$. Here the similarity with (2.10) is obvious and the meaning of $\varrho^{(\ell)} + \bar{\varrho}^{(\ell)}$ becomes straightforward if we set $\varrho^{(\ell)} \equiv 0$, $x \in [\Lambda^c]^{(\ell)}$ and $\bar{\varrho}^{(\ell)} \equiv 0$, $x \in [\Lambda]^{(\ell)}$. Setting

$$U_{\gamma,\beta}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(\Lambda^c)) = \mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(\Lambda^c)) - \mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda)) \tag{4.28}$$

we introduce

$$\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(\Lambda^c), *)$$
$$= \mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(\Lambda^c)) - \tfrac{1}{2} U_{\gamma,\beta}(\varrho^{(\ell)}_{\beta,\sigma}(\Lambda) \mid \varrho^{(\ell)}_{\beta,\sigma}(\Lambda^c)) \tag{4.29}$$

where $\varrho^{(\ell)}_{\beta,\sigma} \equiv \varrho_{\beta,\sigma}$ and $\sigma = +$ or $\sigma = -$. With some abuse of notation we use for $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \varrho^{(\ell)}(\bar{q}^{(\partial\Lambda)}))$ the alternative notations $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{q}^{(\partial\Lambda)})$ or $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(q^{(\partial\Lambda)}))$. The role which is played by the functional (4.27) is clarified in

Lemma 4.6. Consider a $\mathscr{D}^{(\ell)}$ measurable region Λ with the boundary condition $\bar{q}^{(\partial\Lambda)}$ which is a ground state configuration on every connected component of $\partial\Lambda$. Then for $\ell \leqslant \gamma^{-1}$ and any $\varrho^{(\ell)} = (\varrho^{(\ell)}_x)$, $x \in [\Lambda]^{(\ell)}$ such that $\varrho^{(\ell)}_x \leqslant \varrho_{\max}$ one has

$$|\tilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{q}^{(\partial\Lambda)}) - \mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(q^{(\partial\Lambda)}))|$$
$$\leqslant \ell\gamma 5(2^d \varrho^2_{\max} + 2^{3d}\varrho^4_{\max} + \varrho^4_{\max})\, |\Lambda| \tag{4.30}$$

$$|\tilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{q}^{(\partial\Lambda)}, *) - \mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(q^{(\partial\Lambda)}), *)|$$
$$\leqslant \ell\gamma 5(2^d \varrho^2_{\max} + 2^{3d}\varrho^4_{\max} + \varrho^4_{\max})\, |\Lambda| \tag{4.31}$$

Proof. Estimate (4.31) is an obvious consequence of (4.30) and we concentrate on (4.30). The difference between $\tilde{\mathscr{F}}_{\gamma,\beta,\lambda}$ and $\mathscr{F}_{\gamma,\beta,\lambda}$ has two sources. The first one is due to replacement of balls $B_\gamma(\cdot)$ in the definition of $J^{(\cdot)}_\gamma$ by their lattice versions $[B_\gamma(\cdot)]^{(\ell)}$ in the definition of $I^{(\cdot)}_\gamma$. Clearly the difference between $|B_\gamma(\cdot)| = \gamma^{-d}$ and $|[B_\gamma(\cdot)]^{(\ell)}|$ is less than $\gamma^{-d}\ell\gamma$. Hence the error produced by the discretization of $B_\gamma(\cdot)$ can be estimated exactly as in Lemma 4.5.

The second source is due to not properly counted contribution of pairs x_1, x_2 with $x_1 = x_2$ and quadruples x_1, x_2, x_3, x_4 with not all x_{i_j} being different. To estimate from above the absolute value of this error let us consider the following five contributions to the energy of $\varrho^{(\ell)}$.

(i) Self-interaction of $C^{(\ell)}_x$ due to the pair interaction of particles in $C^{(\ell)}_x$.

(ii) Self-interaction of $C^{(\ell)}_x$ due to the four-body interaction of particles in $C^{(\ell)}_x$.

(iii) Interaction between $C^{(\ell)}_{x_1}$ and $C^{(\ell)}_{x_2}$ due to the four-body interaction of two particles in $C^{(\ell)}_{x_1}$ with two particles in $C^{(\ell)}_{x_2}$.

(iv) Interaction between $C_{x_1}^{(\ell)}$ and $C_{x_2}^{(\ell)}$ due to the four-body interaction of three particles in $C_{x_1}^{(\ell)}$ with one particle in $C_{x_2}^{(\ell)}$.

(v) Interaction between $C_{x_1}^{(\ell)}$, $C_{x_2}^{(\ell)}$ and $C_{x_3}^{(\ell)}$ due to the four-body interaction of two particles in $C_{x_1}^{(\ell)}$ with one particle in $C_{x_2}^{(\ell)}$ and one particle in $C_{x_3}^{(\ell)}$.

All five contributions can be estimated by similar arguments. For that reason we present these arguments only for cases (i) and (v).

The strength of self-interaction of $C_x^{(\ell)}$ due to the pair interaction of particles in $C_x^{(\ell)}$ is less than $\gamma^d(\varrho_{\max}\ell^d)^2$. The number of boxes $C_x^{(\ell)}$ in the region Λ is $\ell^{-d}|\Lambda|$. Hence the total contribution is less than $(\ell\gamma)^d\,\varrho_{\max}^2\,|\Lambda|$.

The strength of interaction between $C_{x_1}^{(\ell)}$, $C_{x_2}^{(\ell)}$ and $C_{x_3}^{(\ell)}$ due to the four-body interaction of two particles in $C_{x_1}^{(\ell)}$ with one particle in $C_{x_2}^{(\ell)}$ and one particle in $C_{x_3}^{(\ell)}$ is less than $\gamma^{3d}(\varrho_{\max}\ell^d)^4$. The number of boxes $C_{x_2}^{(\ell)}$ and $C_{x_3}^{(\ell)}$ interacting with given box $C_{x_1}^{(\ell)}$ is less than $(\gamma\ell)^{-2d}$. The number of boxes $C_{x_1}^{(\ell)}$ is $\ell^{-d}|\Lambda|$. Hence the total contribution is less than $(\ell\gamma)^d\,\varrho_{\max}^4\,|\Lambda|$. ∎

Looking for the minima of $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}\mid\bar{\varrho}^{(\ell)}(q^{(\partial\Lambda)}))$ it is simpler to understand $\varrho_x^{(\ell)}$ as continuous variables. Therefore if the minimum of $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}\mid\bar{\varrho}^{(\ell)}(q^{(\partial\Lambda)}))$ is achieved on density configuration $\hat{\varrho}^{(\ell)}$ then it may happen that at least for some $x\in[\Lambda]^{(\ell)}$ the number $\ell^d\hat{\varrho}_x^{(\ell)}$ is not an integer. The solution to this problem is given by

Lemma 4.7. For the density configuration $\tilde{\varrho}_x^{(\ell)}=\ell^{-d}[\ell^d\hat{\varrho}_x^{(\ell)}]$ one has

$$|\mathscr{F}_{\gamma,\beta,\lambda}(\hat{\varrho}^{(\ell)}(\Lambda)\mid\bar{\varrho}^{(\ell)}(q^{(\partial\Lambda)}))-\mathscr{F}_{\gamma,\beta,\lambda}(\tilde{\varrho}^{(\ell)}(\Lambda)\mid\bar{\varrho}^{(\ell)}(q^{(\partial\Lambda)}))|\leqslant c\ell^{-d}\varrho_{\max}|\Lambda| \tag{4.32}$$

(Here $[\,\cdot\,]$ denotes the integer part of a number.)

Proof. We proceed as in the proof of Lemma 4.5. The total number of points $x\in[\Lambda]^{(\ell)}$ is $\ell^{-d}|\Lambda|$. Given point x the absolute value of the difference in the corresponding selfinteraction is less than $\beta^{-1}\log\varrho_{\max}+|\lambda|\,\varrho_{\max}$. The difference in the two-point interaction between points $x_1, x_2\in[\Lambda]^{(\ell)}$ does not exceed in absolute value $2\gamma^d\varrho_{\max}\ell^d$. The number of points x_2 interacting with given x_1 is less than $(\gamma\ell)^{-d}$. The difference in four-point interaction between points $x_1, x_2, x_4, x_4\in[\Lambda]^{(\ell)}$ does not exceed in absolute value $4\gamma^{3d}(\varrho_{\max}\ell^d)^3$. The number of points x_2, x_3, x_4 interacting with given x_1 is less than $(\gamma\ell)^{-3d}$. Combining these estimates one obtains the lemma. ∎

Variational Problem (Dependence on Boundary Condition)

The results of the previous subsection reduce the Peierls estimate to the following variational problem:

(i) Find the minimum of $\mathscr{F}_{\gamma, \beta, \lambda(\beta)}(\varrho^{(\ell_1)}(\mathrm{Supp}(\Gamma)\backslash\delta(\Gamma)) \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_{\sigma}^{(\delta^{\neq}(\Gamma))}, *)$ over the density configurations $\varrho^{(\ell_1)} = (\varrho_x^{(\ell_1)})$, $x \in [\mathrm{Supp}(\Gamma)\backslash \delta(\Gamma)]^{(\ell_1)}$ such that $\varrho^{(\ell_1)}$ belongs to the ground state ensemble of the phase σ.

(ii) Find the minimum of $\mathscr{F}_{\gamma, \beta, \lambda(\beta)}(\varrho^{(\ell_1)}(\mathrm{Supp}(\Gamma)\backslash\delta(\Gamma)) \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_{-\sigma}^{(\delta^{\neq}(\Gamma))}, *)$ over the density configurations $\varrho^{(\ell_1)} = (\varrho_x^{(\ell_1)})$, $x \in [\mathrm{Supp}(\Gamma)\backslash \delta(\Gamma)]^{(\ell_1)}$ such that $\varrho^{(\ell_1)}$ is compatible with η^{Γ}.

(iii) Estimate from below the difference between the minimal value of $\mathscr{F}_{\gamma, \beta, \lambda(\beta)}(\varrho^{(\ell_1)}(\mathrm{Supp}(\Gamma)\backslash\delta(\Gamma)) \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_{-\sigma}^{(\delta^{\neq}(\Gamma))}, *)$ and the minimal value of $\mathscr{F}_{\gamma, \beta, \lambda(\beta)}(\varrho^{(\ell_1)}(\mathrm{Supp}(\Gamma)\backslash\delta(\Gamma)) \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_{\sigma}^{(\delta^{\neq}(\Gamma))}, *)$.

The existence of the minima above is obvious as $0 < \varrho_x^{(\ell_1)} < \varrho_{\max}$. Following the approach of Section 3 define

$$\mathscr{R}_{\gamma, \ell}(x, \varrho^{(\ell)}) = b_{\gamma, \ell}^{-1} \sum_{x_1 \in [B_\gamma(x)]^{(\ell)}} \varrho_{x_1}^{(\ell)} \tag{4.33}$$

and

$$\mathscr{I}_{\gamma, \ell}(x, \varrho^{(\ell)}) = b_{\gamma, \ell}^{-1} \sum_{x_1 \in [B_\gamma(x)]^{(\ell)}} \frac{\varrho_{x_1}^{(\ell)}}{\beta} (\log \varrho_{x_1}^{(\ell)} - 1) \tag{4.34}$$

where we deliberately use a general scale (ℓ) instead of (ℓ_1) as the constructions below are of general origin. In complete similarity with (3.4)

$$\mathscr{F}_{\gamma, \beta, \lambda}(\varrho^{(\ell)}) = \ell^d \sum_{x \in \mathbb{Z}_\ell^d} \Big(\mathscr{I}_{\gamma, \ell}(x, \varrho^{(\ell)}) - \lambda \mathscr{R}_{\gamma, \ell}(x, \varrho^{(\ell)}) - \frac{1}{2!} \mathscr{R}_{\gamma, \ell}(x, \varrho^{(\ell)})^2 + \frac{1}{4!} \mathscr{R}_{\gamma, \ell}(x, \varrho^{(\ell)})^4 \Big) \tag{4.35}$$

implying that $\varrho^{(\ell)}(\mathbb{Z}_\ell^d) \equiv \varrho_{\beta, \lambda, -}$ and $\varrho^{(\ell)}(\mathbb{Z}_\ell^d) \equiv \varrho_{\beta, \lambda, +}$ are the global minimizers for $\mathscr{F}_{\gamma, \beta, \lambda}(\varrho^{(\ell)})$. The local minimizers of $\mathscr{F}_{\gamma, \beta, \lambda}(\cdot)$, i.e., the minima of $\mathscr{F}_{\gamma, \beta, \lambda}(\cdot \mid \bar{\varrho}^{(\ell)}(\Lambda^c))$ or $\mathscr{F}_{\gamma, \beta, \lambda}(\cdot \mid \bar{\varrho}^{(\ell)}(\Lambda^c), *)$, are studied in the lemma below. Note that in this lemma we consider not only $\lambda = \lambda(\beta)$ but all $\lambda \in (\lambda(\beta) - \delta(\beta), \lambda(\beta) + \delta(\beta))$. Though the lemma discusses $\mathscr{F}_{\gamma, \beta, \lambda}(\cdot \mid \bar{\varrho}^{(\ell)}(\Lambda^c))$ the same argument covers the case of $\mathscr{F}_{\gamma, \beta, \lambda}(\cdot \mid \bar{\varrho}^{(\ell)}(\Lambda^c), *)$.

Lemma 4.8. Consider a $\mathscr{D}^{(\ell)}$ measurable region Λ and take a boundary condition $\bar{\varrho}^{(\ell)}(\Lambda^c)$ with $\max_{x\in[\partial\Lambda]^{(\ell)}} |\bar{\varrho}^{(\ell)}_x - \varrho_{\beta,\lambda,\sigma}| \leqslant \zeta$, where σ is one of the phases, $+$ or $-$. Then the unique minimum of $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(\Lambda^c))$ is achieved on the density configuration $\hat{\varrho}^{(\ell)} = (\hat{\varrho}^{(\ell)}_x)$, $x\in[\Lambda]^{(\ell)}$ such that

$$|\hat{\varrho}^{(\ell)}_x - \varrho_{\beta,\lambda,\sigma}| \leqslant \frac{\zeta a(\beta)^{[\gamma\,\mathrm{dist}(x,\Lambda^c)]}}{1-a(\beta)} \tag{4.36}$$

where $[\,\cdot\,]$ denotes the integer part.

Proof. Calculating $(\partial/\partial\varrho^{(\ell)}_x)\,\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(\Lambda^c))$, $x\in[\Lambda]^{(\ell)}$ one obtains the necessary condition for $\hat{\varrho}^{(\ell)}$

$$\begin{aligned} 0 = {} & \frac{1}{\beta}\log\hat{\varrho}^{(\ell)}_x - \lambda - \sum_{x_1} I^{(2)}_\gamma(x,x_1)\,\hat{\varrho}^{(\ell)}_{x_1} \\ & + \frac{1}{3!}\sum_{x_1,x_2,x_3} I^{(4)}_\gamma(x,x_1,x_2,x_3)\,\hat{\varrho}^{(\ell)}_{x_1}\hat{\varrho}^{(\ell)}_{x_2}\hat{\varrho}^{(\ell)}_{x_3} \end{aligned} \tag{4.37}$$

It is clear that for $\bar{\varrho}^{(\ell)}_x \equiv \varrho_{\beta,\lambda,\sigma}$, $x\in[\Lambda^c]^{(\ell)}$ one has $\hat{\varrho}^{(\ell)}_x \equiv \varrho_{\beta,\lambda,\sigma}$, $x\in[\Lambda]^{(\ell)}$. Introduce an auxiliary parameter $t\in[0,1]$ and an *interpolated boundary condition*

$$\bar{\varrho}^{(\ell)}_x(t) = (1-t)\,\varrho_{\beta,\lambda,\sigma} + t\bar{\varrho}^{(\ell)}_x, \qquad x\in[\Lambda^c]^{(\ell)} \tag{4.38}$$

Let $\hat{\varrho}^{(\ell)}(t)$ be the solution of (4.37) with the boundary condition $\bar{\varrho}^{(\ell)}(t)$. Then

$$\begin{aligned} \hat{\varrho}^{(\ell)}_x(t) = \exp\Bigg(& \beta\lambda + \beta\sum_{x_1\in[B_\gamma(x)]^{(\ell)}} I^{(2)}_\gamma(x,x_1)\,\hat{\varrho}^{(\ell)}_{x_1}(t) \\ & - \frac{\beta}{3!}\sum_{x_1,x_2,x_3\in[B_\gamma(x)]^{(\ell)}} I^{(4)}_\gamma(x,x_1,x_2,x_3)\,\hat{\varrho}^{(\ell)}_{x_1}(t)\,\hat{\varrho}^{(\ell)}_{x_2}(t)\,\hat{\varrho}^{(\ell)}_{x_3}(t)\Bigg) \end{aligned} \tag{4.39}$$

for all $x\in[\Lambda]^{(\ell)}$. Taking the derivative with respect to t one obtains

$$\begin{aligned} \frac{d}{dt}\hat{\varrho}^{(\ell)}_x(t) = \beta\hat{\varrho}^{(\ell)}_x(t)\Bigg(& \sum_{x_1\in[B_\gamma(x)]^{(\ell)}} I^{(2)}_\gamma(x,x_1)\,\frac{d}{dt}\hat{\varrho}^{(\ell)}_{x_1}(t) \\ & - \frac{1}{2!}\sum_{x_1,x_2,x_3\in[B_\gamma(x)]^{(\ell)}} I^{(4)}_\gamma(x,x_1,x_2,x_3)\,\frac{d}{dt}\hat{\varrho}^{(\ell)}_{x_1}(t)\,\hat{\varrho}^{(\ell)}_{x_2}(t)\,\hat{\varrho}^{(\ell)}_{x_3}(t)\Bigg) \end{aligned} \tag{4.40}$$

Set

$$I_\gamma^{(2)}(x_1, x_2 \mid \hat{\varrho}^{(\ell)}(t)) = \sum_{x_3, x_4} I_\gamma^{(4)}(x_1, x_2, x_3, x_4)\, \hat{\varrho}_{x_3}^{(\ell)}(t)\, \hat{\varrho}_{x_4}^{(\ell)}(t) \tag{4.41}$$

It is clear that

$$\begin{aligned} I_\gamma^{(2)}(x_1, x_2)(\varrho_{\beta,\lambda,\sigma} - \zeta)^2 &\leqslant I_\gamma^{(2)}(x_1, x_2 \mid \hat{\varrho}^{(\ell)}(t)) \\ &\leqslant I_\gamma^{(2)}(x_1, x_2)(\varrho_{\beta,\lambda,\sigma} + \zeta)^2 \end{aligned} \tag{4.42}$$

as soon as $\max_x |\hat{\varrho}_x^{(\ell)}(t) - \varrho_{\beta,\lambda,\sigma}| \leqslant \zeta$.

Introduce a symmetric matrix

$$A(x_1, x_2 \mid \hat{\varrho}^{(\ell)}(t)) = -I_\gamma^{(2)}(x_1, x_2) + \tfrac{1}{2} I_\gamma^{(2)}(x_1, x_2 \mid \hat{\varrho}^{(\ell)}(t)) \tag{4.43}$$

and a diagonal matrix

$$D(x_1, x_2 \mid \hat{\varrho}^{(\ell)}(t)) = (\beta \hat{\varrho}_{x_1}^{(\ell)}(t))^{-1}\, \mathbb{1}_{x_1 = x_2} \tag{4.44}$$

where $x_1, x_2 \in [\Lambda]^{(\ell)}$. It is not hard to see that the inverse matrix $B = (D - A)^{-1}$ exists if $\max_{x \in [\Lambda]^{(\ell)}} |\hat{\varrho}_x^{(\ell)}(t) - \varrho_{\beta,\lambda,\sigma}| \leqslant \zeta$. Indeed, for such $\hat{\varrho}_x^{(\ell)}(t)$ this matrix is given by a convergent series

$$B = \left(\sum_{n=0}^{\infty} (D^{-1}A)^k \right) D^{-1} \tag{4.45}$$

because

$$\begin{aligned} \|D^{-1}A\| &= \max_{x_1} \left(\beta \hat{\varrho}_{x_1}^{(\ell)}(t) \sum_{x_2} |A(x_1, x_2 \mid \hat{\varrho}^{(\ell)}(t))| \right) \\ &\leqslant a(\beta) < 1 \end{aligned} \tag{4.46}$$

where $a(\beta)$ is defined in (3.16). Moreover, the representation (4.45) and the fact that $A(x_1, x_2 \mid \hat{\varrho}^{(\ell)}(t)) = 0$ if $\mathrm{dist}(x_1, x_2) > \gamma^{-1}$ imply that

$$|B(x_1, x_2 \mid \hat{\varrho}^{(\ell)}(t))| \leqslant \frac{\beta(\varrho_{\beta,\lambda,\sigma} + \zeta)}{1 - a(\beta)}\, a(\beta)^{[\gamma\, \mathrm{dist}(x_1, x_2)]} \tag{4.47}$$

Iterating (4.40) and observing that $(d/dt)\, \bar{\varrho}_x^{(\ell)}(t) = -\varrho_{\beta,\lambda,\sigma} + \bar{\varrho}_x^{(\ell)}$ for $x \in [\Lambda^c]^{(\ell)}$ we rewrite (4.40) as

$$\frac{d}{dt} \hat{\varrho}_x^{(\ell)}(t) = \sum_{x_1 \in [\Lambda]^{(\ell)}} \sum_{x_2 \in [\Lambda^c]^{(\ell)}} B(x, x_1 \mid \hat{\varrho}^{(\ell)}(t))\, A(x_1, x_2 \mid \hat{\varrho}^{(\ell)}(t))(\bar{\varrho}_{x_2}^{(\ell)} - \varrho_{\beta,\lambda,\sigma}) \tag{4.48}$$

The right hand side of (4.48) is an absolutely convergent series in terms of $\hat{\varrho}_x^{(\ell)}(t)$, $x \in [\Lambda]^{(\ell)}$ and solving this differential equation one finds $\hat{\varrho}_x^{(\ell)}(t)$. The solution exists at least up to $t = t_1$ at which the condition

$$\max_{x \in [\Lambda]^{(\ell)}} \left| \frac{d}{dt} \hat{\varrho}_x^{(\ell)}(t) - \varrho_{\beta, \lambda, \sigma} \right| < \zeta \tag{4.49}$$

is violated. Suppose that $t_1 < 1$, i.e.,

$$\max_{x \in [\Lambda]^{(\ell)}} \left| \frac{d}{dt} \hat{\varrho}_x^{(\ell)}(t) - \varrho_{\beta, \lambda, \sigma} \right| < \zeta \tag{4.50}$$

for $t < t_1$ and

$$\left| \frac{d}{dt} \hat{\varrho}_{x_1}^{(\ell)}(t_1) - \varrho_{\beta, \lambda, \sigma} \right| = \zeta \tag{4.51}$$

for some $x_1 \in [\Lambda]^{(\ell)}$. Then the representation (4.48) is valid for $t \in [0, t_1]$ and

$$\max_{x \in [\Lambda]^{(\ell)}} |\hat{\varrho}_x^{(\ell)}(t) - \varrho_{\beta, \lambda, \sigma}| < \zeta \tag{4.52}$$

for $t \leqslant t_1$ as follows from the obvious identity

$$\hat{\varrho}_x^{(\ell)}(t) = \varrho_{\beta, \lambda, \sigma} + \int_0^t \frac{d}{ds} \hat{\varrho}_x^{(\ell)}(s)\, ds \tag{4.53}$$

Plugging (4.50)–(4.52) into (4.40) and using (3.17) one concludes that

$$\max_{x \in [\Lambda]^{(\ell)}} \left| \frac{d}{dt} \hat{\varrho}_x^{(\ell)}(t) - \varrho_{\beta, \lambda, \sigma} \right| < \zeta \tag{4.54}$$

for $t \leqslant t_1$ which contradicts (4.51). Hence

$$\max_{x \in [\Lambda]^{(\ell)}} \left| \frac{d}{dt} \hat{\varrho}_x^{(\ell)}(t) - \varrho_{\beta, \lambda, \sigma} \right| \leqslant \zeta \tag{4.55}$$

and

$$\max_{x \in [\Lambda]^{(\ell)}} |\hat{\varrho}_x^{(\ell)}(t) - \varrho_{\beta, \lambda, \sigma}| \leqslant \zeta \tag{4.56}$$

for all $t \in [0, 1]$. Thus representation (4.48) and estimate (4.47) are valid and being joined with (4.53) they give (4.36).

The density configuration $\hat{\varrho}(\Lambda)$ was defined as the solution of (4.37) and we just checked that such a solution exists and satisfies (4.36). On the other hand, it follows from (4.42) and (3.17) that the Hessian matrix $(\partial^2/\partial\varrho_x^{(\ell)}\,\partial\varrho_y^{(\ell)})\,\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell)}(\Lambda) \mid \bar{\varrho}^{(\ell)}(\Lambda^c))$, $x, y \in [\Lambda]^{(\ell)}$ is positive for any $\varrho^{(\ell)}(\Lambda) \in (\varrho_{\beta,\lambda,\sigma} - \zeta, \varrho_{\beta,\lambda,\sigma} + \zeta)^{\Lambda}$ with the mass bounded away from 0 independently on $\varrho^{(\ell)}(\Lambda)$ and $\bar{\varrho}^{(\ell)}(\Lambda^c)$. Thus the function $\mathscr{F}_{\gamma,\beta,\lambda}(\,\cdot \mid \bar{\varrho}^{(\ell)}(\Lambda^c))$ with convex domain $(\varrho_{\beta,\lambda,\sigma} - \zeta, \varrho_{\beta,\lambda,\sigma} + \zeta)^{\Lambda}$ is convex and therefore $\hat{\varrho}(\Lambda)$ is its unique minimum. ▌

Variational Problem (Comparison of Two Minima)

In this subsection we return back to the case $\lambda = \lambda(\beta)$. For any contour Γ the set

$$\mathrm{S} = \partial^{(\ell_3)}\,\mathrm{Ext}(\Gamma) \cup \Bigg(\bigcup_{m:\,\sigma_m(\Gamma) = \sigma(\Gamma)} \partial^{(\ell_3)}\mathrm{Int}_m(\Gamma) \Bigg) \tag{4.57}$$

is occupied by the ground state of the phase $\sigma = \sigma(\Gamma)$. Consider a strip $\tilde{\delta}^{=}(\Gamma)$ of width γ^{-1} situated in the middle of S. According to Lemma 4.8 inside $\tilde{\delta}^{=}(\Gamma)$ the density configurations minimizing $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_\sigma^{(\delta^{\neq}(\Gamma))}, *)$ and $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_{-\sigma}^{(\delta^{\neq}(\Gamma))}, *)$ differ from $\varrho_{\beta,\sigma}$ by at most $(1 - a(\beta))^{-1}\,\zeta a(\beta)^{[\gamma^{1/3}\gamma^{-1-\alpha_3}]}$. Therefore the density configuration which minimizes $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_\sigma^{(\delta^{\neq}(\Gamma))}, *)$ under the additional condition that this configuration coincides with $\varrho_{\beta,\sigma}$ in $\tilde{\delta}^{=}(\Gamma)$ gives the value of the true minimum of $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_\sigma^{(\delta^{\neq}(\Gamma))}, *)$ up to an nonessential error. The same is true for the density configuration minimizing $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_{-\sigma}^{(\delta^{\neq}(\Gamma))}, *)$ with the same additional condition. Obviously these modified minimizing configurations coincide with each other not only in $\tilde{\delta}^{=}(\Gamma)$, where they both are equal to $\varrho_{\beta,\sigma}$, but in the whole part of S stretching from $\delta^{=}(\Gamma)$ to $\tilde{\delta}^{=}(\Gamma)$, i.e., in

$$\mathrm{S}_1 = \partial^{(\ell_3/2)}\,\mathrm{Ext}(\Gamma) \cup \Bigg(\bigcup_{m:\,\sigma_m(\Gamma) = \sigma(\Gamma)} \partial^{(\ell_3/2)}\,\mathrm{Int}_m(\Gamma) \Bigg) \cup \tilde{\delta}^{=}(\Gamma) \tag{4.58}$$

Estimating from below the difference between the minimal value of $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_{-\sigma}^{(\delta^{\neq}(\Gamma))}, *)$ and the minimal value of $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)} \mid \bar{q}^{(\delta^{=}(\Gamma))}, q_\sigma^{(\delta^{\neq}(\Gamma))}, *)$ the contributions corresponding to S_1 cancel each other. Hence the initial variational problem in $\mathrm{Supp}(\Gamma)\backslash\delta(\Gamma)$ with a general boundary condition imposed on $\delta(\Gamma)$ is reduced to a similar problem in a smaller volume $\mathrm{Supp}(\Gamma)\backslash\mathrm{S}_1$.

For the sake of notational simplicity we suppose from now on that the boundary condition is a standard one already for the initial problem, i.e., it is equal to $\varrho_{\beta,\sigma}$ in $\delta^{=}(\Gamma)$. Thus all we need to finish the Peierls estimate is the lower bound for

$$\Delta\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)},\Gamma)=\ell_1^d\sum_{x\in[\mathrm{Supp}(\Gamma)\setminus\delta(\Gamma)]^{(\ell_1)}}\bigg(\mathscr{I}_{\gamma,\ell_1}(x,\varrho^{(\ell_1)})-\lambda\mathscr{R}_{\gamma,\ell_1}(x,\varrho^{(\ell_1)}) \\ -\frac{1}{2!}\mathscr{R}_{\gamma,\ell_1}(x,\varrho^{(\ell_1)})^2+\frac{1}{4!}\mathscr{R}_{\gamma,\ell_1}(x,\varrho^{(\ell_1)})^4-F_{\beta,\lambda}(\varrho_{\beta,\sigma})\bigg) \tag{4.59}$$

We note that for x such that $\mathrm{dist}(x,\delta(\Gamma))\leqslant\gamma^{-1}$ the values of $\mathscr{I}_{\gamma,\ell}(x,\varrho^{(\ell)})$ and $\mathscr{R}_{\gamma,\ell}(x,\varrho^{(\ell)})$ depend on the boundary condition which is equal to $\varrho_{\beta,\sigma}$ in $\delta^{=}(\Gamma)$ and $\varrho_{\beta,-\sigma}$ in $\delta^{\neq}(\Gamma)$.

Lemma 4.9. For any contour $\Gamma=(\mathrm{Supp}(\Gamma),\eta^{\Gamma})$

$$\min_{\varrho^{(\ell_1)}:\,\eta(\varrho^{(\ell_1)})=\eta^{\Gamma}}\Delta\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)},\Gamma)\geqslant c\ell_2^d\ell_3^{-d}\zeta^2\,|\mathrm{Supp}(\Gamma)| \tag{4.60}$$

Proof. Rewrite $\Delta\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)},\Gamma)$ in the form

$$\Delta\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_1)},\Gamma)=\ell_1^d\sum_{x\in[\mathrm{Supp}(\Gamma)\setminus\delta(\Gamma)]^{(\ell_1)}}\bigg(F_{\beta,\lambda}(\mathscr{R}_{\gamma,\ell_1}(x,\varrho^{(\ell_1)}))-F_{\beta,\lambda}(\varrho_{\beta,\sigma}) \\ +b_{\gamma,\ell_1}^{-1}\sum_{x_1\in[B_\gamma(x)]^{(\ell_1)}}\frac{\varrho_{x_1}^{(\ell_1)}}{\beta}\log\varrho^{(\ell_1)}{}_{x_1} \\ -\frac{\mathscr{R}_{\gamma,\ell_1}(x,\varrho^{(\ell_1)})}{\beta}\log\mathscr{R}_{\gamma,\ell_1}(x,\varrho^{(\ell_1)})\bigg) \tag{4.61}$$

and observe that

$$F_{\beta,\lambda}(\mathscr{R}_{\gamma,\ell_1}(x,\varrho^{(\ell_1)}))-F_{\beta,\lambda}(\varrho_{\beta,\sigma})\geqslant 0 \tag{4.62}$$

by definition of $\varrho_{\beta,\sigma}$ and

$$b_{\gamma,\ell_1}^{-1}\sum_{x_1\in[B_\gamma(x)]^{(\ell_1)}}\frac{\varrho_{x_1}^{(\ell_1)}}{\beta}\log\varrho_{x_1}^{(\ell_1)}\geqslant\frac{\mathscr{R}_{\gamma,\ell_1}(x,\varrho^{(\ell_1)})}{\beta}\log\mathscr{R}_{\gamma,\ell_1}(x,\varrho^{(\ell_1)}) \tag{4.63}$$

by convexity.

We call $C_x^{(\ell_2)} \in \mathrm{Supp}(\Gamma)$ a *wrong box* if either $\eta_x^\Gamma = 0$ or $\eta_x^\Gamma \eta_{x_1}^\Gamma = -1$ for at least one cube $C_{x_1}^{(\ell_2)}$ adjacent to $C_x^{(\ell_2)}$. According to the definition of the contour there exist at least $(3\ell_3)^{-d} |\mathrm{Supp}(\Gamma)|$ wrong boxes $C_x^{(\ell_2)} \in \mathrm{Supp}(\Gamma)$ such that they are at a distance greater than $5\gamma^{-1}$ from each other. In particular $C_{x_1}^{(2\gamma^{-1})} \cap C_{x_2}^{(2\gamma^{-1})} = \varnothing$ for any two such boxes $C_{x_1}^{(\ell_2)}$ and $C_{x_2}^{(\ell_2)}$.

Consider a wrong box $C_x^{(\ell_2)}$ for which $\eta_x^\Gamma = 0$. If inside $C_x^{(2\gamma^{-1})}$ there exist at least $\ell_1^{-d} \ell_2^d$ points x_1 with

$$\mathscr{R}_{\gamma, \ell_1}(x_1, \varrho^{(\ell_1)}) \notin \left(\varrho_{\beta, -} - \frac{\zeta}{2}, \varrho_{\beta, -} + \frac{\zeta}{2} \right) \cup \left(\varrho_{\beta, +} - \frac{\zeta}{2}, \varrho_{\beta, +} + \frac{\zeta}{2} \right) \tag{4.64}$$

then $C_x^{(2\gamma^{-1})}$ contributes to $\Delta \mathscr{F}_{\gamma, \beta, \lambda}(\varrho^{(\ell_1)}, \Gamma)$ by at least

$$c \ell_2^d \zeta^2 \min(F''_{\beta, \lambda}(\varrho_{\beta, -}), F''_{\beta, \lambda}(\varrho_{\beta, +})) \tag{4.65}$$

This contribution comes from the terms

$$\ell_1^d F_{\beta, \lambda}(\mathscr{R}_{\gamma, \ell_1}(x_1, \varrho^{(\ell_1)})) - \ell_1^d F_{\beta, \lambda}(\varrho_{\beta, \sigma}) \tag{4.66}$$

in (4.61).

In the opposite situation when

$$\mathscr{R}_{\gamma, \ell_1}(x_1, \varrho^{(\ell_1)}) \notin \left(\varrho_{\beta, -} - \frac{\zeta}{2}, \varrho_{\beta, -} + \frac{\zeta}{2} \right) \cup \left(\varrho_{\beta, +} - \frac{\zeta}{2}, \varrho_{\beta, +} + \frac{\zeta}{2} \right) \tag{4.67}$$

for not more than $\ell_1^{-d} \ell_2^d$ points $x_1 \in C_x^{(2\gamma^{-1})}$ we will extract a contribution similar to (4.65) from the terms

$$\ell_1^d b_{\gamma, \ell_1}^{-1} \sum_{x_2 \in [B_\gamma(x_1)]^{(\ell_1)}} \frac{\varrho_{x_2}^{(\ell_1)}}{\beta} \log \varrho_{x_2}^{(\ell_1)} - \ell_1^d \frac{\mathscr{R}_{\gamma, \ell_1}(x_1, \varrho^{(\ell_1)})}{\beta} \log \mathscr{R}_{\gamma, \ell_1}(x_1, \varrho^{(\ell_1)}) \tag{4.68}$$

in (4.61).

Indeed, consider $x_1 \in C_x^{(2\gamma^{-1})}$ such that $C_x^{(\ell_2)} \in [B_\gamma(x_1)]^{(\ell_1)}$ and observe that

$$\varrho_x^{(\ell_2)} = \| [C_x^{(\ell_2)}]^{(\ell_1)} \|^{-1} \sum_{x_2 \in [C_x^{(\ell_2)}]^{(\ell_1)}} \varrho_{x_2}^{(\ell_1)} \tag{4.69}$$

does not belong to $(\varrho_{\beta, -} - \zeta, \varrho_{\beta, -} + \zeta) \cup (\varrho_{\beta, +} - \zeta, \varrho_{\beta, +} + \zeta)$ as $\eta_x^\Gamma = 0$. Denote

$$\mathscr{R}_{x_1} = \| [B_\gamma(x_1)]^{(\ell_1)} \backslash [C_x^{(\ell_2)}]^{(\ell_1)} \|^{-1} \sum_{x_2 \in [B_\gamma(x_1)]^{(\ell_1)} \backslash [C_x^{(\ell_2)}]^{(\ell_1)}} \varrho_{x_2}^{(\ell_1)} \tag{4.70}$$

Then by convexity

$$\sum_{x_2 \in [B_\gamma(x_1)]^{(\ell_1)}} \varrho_{x_2}^{(\ell_1)} \log \varrho_{x_2}^{(\ell_1)} \geqslant \|[B_\gamma(x_1)]^{(\ell_1)} \backslash [C_x^{(\ell_2)}]^{(\ell_1)}\| \mathscr{R}_{x_1} \log \mathscr{R}_{x_1} + \|[C_x^{(\ell_2)}]^{(\ell_1)}\| \varrho_x^{(\ell_2)} \log \varrho_x^{(\ell_2)} \tag{4.71}$$

implying the lower bound

$$\frac{\|[B_\gamma(x_1)]^{(\ell_1)} \backslash [C_x^{(\ell_2)}]^{(\ell_1)}\|}{\beta b_{\gamma,\ell_1}} \mathscr{R}_{x_1} \log \mathscr{R}_{x_1} + \frac{\|[C_x^{(\ell_2)}]^{(\ell_1)}\|}{\beta b_{\gamma,\ell_1}} \varrho_x^{(\ell_2)} \log \varrho_x^{(\ell_2)} - \frac{\mathscr{R}_{\gamma,\ell_1}(x_1, \varrho^{(\ell_1)})}{\beta} \log \mathscr{R}_{\gamma,\ell_1}(x_1, \varrho^{(\ell_1)}) \tag{4.72}$$

for (4.68). Now we apply the inequality

$$(1-\alpha)\, a \log a - \alpha b \log b - ((1-\alpha)\, a + \alpha b) \log((1-\alpha)\, a + \alpha b) \geqslant \frac{\alpha}{2} \frac{(a-b)^2}{\max(a,b)} - \alpha^2 \frac{(a-b)^2}{a} \tag{4.73}$$

which is true for any $a, b > 0$ and $0 < \alpha < 1$. This leads to the lower bound

$$\frac{\|[C_x^{(\ell_2)}]^{(\ell_1)}\|}{2\beta b_{\gamma,\ell_1}} \frac{(\mathscr{R}_{x_1} - \varrho_x^{(\ell_2)})^2}{\max(\mathscr{R}_{x_1}, \varrho_x^{(\ell_2)})} - \left(\frac{\|[C_x^{(\ell_2)}]^{(\ell_1)}\|}{\beta b_{\gamma,\ell_1}}\right)^2 \frac{(\mathscr{R}_{x_1} - \varrho_x^{(\ell_2)})^2}{\mathscr{R}_{x_1}} \tag{4.74}$$

for (4.72). Observing that $(\beta b_{\gamma,\ell_1})^{-1} \|[C_x^{(\ell_2)}]^{(\ell_1)}\|$ is of order $\gamma^{\alpha_2 d}$ we conclude that (4.74) exceeds

$$c \frac{\|[C_x^{(\ell_2)}]^{(\ell_1)}\|}{\beta b_{\gamma,\ell_1} \varrho_{\beta,-}} (\mathscr{R}_{\gamma,\ell_1}(x_1, \varrho^{(\ell_1)}) - \varrho_x^{(\ell_2)})^2 \geqslant c\gamma^{\alpha_2 d} \frac{\zeta^2}{\beta \varrho_{\beta,-}} \tag{4.75}$$

for γ sufficiently small. The number of points $x_1 \in C_x^{(2\gamma^{-1})}$ for which (4.75) is true is not less than $b_{\gamma,\ell_1}/2$. Remind that for these points

$$\mathscr{R}_{\gamma,\ell_1}(x_1, \varrho^{(\ell_1)}) \in \left(\varrho_{\beta,-} - \frac{\zeta}{2}, \varrho_{\beta,-} + \frac{\zeta}{2}\right) \cup \left(\varrho_{\beta,+} - \frac{\zeta}{2}, \varrho_{\beta,+} + \frac{\zeta}{2}\right) \tag{4.76}$$

and $[B_\gamma(x_1)]^{(\ell_1)} \ni C_x^{(\ell_2)}$. Thus (4.75) again gives us a lower bound

$$c\ell_2^d \zeta^2 (\beta \varrho_{\beta,-})^{-1} \tag{4.77}$$

similar to (4.65).

Finally for the wrong box $C_x^{(\ell_2)}$ of the second type, i.e., when $\eta_x^\Gamma \eta_{x_1}^\Gamma = -1$ for an adjacent box $C_{x_1}^{(\ell_2)}$, we can consider $C_x^{(\ell_2)} \cup C_{x_1}^{(\ell_2)}$ instead of $C_x^{(\ell_2)}$ and repeat all the arguments above. They will work perfectly because

$$\|[C_x^{(\ell_2)}]^{(\ell_1)} \cup [C_{x_1}^{(\ell_2)}]^{(\ell_1)}\|^{-1} \sum_{x_2 \in [C_x^{(\ell_2)}]^{(\ell_1)} \cup [C_{x1}^{(\ell_2)}]^{(\ell_1)}} \varrho_{x_2}^{(\ell_1)} \tag{4.78}$$

is again outside $(\varrho_{\beta,-} - \zeta, \varrho_{\beta,-} + \zeta) \cup (\varrho_{\beta,+} - \zeta, \varrho_{\beta,+} + \zeta)$. ∎

To finish the proof of the Peierls estimate we need to compare all the errors which are at most of the order $c\ell_1 \gamma\, |\mathrm{Supp}(\Gamma)| = c\gamma^{\alpha_1}\, |\mathrm{Supp}(\Gamma)|$ (see (4.22) and (4.23)) with the contribution coming from Lemma 4.9. The last is of order $c\ell_2^d \ell_3^{-d}\, |\mathrm{Supp}(\Gamma)| = c\gamma^{d\alpha_2 + d\alpha_3}\, |\mathrm{Supp}(\Gamma)|$ and dominates $c\gamma^{\alpha_1}\, |\mathrm{Supp}(\Gamma)|$ since $d\alpha_2 + d\alpha_3 < \alpha_1$.

5. AUXILIARY MODEL

In this section we study metastable models and we prove that the corresponding measures satisfy the Dobrushin uniqueness condition and hence exhibit an exponential decay of correlations. We note again that the configuration $q^{(\Lambda)}$ in a $\mathscr{D}^{(\ell_2)}$ measurable region Λ belongs to the ground state ensemble of the phase σ iff in every cube $C_x^{(\ell_2)} \in \Lambda$ the density of particles

$$\varrho_x^{(\ell_2)}(q^{(\Lambda)}) = \ell_2^{-d}\, |q^{(\Lambda)} \cap C_x^{(\ell_2)}| \tag{5.1}$$

belongs to the interval

$$(\varrho_{\beta,\sigma} - \zeta, \varrho_{\beta,\sigma} + \zeta) \tag{5.2}$$

The partition function of the auxiliary model is given by (3.37), i.e., it is the integral over ground state configurations of the corresponding contour partition function.

In the previous section we have shown that some partition functions initially defined in terms of particle configurations can be approximated by partition functions written in terms of density configurations related to some scale ℓ. Now we go further and show that the auxiliary model for each of the two phases can be *equivalently* rewritten in terms of density configurations. Such an equivalent model is defined on the lattice $\mathbb{Z}_{\ell_2}^d$ with the density variables $\varrho_x^{(\ell_2)}$ taking discrete values $n\ell_2^{-d}$, $n = 1, 2...$ from the bounded interval (5.2). The corresponding Hamiltonian is of infinite range but with sufficiently fast decaying interactions. Quite naturally this Hamiltonian is close to (4.27) (with $\ell = \ell_2$) and it can be understood as a small perturbation of a positive definite quadratic form. The treatment of the equivalent model is based on a specific approach [COPP] to the Dobrushin uniqueness theorem developed initially for unbounded lattice spin systems.

From now on we fix ℓ_2 as the scale at which we define density configurations. Thus all regions are assumed to be $\mathscr{D}^{(\ell_2)}$ measurable and we often drop the superscript (ℓ_2) from notations.

In the next subsection we construct the effective Hamiltonian, the exact form of which is stated in Lemma 5.2 at the very end of the next subsection. Then in the last two subsections we prove Lemma 5.3 saying that the effective Hamiltonian satisfies the Dobrushin uniqueness condition.

Reduction to Density Model

The technical part of the reduction is based on the cluster or polymer expansion technique. For the convenience of the reader Section 7 quotes a version of the general cluster expansion theorem which is suitable for our purposes.

Consider a region Λ with the boundary condition $\bar{q}^{(\Lambda^c)}$ belonging to the ground state ensemble of the phase σ. For the partition function $Z^{\Lambda}_{\gamma, \beta, \lambda}(\Lambda \mid \bar{q}^{(\Lambda^c)})$ we decompose the integral in (3.37) into a sum of integrals. In this decomposition the external sum is taken over all density configurations ϱ_x, $x \in [\Lambda]^{(\ell_2)}$ satisfying (5.2). Given such a density configuration $\varrho(\Lambda)$ an internal integral is taken over all particle configurations $q^{(\Lambda)}$ such that $\varrho_x^{(\ell_2)}(q^{(\Lambda)}) = \varrho_x^{(\ell_2)}(\Lambda)$ for all $x \in [\Lambda]^{(\ell_2)}$. The reduction we perform is nothing but the calculation of

$$\log \int_{\mathscr{Q}^{(\Lambda)}} dq^{(\Lambda)}\, \mathbb{1}_{\varrho^{(\ell_2)}(q) = \varrho(\Lambda)}\, e^{-\beta H_{\gamma, \lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})} \sum_{\{\Gamma_i\}^{\sigma} \in \Lambda} \prod_i W^T(\eta^{\Gamma_i} \mid q^{(\delta^=(\Gamma_i))}) \tag{5.3}$$

as a function of $\varrho_x(\Lambda)$. Denote by $q_{x,i}$ the particles of $q^{(\Lambda)}$ situated inside $C_x^{(\ell_2)}$ and set $n_x = \ell_2^d \varrho_x = |q^{(\Lambda)} \cap C_x^{(\ell_2)}|$. Then $\int_{\mathscr{Q}^{(\Lambda)}} dq$ is

$$\left(\prod_{x \in [\Lambda]^{(\ell_2)}} \frac{\ell_2^{dn_x}}{n_x!}\, \ell_2^{-dn_x} \right) \int \cdots \int \left(\prod_{x \in [\Lambda]^{(\ell_2)}} \prod_{i=1}^{n_x} \mathbb{1}_{q_{x,i} \in C_x^{(\ell_2)}}\, dq_{x,i} \right) e^{-\beta H_{\gamma, \lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})}$$

$$\times \sum_{\{\Gamma_i\}^{\sigma} \in \Lambda} \prod_i W^T(\eta^{\Gamma_i} \mid q^{(\delta^=(\Gamma_i))}) \tag{5.4}$$

where the integral can be understood as an expectation with respect to the system of independent particles $q_{x,i}$ uniformly distributed in the corresponding boxes $C_x^{(\ell_2)}$.

The Hamiltonian $H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})$ can be decomposed into the sum of a ϱ-dependent Hamiltonian

$$H_{\gamma,\lambda}(\varrho^{(\ell_2)}(Q^{(\Lambda)}) \mid \bar{q}^{(\Lambda^c)}) = H_{\gamma,\lambda}([q^{(\Lambda)}]^{(\ell_2)} \mid \bar{q}^{(\Lambda^c)}) = H_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}) \quad (5.5)$$

and error terms

$$\Delta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)}) = H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)}) - H_{\gamma,\lambda}(\varrho^{(\ell_2)}(q^{(\Lambda)}) \mid \bar{q}^{(\Lambda^c)}) \quad (5.6)$$

Clearly $\exp(-\beta H_{\gamma,\lambda}(\varrho \mid \bar{q}^{(\Lambda^c)}))$ can be taken outside the integral in (5.4) leaving us with the calculation of the log of the partition function

$$\left(\prod_{x\in[\Lambda]^{(\ell_2)}} \ell_2^{-dn_x}\right)\int\cdots\int\left(\prod_{x\in[\Lambda]^{(\ell_2)}}\prod_{i=1}^{n_x} \mathbb{1}_{q_{x,i}\in C_x^{(\ell_2)}}\, dq_{x,i}\right) e^{-\beta\,\Delta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})}$$
$$\times \sum_{\{\Gamma_i\}^\sigma \in \Lambda} \prod_i W^T(\eta^{\Gamma_i} \mid q^{(\delta^=(\Gamma_i))}) \quad (5.7)$$

where the first product is included for the convenience of treating this partition function as an expectation over a system of independent particles.

Observe that the error part of the Hamiltonian is given by

$$\Delta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})$$
$$= -\sum_{q_{i_1},q_{i_2}\in q} \Delta J_\gamma^{(2)}(q_{i_1},q_{i_2}) + \sum_{q_{i_1},q_{i_2},q_{i_3},q_{i_4}\in q} \Delta J_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4}) \quad (5.8)$$

where $\Delta J_\gamma^{(2)}(q_{i_1},q_{i_2})$ and $\Delta J_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4})$ are much smaller than 1. The estimates

$$0 < \Delta J_\gamma^{(2)}(q_{i_1},q_{i_2}) < c\gamma^{\alpha_2} J_\gamma^{(2)}(q_{i_1},q_{i_2}) \quad (5.9)$$

and

$$0 < \Delta J_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4}) < c\gamma^{\alpha_2} J_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4}) \quad (5.10)$$

are true unless some of the interacting particles are at the distance larger than $\gamma^{-1}-\gamma^{-1+\alpha_2}$ from each other. In the last case $\Delta J_\gamma^{(2)}(q_{i_1},q_{i_2})$ and $\Delta J_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4})$ are extremely small

$$0 < \gamma^{-d}\,\Delta J_\gamma^{(2)}(q_{i_1},q_{i_2}), \qquad \gamma^{-3d}\,\Delta J_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4}) < c\gamma^{3d/2-\alpha_2/2} \quad (5.11)$$

Denoting

$$w_\gamma^{(2)}(q_{i_1}, q_{i_2}) = e^{\beta\, \Delta J_\gamma^{(2)}(q_{i_1}, q_{i_2})} - 1 \tag{5.12}$$

and

$$w_\gamma^{(4)}(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}) = e^{\beta\, \Delta J_\gamma^{(4)}(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4})} - 1 \tag{5.13}$$

we have

$$e^{-\beta\, \Delta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar q^{(\Lambda^c)})} = \prod_{q_{i_1}, q_{i_2} \in q} (1 + w_\gamma^{(2)}(q_{i_1}, q_{i_2})) \times \prod_{q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4} \in q} (1 + w_\gamma^{(4)}(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4})) \tag{5.14}$$

First we obtain the polymer expansion for the log of the partition function

$$\Bigg(\prod_{x \in [\Lambda]^{(\ell_2)}} \ell_2^{-dn_x}\Bigg) \int \cdots \int \Bigg(\prod_{x \in [\Lambda]^{(\ell_2)}} \prod_{i=1}^{n_x} \mathbb{1}_{q_{x,i} \in C_x^{(\ell_2)}}\, dq_{x,i}\Bigg)\, e^{-\beta\, \Delta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar q^{(\Lambda^c)})} \tag{5.15}$$

containing no contours. Opening all brackets in (5.14) we rearrange the expression under the integral in (5.15) in the following way.

Let a *2-link*, $L^{(2)} = (q_1, q_2)$, be a couple of particles q_1, q_2 such that $W(L^{(2)}) = w_\gamma^{(2)}(q_1, q_2) \neq 0$. Similarly a *4-link*, $L^{(4)} = (q_1, q_2, q_3, q_4)$, is a quadruple of particles q_1, q_2, q_3, q_4 such that $W(L^{(4)}) = w_\gamma^{(4)}(q_1, q_2, q_3, q_4) \neq 0$. The quantities $W(L^{(2)})$ and $W(L^{(4)})$ are called the statistical weights of the 2-link and 4-link respectively. Two links are *connected* if they have a common particle. We stress that if two links have no common particles but have a common space point occupied by one particle from the first link and another particle from the second link then these links are not connected. A *pre-diagram*, θ, is a connected set of links. Denote by $q(\theta) = (q_i(\theta))$ the particles of $q^{(\Lambda)}$ which influence θ, i.e., the endpoints of links of θ. The *statistical weight*, $w(\theta)$, of the pre-diagram is the product of the statistical weights of the contributing links. Two pre-diagrams are *compatible* if they are not connected. Finally, a *compatible collection of pre-diagrams* consists of mutually compatible pre-diagrams.

The definitions above justify the representation for (5.15) of the form

$$\Bigg(\prod_{x \in [\Lambda]^{(\ell_2)}} \ell_2^{-dn_x}\Bigg) \int \cdots \int \Bigg(\prod_{x \in [\Lambda]^{(\ell_2)}} \prod_{i=1}^{n_x} \mathbb{1}_{q_{x,i} \in C_x^{(\ell_2)}}\, dq_{x,i}\Bigg) \sum_{\{\theta_j\} \notin \Lambda^c} \prod_j w(\theta_j) \tag{5.16}$$

where the sum goes over compatible collections, $\{\theta_j\}$, of pre-diagrams and $\{\theta_j\} \notin \Lambda^c$ means that every θ_j has at least one particle inside Λ.

We call two pre-diagrams *equivalent* if they can be transformed one into another by shifting some particles such that every shifted particle $q_{x,i}$ remains in its initial box $C_x^{(\ell_2)}$. The corresponding equivalence classes, Θ, are called *diagrams*. To have a geometrical interpretation of the diagram Θ we identify it with the pre-diagram $\theta \in \Theta$ having all particles at the centers of the corresponding cubes $C_x^{(\ell_2)}$. We say that two diagrams Θ_1 and Θ_2 are compatible if any $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$ are compatible. Setting

$$W(\Theta) = \ell_2^{-d\,|q(\Theta)|} \int \cdots \int w(\theta) \prod_{i=1}^{|q(\Theta)|} \mathbb{1}_{q_i(\theta) \in C^{(\ell_2)}_{q_i(\Theta)}}\, dq_i(\theta) \tag{5.17}$$

we rewrite the partition function (5.15) in the so called cluster form

$$\sum_{\{\Theta_j\} \notin \Lambda^c} \prod_j W(\Theta_j) \tag{5.18}$$

where the sum is extended to all compatible collections of diagrams. The transition from (5.16) to (5.18) relies on the fact that for compatible θ_{j_1} and θ_{j_2} the corresponding sets of particles $q(\theta_{j_1})$ and $q(\theta_{j_2})$ do not intersect each other.

Lemma 5.1. Let

$$a(\Theta) = |q(\Theta)| \tag{5.19}$$

Then

$$\sum_{\Theta' \nsim \Theta} |W(\Theta')|\, e^{a(\Theta')} \leqslant a(\Theta) \tag{5.20}$$

where $\Theta' \nsim \Theta$ denotes a diagram Θ' not compatible with a given diagram Θ.

Proof. We say that a diagram Θ is not compatible with a particle q and we denote it $\Theta \nsim q$ if $q \in q(\Theta)$. From our definition of compatibility of diagrams it is clear that (5.20) follows from

$$\sum_{\Theta \nsim q} |W(\Theta)|\, e^{a(\Theta)} \leqslant 1 \tag{5.21}$$

Note that for each box $C_x^{(\ell_2)}$, $x \in [\Lambda^{(\ell_2)}]$ the number of particles inside $C_x^{(\ell_2)}$ is fixed once the density ϱ of the configuration is fixed. Hence when ϱ

belongs to the interval (5.2) there are at most $c(\varrho_{\beta,\lambda,\sigma}+\zeta)\,\gamma^{-d}$ 2-links and at most $c(\varrho_{\beta,\lambda,\sigma}+\zeta)^3\,\gamma^{-3d}$ 4-links passing through any given particle. The statistical weights of links satisfy

$$|W(L^{(2)})| \leqslant c\gamma^{\alpha_2}\gamma^{d} \tag{5.22}$$

and

$$|W(L^{(4)})| \leqslant c\gamma^{\alpha_2}\gamma^{3d} \tag{5.23}$$

as follows from (5.9)–(5.11) and (2.3)–(2.4).

We provide a diagram with an abstract tree structure according to the following algorithm. The root of the tree is the particle q_1. Links which start from q_1 are called links of the first level. Links which start at endpoints of the links of the first level and are different from them are the links of the second level. Generally, links which starts at the endpoints of nth level links and are different from all links of levels $1, 2,..., n$ are called links of level $n+1$.

Denote by $n(\Theta)$ the maximal level of links in Θ. It is clear that for γ small enough

$$\sum_{\Theta \nsim q:\, n(\Theta)=1} |W(\Theta)|\, e^{a(\Theta)}$$

$$\leqslant \prod_{L^{(2)}\ni q} (1+e^2\,|W(L^{(2)})|) \prod_{L^{(4)}\ni q} (1+e^4\,|W(L^{(4)})|)-1 \leqslant 1 \tag{5.24}$$

By induction suppose that

$$\sum_{\Theta \nsim q:\, n(\Theta)\leqslant N} |W(\Theta)|\, e^{a(\Theta)} \leqslant 1 \tag{5.25}$$

and consider Θ with $n(\Theta)\leqslant N+1$. Take a link of the first level in such Θ. From every non root endpoint of this link "grows" a subdiagram Θ_1 with $n(\theta_1)\leqslant N$. Hence

$$\sum_{\Theta \nsim q:\, n(\Theta)\leqslant N+1} |W(\Theta)|\, e^{a(\Theta)} \leqslant \prod_{L^{(2)}\ni q} (1+|W(L^{(2)})|\,(e+1)^2)$$

$$\times \prod_{L^{(4)}\ni q} (1+|W(L^{(4)})|\,(e+1)^4)-1$$

$$\leqslant 1 \tag{5.26}$$

Here e correspond to the case when nothing is "growing" from a given endpoint of the first level link while 1 is the inductive estimate for the case when nonempty subdiagram Θ_1 with $n(\theta_1) \leqslant N$ is "growing" from this endpoint. ▮

From Lemma 5.1 applying Theorem 7.1 one obtains the polymer expansion

$$\sum_{\pi \notin \Lambda^c} W(\pi) \tag{5.27}$$

for the log of (5.18). The precise definition of the polymer $\pi = [\Theta_j^{\varepsilon_j}]$ and its statistical weight $W(\pi)$ can be found in Section 7. Geometrically a polymer is again a diagram-like object probably with some links entering it more than once. We underline that constructing pre-diagrams, diagrams and polymers one considers all particles entering these object as distinct, say having unique indices or labels. For that reason we call polymers from (5.27) *labeled polymers*. To clarify the dependence of (5.27) on ϱ_x we perform another factorization and define unlabeled polymers.

Suppose that from the total $n_x = \varrho_x \ell_2^d$ particles situated inside a box $C_x^{(\ell_2)}$ exactly $k(\pi)$ particles contribute to the labeled polymer π. Replacing these $k(\pi)$ particles with another $k(\pi)$ particles from the same box $C_x^{(\ell_2)}$ one obtains different labeled polymer with the same statistical weight. Two labeled polymers which can be transformed one into another after several replacements, possibly taking place in different boxes, are called *equivalent*. The corresponding equivalence classes are called *unlabeled polymers* and are denoted by τ. In other words, an unlabeled polymer is obtained from a labeled one by dropping the labels of the particles.

Denote by $X(\tau) \subseteq [\Lambda]^{(\ell_2)}$ the set of the centers of all boxes $C_x^{(\ell_2)}$ containing particles from τ. For $x \in X(\tau)$ let $k_x(\tau)$ be the number of particles from $C_x^{(\ell_2)}$ contributing to τ. Then the total number of different labeled polymers $\pi \in \tau$ is a polynomial function of $\varrho_x \ell_2^d$, $x \in X(\tau)$

$$0 < P(\tau) \leqslant \prod_{x \in X(\tau)} (\varrho_x \ell_2^d)^{k_x(\tau)} \tag{5.28}$$

Setting $W(\tau) = W(\pi)$, where π is an arbitrary labeled polymer from τ, we obtain the expression for the log of the partition function (5.15)

$$\sum_{\tau \in \Lambda} W(\tau)\, P(\tau) \tag{5.29}$$

written in terms of ϱ_x. Despite its involved structure we need only a few simple estimates on this sum.

It follows from Corollary 7.2 in Section 7 that the sum of statistical weights of all labeled (and hence unlabeled) polymers passing through a given particle and containing not less than k links does not exceed $\gamma^{-k\alpha_2/2}$. Hence the sum of the statistical weights of all labeled (or unlabeled) polymers containing two given particles q_1 and q_2 at a distance $r > \gamma^{-1}$ from each other satisfies the bound

$$\sum_{\pi \ni q_1, q_2} |W(\pi)| = \sum_{\tau \ni q_1, q_2} |W(\tau)| \, P(\tau) \leqslant \gamma^{-[\gamma r] \, \alpha_2/2} \tag{5.30}$$

Similarly for any given particle q and sufficiently large absolute constant c

$$\sum_{\pi \ni q:\, L(\tau) \geqslant c} |W(\pi)| = \sum_{\tau \ni q:\, L(\tau) \geqslant c} |W(\tau)| \, P(\tau) \leqslant \gamma^{4d} \tag{5.31}$$

where $L(\tau)$ denotes the number of links contributing to π or τ.

Polymer sum (5.29) can be viewed as a Hamiltonian which we separate into two parts. The first one is

$$\varDelta H^{(1)}_{\gamma,\lambda}(\varrho \mid \bar{q}^{(\varLambda^c)}) = \sum_{\tau \notin \varLambda^c:\, L(\tau) < c} W(\tau) \, P(\tau) \tag{5.32}$$

and the second is

$$\varDelta H^{(2)}_{\gamma,\lambda}(\varrho \mid \bar{q}^{(\varLambda^c)}) = \sum_{\tau \notin \varLambda^c:\, L(\tau) \geqslant c} W(\tau) \, P(\tau) \tag{5.33}$$

It is not hard to see that $\varDelta H^{(1)}_{\gamma,\lambda}(\varrho \mid \bar{q}^{(\varLambda^c)})$ is simply a finite radius Hamiltonian of the polynomial type

$$\varDelta H^{(1)}_{\gamma,\lambda}(\varrho \mid \bar{q}^{(\varLambda^c)}) = \ell_2^d \sum_{D \notin \varLambda^c} W(D) \prod_{q \in D} \varrho_q \tag{5.34}$$

Here the sum is taken over connected sets of links (with no restriction for a given link to enter this set more than once) containing less than $c_{(5.31)}$ links and the product is over all endpoints of the links. The notation ϱ_q instead of ϱ_x is not ambiguous as all endpoints of the links are assumed to be at the centers of the corresponding boxes $C_x^{(\ell_2)}$. The statistical weights $W(D)$ are obtained by resummation from (5.32).

From (5.15) we pass to partition function (5.7) containing contours. For the log of the ratio between (5.7) and (5.15) we also obtain a polymer expansion exploiting the theory of contour models with interactions [DS], [BKL]. This expansion has the form

$$\sum_{\xi \notin \Lambda^c} W(\xi) \tag{5.35}$$

where ξ are other polymers constructed from labeled polymers π and contours Γ in the same way as polymers π are constructed from diagrams Θ. The statistical weights $W(\xi)$ are local functions of ϱ_x and we interpret the whole sum as the Hamiltonian

$$\Delta H^{(3)}_{\gamma,\lambda}(\varrho \mid \bar{q}^{(\Lambda^c)}) = \sum_{\xi \notin \Lambda^c} W(\xi) \tag{5.36}$$

The only property of this Hamiltonian used later is the estimate

$$\sum_{\xi \ni q} |W(\xi)| \leqslant \gamma^{4d} \tag{5.37}$$

Since all details can be found in [DS] and [BKL] we give only a sketch of the proofs pointing out few technically important points.

Denote by $\nu(\,\cdot \mid q^{(\Lambda^c)})$ the Gibbs distribution of $\sum_{x \in [\Lambda]^{(\ell_2)}} \ell_2^d \varrho_x$ particles given by the Hamiltonian $\Delta H_{\gamma,\lambda}(q^{(\Lambda)} \mid \bar{q}^{(\Lambda^c)})$ with the corresponding partition function (5.15). For every contour Γ consider region $R(\Gamma) = \partial^{(\ell_3/3)} \operatorname{Supp}(\Gamma)$ with empty boundary condition $\varnothing^{(R(\Gamma)^c)}$ and define a modified statistical weight

$$\tilde{W}^T(\eta^\Gamma \mid q^{(\delta^=(\Gamma))}) = \nu(q^{(\delta^=(\Gamma))} \mid \varnothing^{(R(\Gamma)^c)})\, W^T(\eta^\Gamma \mid q^{(\delta^=(\Gamma))}) \tag{5.38}$$

Then the ratio of partition functions (5.7) and (5.15) can be rewritten as

$$\sum_{\{\Gamma_i\}^\sigma \in \Lambda} \int \prod_i dq^{(\delta^=(\Gamma_i))} \exp\Bigg(\sum_{\substack{\pi \notin \Lambda^c:\ \exists i,\ \pi \cap \delta^=(\Gamma_i) \neq \varnothing \\ \pi \cap R(\Gamma_i)^c \neq \varnothing}} \big(W(\pi \mid q^{(\cup_i \delta^=(\Gamma_i))}) - W(\pi)\big)\Bigg) \times \prod_i \tilde{W}^T(\eta^{\Gamma_i} \mid q^{\delta^=(\Gamma_i))}) \tag{5.39}$$

Here the statistical weight $W(\pi \mid q^{(\cup_i \delta^=(\Gamma_i))})$ is defined respecting an additional boundary condition $q^{(\cup_i \delta^=(\Gamma_i))}$ which is imposed in $\bigcup_i \delta^=(\Gamma_i)$. We note that only $\bar{q}^{(\Lambda^c)}$ affects $W(\pi)$ if $\pi \cap \Lambda^c \neq \varnothing$.

The polymer sum in (5.39) describes the interaction between contours Γ_i. It is important that every polymer contributing to (5.39) is sufficiently long and contains at least $[\gamma \ell_3/3]$ links. In view of (5.30) such a polymer has a very small statistical weight. Therefore, in complete similarity with (5.12)–(5.14) expanding

$$e^{W(\pi \mid q^{(\cup_i \delta^{=}(\Gamma_i))}) - W(\pi)} = 1 + (e^{W(\pi \mid q^{(\cup_i \delta^{=}(\Gamma_i))}) - W(\pi)} - 1) \tag{5.40}$$

and integrating over $\prod_i dq^{(\delta^{=}(\Gamma_i))}$ one can derive for (5.39) a representation analogous to (5.18). In this representation newly defined diagrams are constructed from contours Γ connected via polymers π. Taking logarithms and applying Theorem 7.1 one obtains (5.35) with correspondingly defined polymers ξ. The key fact ensuring the condition (7.3) of Theorem 7.1 is that for γ small enough the sum of absolute values of statistical weights of all polymers π containing a given particle and being longer than $\ell_3/3$ is much smaller than the quantity $c\ell_2^d \ell_3^{-d} \zeta$ entering the Peierls estimate (3.44).

The route to (5.35) looks rather involved and tedious but it is a standard one in the cluster expansion technique. On the other hand we need only minor knowledge about ξ, namely (5.37). This estimate is obtained by the methods of [DS] and [BKL] along the following way.

The sum over all polymers ξ containing a given particle q_1 is equal to

$$\sum_{\Gamma:\, \delta^{=}(\Gamma) \ni q_1} \sum_{\xi:\, \xi \ni \Gamma} |W(\xi)| + \sum_{\pi:\, \pi \ni q_1,\, L(\pi) \geqslant c_{(5.31)}} \sum_{\xi:\, \xi \ni \pi} |W(\xi)| \tag{5.41}$$

By (7.6) the first internal sum does not exceed $cW^T(\eta^\Gamma \mid q^{(\delta^{=}(\Gamma))})$ and the second internal sum does not exceed $c\,|W(\pi)|$. In turn

$$\sum_{\Gamma:\, \delta^{=}(\Gamma) \ni q_1} cW^T(\eta^\Gamma \mid q^{(\delta^{=}(\Gamma))}) + \sum_{\pi:\, \pi \ni q_1,\, L(\pi) \geqslant c_{(5.31)}} c\,|W(\pi)| \leqslant \gamma^{4d} \tag{5.42}$$

because of (3.38) and (5.31).

The final result of this subsection can be stated now as

Lemma 5.2. The expression (5.3) is equal to

$$\sum_{x \in [\Lambda]^{(\ell_2)}} \log\left(\frac{\ell_2^{dn_x}}{n_x!}\right) - \beta H_{\gamma,\lambda}(\varrho \mid \bar{q}^{(\Lambda^c)}) + \sum_{i=1}^{3} \Delta H_{\gamma,\lambda}^{(i)}(\varrho \mid \bar{q}^{(\Lambda^c)}) \tag{5.43}$$

Expression (5.43) without the last sum is nothing but $\beta\tilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho^{(\ell_2)} \mid \bar{q}^{(\Lambda^c)})$ defined in (4.21) with its main part $\beta\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_2)} \mid \bar{q}^{(\Lambda^c)})$ given by (4.26)–(4.27).

Dobrushin Uniqueness (Basic Calculation)

Lemma 5.3. The effective Hamiltonian (5.43) satisfies the Dobrushin uniqueness condition.

Proof. For the sake of simplicity we first check that the Dobrushin uniqueness condition is true for the Hamiltonian $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_2)})$. Then in the next subsection we show that only minor modifications are necessary to treat (5.43).

Take any site $x \in \mathbb{Z}^d_{\ell_2}$ and consider two boundary conditions $\bar{\varrho}^{(0)}$ and $\bar{\varrho}^{(1)}$ on $\mathbb{Z}^d_{\ell_2}\backslash x$ such that they both belong to the interval (5.2) and differ only at a site $y \in \mathbb{Z}^d_{\ell_2}$. For definiteness we assume that $\bar{\varrho}^{(1)}_y > \bar{\varrho}^{(0)}_y$. Denote by $v^{(0)}(d\varrho_x)$ and $v^{(1)}(d\varrho_x)$ conditional Gibbs distributions defined by the Hamiltonians $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(0)})$ and $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(1)})$ respectively. The *Vasserstein distance* between $v^{(0)}(d\varrho_x)$ and $v^{(1)}(d\varrho_x)$ is

$$R(v^{(0)}, v^{(1)}) = \int_{-\infty}^{\infty} dz \left| \int_{-\infty}^{z} (v^{(0)}(d\varrho_x) - v^{(1)}(d\varrho_x)) \right|$$

$$= \int_{0}^{\infty} dz \left| \int_{0}^{z} (v^{(0)}(d\varrho_x) - v^{(1)}(d\varrho_x)) \right| \tag{5.44}$$

where the last equality utilizes the positivity of ϱ_x.

The *Dobrushin uniqueness condition* is satisfied if one is able to find a function r_{xy} such that

$$R(v^{(0)}, v^{(1)}) \leqslant r_{xy} \, |\bar{\varrho}^{(0)}_y - \bar{\varrho}^{(1)}_y| \tag{5.45}$$

and

$$\sum_y r_{xy} < 1 \tag{5.46}$$

To check this condition we follow the strategy of [COPP] and define $v_t(d\varrho_x)$, $t \in [0, 1]$ as a Gibbs measure corresponding to the interpolated Hamiltonian

$$\mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid t) = \mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(0)}) + t(\mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(1)}) - \mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(0)})) \tag{5.47}$$

This measure has the density

$$p(\varrho_x \mid t) = \frac{\exp(-\beta \mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid t))}{\int_0^\infty \exp(-\beta \mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid t)) \, d\varrho_x} \tag{5.48}$$

which is differentiable in t and therefore from (5.44) we have

$$R(v^{(0)}, v^{(1)}) \leqslant \int_0^1 dt \int_0^\infty dz \left| \int_0^z \frac{\partial}{\partial t} p(\varrho_x \mid t) \, d\varrho_x \right| \tag{5.49}$$

Denoting

$$\Delta\mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(0)}, \bar{\varrho}^{(1)}) = (\mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(1)}) - \mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(0)})) \tag{5.50}$$

one has

$$\frac{\partial}{\partial t} p(\varrho_x \mid t) = \beta p(\varrho_x \mid t)(\Delta\mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(0)}, \bar{\varrho}^{(1)}) - \langle \Delta\mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(0)}, \bar{\varrho}^{(1)})\rangle_t) \tag{5.51}$$

where $\langle \cdot \rangle_t$ denotes the expectation with respect to $p(\varrho_x \mid t)\, d\varrho_x$. Observe that

$$\Delta\mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(0)}, \bar{\varrho}^{(1)}) = \ell_2^d \left(I_\gamma^{(2)}(x, y) - \frac{1}{2!} I_\gamma^{(2)}(x, y \mid \bar{\varrho}) \right) (\bar{\varrho}_y^{(1)} - \bar{\varrho}_y^{(0)})\, \varrho_x \tag{5.52}$$

where $I_\gamma^{(2)}(x, y \mid \bar{\varrho})$ is defined by (4.41) (with $\bar{\varrho}(\Lambda \setminus x \setminus y) = \bar{\varrho}^{(0)}(\Lambda \setminus x \setminus y) = \bar{\varrho}^{(1)}(\Lambda \setminus x \setminus y)$ instead of $\hat{\varrho}$) and satisfies (4.42). by direct calculation

$$\int_0^\infty dz \left| \int_0^z (\varrho_x - \langle \varrho_x \rangle_t)\, p(\varrho_x \mid t)\, d\varrho_x \right| = \langle \varrho_x^2 \rangle_t - \langle \varrho_x \rangle_t^2 \tag{5.53}$$

For sufficiently small γ and therefore sufficiently large ℓ_2^d the last expression can be estimated by the Laplace method and it is equal to

$$\ell_2^{-d} \varrho_*(t) + O(\ell_2^{-2d}) \tag{5.54}$$

where $\ell_2^d \varrho_*(t)^{-1}$ is the value of $\beta(\partial^2/\partial\varrho_x^2)\, \mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid t)$ at the point $\varrho_*(t)$ of the minimum of $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho_x \mid t)$. As we know from Lemma 4.8 such a minimum exists, is unique and lies strictly inside the interval (5.2). Hence $O(\ell_2^{-2d})$ in (5.54) is uniform in t (see [F]).

Combining (5.49)–(5.54) we obtain that

$$R(v^{(0)}, v^{(1)}) \leqslant \int_0^1 dt(\beta\varrho_* + O(\ell_2^{-d})) \left(I_\gamma^{(2)}(x, y) - \frac{1}{2!} I_\gamma^{(2)}(x, y \mid \bar{\varrho}) \right) (\bar{\varrho}_y^{(1)} - \bar{\varrho}_y^{(0)}) \tag{5.55}$$

Setting

$$r_{xy} = \max_{\bar{\varrho}} \left| I_\gamma^{(2)}(x, y) - \frac{1}{2!} I_\gamma^{(2)}(x, y \mid \bar{\varrho}) \right| \int_0^1 dt(\beta\varrho_*(t) + O(\ell_2^{-d})) \tag{5.56}$$

one concludes that (5.46) is true for sufficiently small γ because of (3.17), (4.42) and the fact that $\sum_y I_\gamma^{(2)}(x, y) = 1$.

Dobrushin Uniqueness (General Case)

We turn now to the complete effective Hamiltonian (5.43) and show that only nonessential modifications of the calculation of the previous subsection are required to cover this case.

The first correction is due to the difference between $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho^{(\ell_2)} \mid \bar{q}^{(\Lambda^c)})$ and $\widetilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho^{(\ell_2)} \mid \bar{q}^{(\Lambda^c)})$. This difference is discussed in detail in Lemma 4.6. The corresponding modifications of our previous arguments are the following.

The difference between volumes of $B_\gamma(\cdot)$ and $[B_\gamma(\cdot)]^{(\ell_2)}$ produces a factor $(1+O(\gamma^\alpha))$, $\alpha>0$, in front of $I_\gamma^{(2)}(\cdot,\cdot)$ and $I_\gamma^{(4)}(\cdot,\cdot,\cdot,\cdot)$ which clearly is not essential.

The self interactions like $I_\gamma^{(2)}(\varrho_x,\varrho_x)$ or $I_\gamma^{(4)}(\varrho_x,\varrho_x,\varrho_y,\varrho_z)$ produce nonlinear terms in $\widetilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho_x \mid t)$ which is defined similarly to (5.47). Their contribution to $\widetilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho_x \mid t)$ has the form

$$c_2\varrho_x^2+c_3\varrho_x^3+c_4\varrho_x^4+t(c_5\varrho_x^2+c_6\varrho_x^3)(\bar{\varrho}_y^{(1)}-\bar{\varrho}_y^{(0)}) \tag{5.57}$$

where c_i does not depend on t, ϱ_x and $\varrho_y^{(i)}$. Only the last, t-dependent, part from (5.57) contributes to $\Delta\widetilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho_x \mid \bar{\varrho}^{(0)},\bar{\varrho}^{(1)})$ which is defined similarly to (5.50). The corresponding effect on $\int_0^\infty dz\, \left|\int_0^z (\partial/\partial t)\, p(\varrho_x \mid t)\, d\varrho_x\right|$ result in terms of the form

$$\begin{aligned}\beta c_5(\bar{\varrho}_y^{(1)}-\bar{\varrho}_y^{(0)}) &\int_0^\infty dz\left|\int_0^z (\varrho_x^2-\langle\varrho_x^2\rangle_t)\, p(\varrho_x \mid t)\, d\varrho_x\right|\\ &=\beta c_5(\bar{\varrho}_y^{(1)}-\bar{\varrho}_y^{(0)})(\langle\varrho_x^3\rangle_t-\langle\varrho_x^2\rangle_t\langle\varrho_x\rangle_t)\end{aligned} \tag{5.58}$$

and

$$\begin{aligned}\beta c_6(\bar{\varrho}_y^{(1)}-\bar{\varrho}_y^{(0)}) &\int_0^\infty dz\left|\int_0^z (\varrho_x^3-\langle\varrho_x^3\rangle_t)\, p(\varrho_x \mid t)\, d\varrho_x\right|\\ &=\beta c_6(\bar{\varrho}_y^{(1)}-\bar{\varrho}_y^{(0)})(\langle\varrho_x^4\rangle_t-\langle\varrho_x^3\rangle_t\langle\varrho_x\rangle_t)\end{aligned} \tag{5.59}$$

in addition to $\ell_2^d\beta(I_\gamma^{(2)}(x,y)-(1/2!)\, I_\gamma^{(2)}(x,y \mid \bar{\varrho}))(\bar{\varrho}_y^{(1)}-\bar{\varrho}_y^{(0)})(\langle\varrho_x^2\rangle_t-\langle\varrho_x\rangle_t^2)$. Clearly

$$|c_i|\leqslant \ell_2^d\gamma^\alpha \tag{5.60}$$

with some $\alpha>0$.

Applying the Laplace method one has

$$\langle \varrho_x^{k+1} \rangle_t - \langle \varrho_x^k \rangle_t \langle \varrho_x \rangle_t = \frac{k \varrho_*(t)^k}{\beta(\partial^2/\partial \varrho_x^2)\, \tilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho_x \mid t)|_{\varrho_*(t)}} + O(\ell_2^{-2d}) \tag{5.61}$$

where again $\varrho_*(t)$ is the minimum of $\tilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho_x \mid t)$. Since

$$\beta \frac{\partial^2}{\partial \varrho_x^2} \tilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho_x \mid t) = \ell_2^d \varrho_x^{-1} + \beta 2c_2 + \beta 6 c_3 \varrho_x$$
$$+ \beta 12 c_4 \varrho_x^2 + \beta t (2c_5 + 6c_6 \varrho_x)(\bar{\varrho}_y^{(1)} - \bar{\varrho}_y^{(0)}) \tag{5.62}$$

and because of (5.60) all corrections above do not destroy the arguments of the previous subsection.

As the next step we incorporate in our calculation $\Delta H^{(1)}_{\gamma,\lambda}(\varrho \mid \bar{q}^{(\Lambda^c)})$. According to (5.34) this is the same type of a polynomial correction which was just discussed. Thus the same arguments work. The necessary smallness, like in (5.60), of coefficient $W(D)$ is a consequence of the smallness of the statistical weights of links (5.22)–(5.23) and the definition of $W(D)$ (see also (5.30)).

Finally, to treat $\sum_{i=2,3} \Delta H^{(i)}_{\gamma,\lambda}(\varrho \mid \bar{q}^{(\Lambda^c)})$ we simply observe that for any boundary condition $\bar{\varrho}$ given on $\mathbb{Z}^d_{\ell_2} \backslash x$

$$R(v(\varrho_x \mid \bar{\varrho}), v_{2,3}(\varrho_x \mid \bar{\varrho})) < \gamma^d \tag{5.63}$$

where $v(\varrho_x \mid \bar{\varrho})$ is the conditional Gibbs distribution given by the whole Hamiltonian (5.43) while $v_{2,3}(\varrho_x \mid \bar{\varrho})$ is a similar distribution given by the Hamiltonian (5.43) without $\sum_{i=2,3} \Delta H^{(i)}_{\gamma,\lambda}(\varrho \mid \bar{q}^{(\Lambda^c)})$. This estimate is obvious in view of definition (5.44) and bounds (5.31) and (5.37). One can comment that the contribution of contours or long polymers to the free energy of the auxiliary model is too small to affect anything at all. ∎

6. PROPERTIES OF AUXILIARY MODEL

In this section we use the Dobrushin uniqueness result established for the auxiliary model in the previous section to prove Statements 3.3 and 3.5. We begin with a construction which is necessary for the proof of both statements. Namely, given a phase σ and a particle configuration $\bar{q}$ from the ground state ensemble of the phase σ we derive an appropriate representation for the logarithm of partition function (3.37). Lemma 5.2 gives

$$Z^{A}_{\gamma,\beta,\lambda}(\Lambda \mid \bar{q}^{(\Lambda^c)}) = \sum_{\varrho^{(\ell_2)}(\Lambda)} \exp\left(-\beta\widetilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho^{(\ell_2)}(\Lambda) \mid \bar{q}^{(\Lambda^c)}) - \sum_{i=1}^{3} \Delta H^{(i)}_{\gamma,\lambda}(\varrho^{(\ell_2)}(\Lambda) \mid \bar{q}^{(\Lambda^c)})\right) \tag{6.1}$$

Below we consider density configurations, lattice volumes, etc. related to the scale ℓ_2 and in most cases we omit the superscript (ℓ_2) from the notations. On few occasions when we need other scales we specify them explicitly.

In Section 4 we studied in detail the minimizers of $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)})$ for bounded Λ and $\Lambda = \mathbb{Z}^d_{\ell_2}$. Now we need to study minimizers for

$$\widetilde{H}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}) = \beta\widetilde{\mathscr{F}}_{\gamma,\beta,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}) + \Delta H^{(1)}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}) \tag{6.2}$$

where both $\bar{q}^{(\Lambda^c)}$ and $\varrho(\Lambda)$ belong to the ground state ensemble of the phase σ. This is a finite range translation-invariant Hamiltonian similar to $\mathscr{F}_{\gamma,\beta,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)})$ but not allowing a simple representation of the type (4.35). For a bounded Λ the existence of at least one minimizer of $\widetilde{H}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)})$ follows from the compactness of its domain, $(\varrho_{\beta,\sigma} - \zeta, \varrho_{\beta,\sigma} + \zeta)^{\Lambda}$. This domain is convex and the minimizer is unique because the function $\widetilde{H}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)})$ (of the finite number of variables ϱ_x, $x \in \Lambda$) is also convex. This is a consequence of estimate (3.17) from which it is not hard to see that for any $\bar{q}^{(\Lambda^c)}$ and $\varrho(\Lambda)$ the Hessian matrix of $\widetilde{H}_{\gamma,\lambda}(\,\cdot \mid \bar{q}^{(\Lambda^c)})$ calculated at $\varrho(\Lambda)$ is positive definite with the mass bounded from 0 independently of $\bar{q}^{(\Lambda^c)}$ and $\varrho(\Lambda)$. Depending on the context we denote the unique minimizer of $\widetilde{H}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)})$ by $\hat{\varrho}$, $\hat{\varrho}(\Lambda)$ or $\hat{\varrho}^{\bar{q}^{(\Lambda^c)}}$.

To clarify the structure and convexity of $\widetilde{H}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)})$ we introduce

$$\Delta\widetilde{H}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}) = \widetilde{H}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}) - \widetilde{H}_{\gamma,\lambda}(\hat{\varrho}(\Lambda) \mid \bar{q}^{(\Lambda^c)}) \tag{6.3}$$

It is not hard to check that

$$\Delta\widetilde{H}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}) = \ell_2^d \sum_{x \in [\Lambda]^{(\ell_2)}} \sum_{n=2}^{\infty} \frac{1}{n(n-1)\,\hat{\varrho}_x^{n-1}} \Delta\varrho_x^n + \ell_2^d \sum_{D \not\subset [\Lambda^c]^{(\ell_2)}} W(D) \sum_{X_1(D), X_2(D)} \prod_{x \in X_1(D)} \hat{\varrho}_x(\Lambda) \prod_{x \in X_2(D)} \Delta\varrho_x \tag{6.4}$$

Here $\Delta\varrho_x = \varrho_x - \hat{\varrho}_x(\Lambda)$ and D is a connected family of links with endpoints forming a set $X(D) \subset \mathbb{Z}^d_{\ell_2}$. The number of links in D is less than $c_{(5.31)}$. The subsets $X_1(D)$ and $X_2(D)$ form a partition of $X(D)$ with at least two elements in $X_2(D)$. The second internal sum is taken over all partitions of that type. For $W(D)$ one has the estimate

$$\sum_{\substack{D:\, X(D) \ni x \\ D \neq L^{(2)},\, L^{(4)}}} |W(D)|\, \varrho_{\max}^{|X(D)|} \leqslant \gamma^{\alpha_2/3} \tag{6.5}$$

as follows from (5.30). Also

$$|\Delta\varrho_x| \leqslant 2\zeta \tag{6.6}$$

The quadratic part of (6.4) is

$$\frac{1}{2!}\,\ell_2^d \sum_{x \in [\Lambda]^{(\ell_2)}} \frac{1}{\hat{\varrho}}\, \Delta\varrho_x^2 - \frac{1}{2!}\,\ell_2^d \sum_{x_1,\, x_2:\, x_1 \cup x_2 \notin [\Lambda^c]^{(\ell_2)}} \tilde{I}^{(2)}(x_1, x_2 \mid \hat{\varrho}(\Lambda))\, \Delta\varrho_{x_1}\, \Delta\varrho_{x_2} \tag{6.7}$$

with

$$\frac{1}{\hat{\varrho}_{x_1}(\Lambda)} - \sum_{x_2} \tilde{I}^{(2)}(x_1, x_2 \mid \hat{\varrho}(\Lambda)) \geqslant m(\beta) > 0 \tag{6.8}$$

for any $x_1 \in [\Lambda]^{(\ell_2)}$. Here $\tilde{I}^{(2)}(x_1, x_2 \mid \hat{\varrho}(\Lambda))$ is an analogue of (4.41) and satisfies the same estimate (4.42) for γ small enough. (By construction $|I^{(2)}(x_1, x_2 \mid \hat{\varrho}(\Lambda)) - \tilde{I}^{(2)}(x_1, x_2 \mid \hat{\varrho}(\Lambda))| \leqslant c\gamma^{\alpha_2} I^{(2)}(x_1, x_2))$. Moreover, in view of (6.5) and assuming that ζ is chosen to be small enough with respect to $m(\beta)$

$$\Delta\tilde{H}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}) = \frac{1}{2!}\,\ell_2^d \sum_{x \in [\Lambda]^{(\ell_2)}} \frac{m(\beta)}{2}\, \Delta\varrho_x^2 + U(\hat{\varrho}(\Lambda), \Delta\varrho(\Lambda)) \tag{6.9}$$

with positive convex $U(\hat{\varrho}(\Lambda), \Delta\varrho(\Lambda))$ having minimum at $\Delta\varrho(\Lambda) \equiv 0$.

Along with the minimizers of $\tilde{H}_{\gamma,\lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)})$ for bounded regions Λ we also need a global minimizer, i.e., an analogue of $\varrho_{\beta,\sigma}$. To find such a minimizer and to show its uniqueness (for given σ) we consider a sufficiently large (with respect to the range of $\tilde{H}_{\gamma,\lambda}(\cdot)$) periodic box Λ and the corresponding Hamiltonian $\tilde{H}_{\gamma,\lambda}(\varrho(\Lambda)$. All convexity considerations remain true for $\tilde{H}_{\gamma,\lambda}(\varrho(\Lambda))$ such that $\tilde{H}_{\gamma,\lambda}(\varrho(\Lambda))$ has a unique minimizer $\hat{\varrho}(\Lambda)$.

Because of the translation invariance of $\tilde{H}_{\gamma,\lambda}(\varrho(\Lambda))$ this minimizer is a constant configuration $\hat{\varrho}_x(\Lambda) \equiv \hat{\varrho}$, $x \in \Lambda$. (Otherwise one is able to construct another minimizers by space translations.) Since $\tilde{H}_{\gamma,\lambda}(\varrho(\Lambda))$ is of finite range we observe that $\hat{\varrho}$ is independent on Λ for all sufficiently large periodic Λ and hence $\hat{\varrho}$ is the global minimizer we are looking for. Note that the specific energy $\tilde{h}_{\gamma,\beta,\lambda}(s) = \lim_{\Lambda \to \mathbb{Z}^d_{\ell_2}} |\Lambda|^{-1} \tilde{H}_{\gamma,\lambda}(\varrho(\Lambda))$ of any constant density configuration $\varrho(\Lambda) \equiv s$ is just a finite sum and $\hat{\varrho}$ is nothing but the value of s at which $\tilde{h}_{\gamma,\beta,\lambda}(s)$ achieves its minimum. Thus given σ such a minimum is unique for $s \in (\varrho_{\beta,\sigma} - \zeta, \varrho_{\beta,\sigma} + \zeta)$ and we have different values $\hat{\varrho}_{\beta,\lambda,-}$ and $\hat{\varrho}_{\beta,\lambda,+}$ corresponding to $\sigma = -1$ and $\sigma = +1$.

Denote by $\hat{\lambda}(\beta,\gamma)$ the value of λ at which $\tilde{h}_{\gamma,\beta,\lambda}(\hat{\varrho}_{\beta,\lambda,-}) = \tilde{h}_{\gamma,\beta,\lambda}(\hat{\varrho}_{\beta,\lambda,+})$ and set $\hat{\varrho}_{\beta,\sigma} = \hat{\varrho}_{\beta,\hat{\lambda}(\beta),\sigma}$. The important consequences of the representation of $\hat{\varrho}_{\beta,\lambda,\sigma}$ via $\tilde{h}_{\gamma,\beta,\lambda}(s)$ are the existence of $\hat{\lambda}(\beta,\gamma)$ and the estimates

$$|\varrho_{\beta,\sigma} - \hat{\varrho}_{\beta,\lambda,\sigma}| \leqslant c\gamma^{\alpha_1/2} \tag{6.10}$$

and

$$|\lambda(\beta) - \hat{\lambda}(\beta,\gamma)| \leqslant c\gamma^{\alpha_1} \tag{6.11}$$

Indeed, let $h_{\gamma,\beta,\lambda}(s)$ be the specific energy of $\varrho \equiv s$ calculated via the Hamiltonian $\tilde{H}_{\gamma,\lambda}(\varrho(\Lambda)) + \Delta H^{(2)}_{\gamma,\lambda}(\varrho(\Lambda))$ with periodic Λ. Then

$$|\tilde{h}_{\gamma,\beta,\lambda}(s) - h_{\gamma,\beta,\lambda}(s)| \leqslant c\gamma^{4d} \tag{6.12}$$

as follows from (5.31).

Another way to calculate $h_{\gamma,\beta,\lambda}(s)$ is the approach of Section 4. Here the starting point is the partition function (5.15) with fixed density configuration $\varrho^{(\ell_2)} \equiv s$. Then one approximate this partition function by using the density configurations defined with respect to the scale $\ell_1 \ll \ell_2$. (Note that the density configuration which is constant in the scale ℓ_2 is not necessarily a constant one in the finer scale ℓ_1.) As the first step of the approximation one shifts all particles to the centers of the corresponding boxes $C^{(\ell_1)}$ for the price of the error (in the value of $\tilde{H}_{\gamma,\lambda}(\varrho^{(\ell_1)}(\Lambda)) + \Delta H^{(2)}_{\gamma,\lambda}(\varrho^{(\ell_1)}(\Lambda)))$ which in absolute value does not exceed $c\gamma^{\alpha_1}|\Lambda|$ (see Lemma 4.5). Lemmas 4.6 and 4.7 then show that up to the same error the Hamiltonian $\tilde{H}_{\gamma,\lambda}(\varrho^{(\ell_1)}(\Lambda)) + \Delta H^{(2)}_{\gamma,\lambda}(\varrho^{(\ell_1)}(\Lambda))$ can be approximated by (4.26).

Now the convexity considerations (see (3.6)) imply that among all density configurations $\varrho^{(\ell_1)}(\Lambda)$ such that $\varrho^{(\ell_2)}(\varrho^{(\ell_1)}(\Lambda)) \equiv s$ the configuration

$\varrho^{(\ell_1)}(\Lambda) \equiv s$ has the minimal value of the mean field functional (4.26). Hence (see (4.19) and above) counting only the contribution of $\varrho^{(\ell_1)}(\Lambda) \equiv s$ produces the error which again in absolute value does not exceed $c\gamma^{\alpha_1} |\Lambda|$ (in fact it is much smaller). The specific energy of the constant density configuration $\varrho^{(\ell_1)} \equiv s$ calculated via (4.26) is nothing but the mean field specific energy $F_{\beta, \lambda}(s)$ and therefore

$$|F_{\beta, \lambda}(s) - h_{\gamma, \beta, \lambda}(s)| \leqslant c\gamma^{\alpha_1} \tag{6.13}$$

Since $\tilde{h}_{\gamma, \beta, \lambda}(s) = F_{\beta, \lambda}(s) + \sum_{k=2}^{K} a_k s^k$ with $|a_k| < c\gamma^{\alpha_2}$, i.e., $\tilde{h}_{\gamma, \beta, \lambda}(s)$ is a small polynomial perturbation of $F_{\beta, \lambda}(s)$, and

$$|F_{\beta, \lambda}(s) - \tilde{h}_{\gamma, \beta, \lambda}(s)| \leqslant c\gamma^{\alpha_1} \tag{6.14}$$

as follows from (6.12) and (6.13), one immediately obtains (6.10) and (6.11). (The existence of $\hat{\lambda}(\beta, \gamma)$ follows from an elementary linear analysis and continuity of $\tilde{h}_{\gamma, \beta, \lambda}(s)$.)

Finally observe that a straightforward modification of the arguments of Lemma 4.8 shows that $|\hat{\varrho}_{\beta, \lambda, \sigma} - \hat{\varrho}^{\bar{q}^{(\Lambda^c)}}|$ satisfy (4.36).

With $\hat{\varrho}_{\beta, \lambda, \sigma}$ and $\hat{\varrho}^{\bar{q}^{(\Lambda^c)}}$ being properly defined we construct now a convenient representation for the log of the partition function (6.1) using the following interpolation trick. For $t \in [0, 1]$ and $s \in [0, 1]$ introduce an interpolated Hamiltonian

$$\begin{aligned} \Delta H_{\gamma, \lambda}(\Delta\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}; t, s) = {} & \ell_2^d \sum_{x \in [\Lambda]^{(\ell_2)}} \sum_{n=2}^{\infty} \frac{1}{n(n-1)\, \hat{\varrho}_x^{n-1}} \Delta\varrho_x^n \\ & + t\ell_2^d \sum_{D \not\subset [\Lambda^c]^{(\ell_2)}} W(D) \sum_{\substack{X_1(D), \\ X_2(D)}} \prod_{x \in X_1(D)} \hat{\varrho}_x(\Lambda) \prod_{x \in X_2(D)} \Delta\varrho_x \\ & + s \sum_{i=2, 3} \Delta H^{(i)}_{\gamma, \lambda}(\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}) \end{aligned} \tag{6.15}$$

The Hamiltonian $\Delta H_{\gamma, \lambda}(\Delta\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}; t, s)$ satisfies the Dobrushin uniqueness condition similarly to $\Delta H_{\gamma, \lambda}(\Delta\varrho(\Lambda) \mid \bar{q}^{(\Lambda^c)}; 1, 1)$. We denote by $\langle \cdot \rangle_{t, s, \Lambda, \bar{q}^{(\Lambda^c)}}$ the expectation with respect to the corresponding Gibbs measure in the finite domain Λ with the boundary condition $\bar{q}^{(\Lambda^c)}$ and we denote by $\langle \cdot \rangle_{t, s}$ the expectation with respect to the corresponding unique limit Gibbs distribution.

From (6.1) and (6.15) by direct calculation one obtains

$$\log Z^{A}_{\gamma,\beta,\lambda}(\Lambda \mid \bar{q}^{(\Lambda^c)})$$
$$= -\tilde{H}_{\gamma,\lambda}(\hat{\varrho}(\Lambda) \mid \bar{q}^{(\Lambda^c)}) + \ell_2^d \sum_{D \not\subset [\Lambda^c]^{(\ell_2)}} W(D) \sum_{\substack{X_1(D),\\ X_2(D)}} \prod_{x \in X_1(D)} \hat{\varrho}_x$$
$$\times \int_0^1 dt \left\langle \prod_{x \in X_2(D)} \Delta\varrho_x \right\rangle_{t,0,\Lambda,\bar{q}^{(\Lambda^c)}}$$
$$+ \sum_{\tau \in \Lambda:\, L(\tau) \geqslant c_{(5.31)}} W(\tau) \int_0^1 ds \langle P(\tau, \Delta\varrho)\rangle_{1,s,\Lambda,\bar{q}^{(\Lambda^c)}}$$
$$+ \sum_{\xi \in \Lambda} \int_0^1 ds \langle W(\xi, \Delta\varrho)\rangle_{1,s,\Lambda,\bar{q}^{(\Lambda^c)}} \tag{6.16}$$

Here $\hat{\varrho}(\Lambda) = \hat{\varrho}^{\bar{q}^{(\Lambda^c)}}$ is the minimizer of $\tilde{H}_{\gamma,\lambda}(\cdot \mid \bar{q}^{(\Lambda^c)})$ and notations $P(\tau, \Delta\varrho)$ and $W(\xi, \Delta\varrho)$ are used instead of $P(\tau)$ and $W(\xi)$ to underline the density configuration $\hat{\varrho}(\Lambda) + \Delta\varrho(\Lambda)$ with respect to which these quantities are calculated. We use two parameters s and t instead of a single one for the technical transparency. In models $\langle \cdot \rangle_{t,0,\Lambda,\bar{q}^{(\Lambda^c)}}$, i.e., with $s = 0$, the range of interaction is finite so it is simpler to control decay of correlations.

Namely, a standard consequence of the Dobrushin uniqueness theorem [D2] is the estimate

$$|\langle \Delta\varrho_x\rangle_{t,0,\Lambda,\bar{q}^{(\Lambda^c)}} - \langle \Delta\varrho_x\rangle_{t,0}| \leqslant 2\zeta \sum_{n=2}^{\infty} \sum_{\substack{y_1 = x\\ y_2,\ldots,y_{n-1} \in [\Lambda]^{(\ell_2)}\\ y_n \in [\Lambda^c]^{(\ell_2)}}} \prod_{i=1}^{n-1} r_{y_i y_{i+1}} \tag{6.17}$$

where r_{xy} are defined by (5.56) with nonessential modification discussed below (5.62). Similarly

$$\left| \left\langle \prod_{x \in X} \Delta\varrho_x \right\rangle_{t,0,\Lambda,\bar{q}^{(\Lambda^c)}} - \langle \Delta\varrho_x\rangle_{t,0} \right|$$
$$\leqslant (2\zeta)^{|X|} \sum_{n=2}^{\infty} \sum_{\substack{y_1 \in X\\ y_2,\ldots,y_{n-1} \in [\Lambda]^{(\ell_2)}\\ y_n \in [\Lambda^c]^{(\ell_2)}}} \prod_{i=1}^{n-1} r_{y_i y_{i+1}} \tag{6.18}$$

Now if follows from definition (5.56) of r_{xy} and estimate (3.17) that

$$|\langle \Delta\varrho_x\rangle_{t,0,\Lambda,\bar{q}^{(\Lambda^c)}} - \langle \Delta\varrho_x\rangle_{t,0}| \leqslant \frac{c\zeta}{1 - a(\beta)} \left(\frac{1 + a(\beta)}{2} \right)^{[\gamma\, \mathrm{dist}(x, \Lambda^c)]} \tag{6.19}$$

and

$$\left| \left\langle \prod_{x \in X} \varDelta\varrho_x \right\rangle_{t,\,0,\,\varLambda,\,\bar{q}^{(\varLambda^c)}} - \langle \varDelta\varrho_x \rangle_{t,\,0} \right| \leqslant |X| (2\zeta)^{|X|} \frac{c\zeta}{1-a(\beta)} \left(\frac{1+a(\beta)}{2} \right)^{[\gamma\, \mathrm{dist}(X,\, \varLambda^c)]} \tag{6.20}$$

which are similar to (4.36). Also given x

$$\sum_{n=2}^{\infty} \sum_{\substack{y_1 = x \\ y_2,\ldots,\, y_n \in \mathbb{Z}^d_{\ell_2}}} \prod_{i=1}^{n-1} r_{y_i y_{i+1}} \leqslant \frac{2}{1-a(\beta)} \tag{6.21}$$

In fact is easily follows from the structure and smallness of $|W(\tau)|$ and $|W(\xi)|$ expressed by (5.31) and (5.37) that the same exponential decay of correlations takes place for models with $s \neq 0$ which include the infinite range part $s \sum_{i=2,\,3} \varDelta H^{(i)}_{\gamma,\,\lambda}(\varrho(\varLambda) \mid \bar{q}^{(\varLambda^c)})$ of the Hamiltonian (6.15)

The final representation of $\log Z^{A}_{\gamma,\,\beta,\,\lambda}(\varLambda \mid \bar{q}^{(\varLambda^c)})$ is

$$\begin{aligned}
\log Z^{A}_{\gamma,\,\beta,\,\lambda}&(\varLambda \mid \bar{q}^{(\varLambda^c)}) \\
&= -\tilde{H}_{\gamma,\,\lambda}(\hat{\varrho}(\varLambda) \mid \bar{q}^{(\varLambda^c)}) \\
&\quad - \ell_2^d \sum_{D \not\subset [\varLambda^c]^{(\ell_2)}} W(D) \sum_{\substack{X_1(D),\\ X_2(D)}} \prod_{x \in X_1(D)} \hat{\varrho}_x \int_0^1 dt \left\langle \prod_{x \in X_2(D)} \varDelta\varrho_x \right\rangle_{t,\,0} \\
&\quad - \sum_{\tau \notin \varLambda^c:\, L(\tau) \geqslant c} W(\tau) \int_0^1 ds \langle P(\tau, \varDelta\varrho) \rangle_{1,\,s} \\
&\quad - \sum_{\xi \notin \varLambda^c} \int_0^1 ds \langle W(\xi, \varDelta\varrho) \rangle_{1,\,s} \\
&\quad + \ell_2^d \sum_{D \not\subset [\varLambda^c]^{(\ell_2)}} W(D) \sum_{\substack{X_1(D),\\ X_2(D)}} \prod_{x \in X_1(D)} \hat{\varrho}_x \int_0^1 dt \\
&\quad \times \left(\left\langle \prod_{x \in X_2(D)} \varDelta\varrho_x \right\rangle_{t,\,0,\,\varLambda,\,\bar{q}^{(\varLambda^c)}} - \left\langle \prod_{x \in X_2(D)} \varDelta\varrho_x \right\rangle_{t,\,0} \right) \\
&\quad + \sum_{\tau \notin \varLambda^c:\, L(\tau) \geqslant c} W(\tau) \int_0^1 ds (\langle P(\tau, \varDelta\varrho) \rangle_{1,\,s,\,\varLambda,\,\bar{q}^{(\varLambda^c)}} - \langle P(\tau, \varDelta\varrho) \rangle_{1,\,s}) \\
&\quad + \sum_{\xi \notin \varLambda^c} \int_0^1 ds (\langle W(\xi, \varDelta\varrho) \rangle_{1,\,s,\,\varLambda,\,\bar{q}^{(\varLambda^c)}} - \langle W(\xi, \varDelta\varrho) \rangle_{1,\,s})
\end{aligned} \tag{6.22}$$

Here speaking about $\langle f(\Delta\varrho_x)\rangle_{t,s,\Lambda,\bar q^{(\Lambda^c)}}$ we assume that $\Delta\varrho_x=\varrho_x-\hat\varrho_x(\Lambda)$ while for $\langle f(\Delta\varrho_x)\rangle_{t,s}$ we mean $\Delta\varrho_x=\varrho_x-\hat\varrho_{\beta,\lambda,\sigma}$ such that $\langle f(\Delta\varrho_x)\rangle_{t,s}$ is truly independent on any boundary conditions.

Proof of Statement 3.3. From the interpolation trick we have a representation for the metastable free energy

$$\begin{aligned}
f^{\Lambda}_{\sigma,\lambda,\gamma}=&\,\ell_2^d\hat\varrho_{\beta,\lambda,\sigma}(\log\hat\varrho_{\beta,\lambda,\sigma}-1)-\ell_2^d\lambda\hat\varrho_{\beta,\lambda,\sigma}-\ell_2^d\frac{1}{2!}\sum_{x\in\mathbb{Z}^d_{\ell_2}}J^{(2)}_\gamma(0,x)\,\hat\varrho^2_{\beta,\lambda,\sigma}\\
&+\ell_2^d\frac{1}{4!}\sum_{x_1,x_2,x_3\in\mathbb{Z}^d_{\ell_2}}J^{(4)}_\gamma(0,x_1,\dots,x_3)\,\hat\varrho^4_{\beta,\lambda,\sigma}\\
&-\ell_2^d\sum_{\substack{D:\,X(D)\ni 0\\ D\neq L^{(2)},\,L^{(4)}}}\frac{W(D)}{|X(D)|}\,\hat\varrho^{|X(D)|}_{\beta,\lambda,\sigma}\\
&-\sum_{\tau:\,X(\tau)\ni 0}\frac{W(\tau)}{|X(\tau)|}P(\tau,\hat\varrho_{\beta,\lambda,\sigma})-\sum_{\xi:\,X(\xi)\ni 0}\frac{W(\xi,\hat\varrho_{\beta,\lambda,\sigma})}{|X(\xi)|}\\
&-\ell_2^d\sum_{\substack{D:\,X(D)\ni 0\\ |X_2(D)|\geqslant 2}}\frac{W(D)}{|X(D)|}\sum_{\substack{X_1(D),\\ X_2(D)}}\hat\varrho^{|X_1(D)|}_{\beta,\lambda,\sigma}\int_0^1 dt\left\langle\prod_{x\in X_2(D)}\Delta\varrho_x\right\rangle_{t,0}\\
&-\sum_{\tau:\,X(\tau)\ni 0}\frac{W(\tau)}{|X(\tau)|}\int_0^1 ds\langle P(\tau,\Delta\varrho)\rangle_{1,s}\\
&-\sum_{\xi:\,X(\xi)\ni 0}\int_0^1 ds\,\frac{\langle W(\xi,\Delta\varrho)\rangle_{1,s}}{|X(\xi)|}
\end{aligned}\tag{6.23}$$

Here the definition of $X(\tau)$ and $X(\xi)$ is similar to that of $X(D)$ and $\Delta\varrho$ is defined with respect to $\hat\varrho_{\beta,\lambda,\sigma}$.

The difference $g_\sigma(\Lambda\mid\bar q^{(\Lambda^c)})$ between RHS of (6.22) and $f^{\Lambda}_{\sigma,\lambda,\gamma}\,|\Lambda|$ can be estimated as

$$|g_\sigma(\Lambda\mid\bar q^{(\Lambda^c)})|\leqslant c\ell_2^d\,\frac{|\bar\Lambda\cap\bar\Lambda^c|}{\ell_2^{d-1}}\,\gamma^{-\alpha_2}\leqslant c\gamma^{-1}\,|\bar\Lambda\cap\bar\Lambda^c|\tag{6.24}$$

Here $|\bar\Lambda\cap\bar\Lambda^c|$ is the $(d-1)$-dimensional volume (area) of hypersurface separating Λ and Λ^c and $|\bar\Lambda\cap\bar\Lambda^c|\,\ell_2^{-d+1}\gamma^{-\alpha_2}$ is the number of lattice points in $[\delta\Lambda]^{(\ell_2)}$. The first factor $c\ell_2^d$ is the upper estimate for the sum of the absolute values of the error terms associated with given site $x\in[\partial\Lambda]^{(\ell_2)}$. To be more precise, the contribution to $g_\sigma(\Lambda\mid\bar q^{(\Lambda^c)})$ is given by terms in (6.22) and (6.23) crossing the boundary of Λ directly or indirectly. Directly crossing terms are D's with $X(D)$ intersecting both Λ and Λ^c,

τ's with $X(\tau)$ intersecting both Λ and Λ^c, ξ's with $X(\xi)$ intersecting both Λ and Λ^c and links with endpoints in both Λ and Λ^c. Indirectly intersection terms come from estimate (6.18) for the difference terms in the last three lines of (6.22). Graphically those terms can be represented by chains of r's joining something inside Λ, say some $D \in \Lambda$, with the sites outside Λ. It is important for us that geometrically any of the terms crossing the boundary passes through some site $x \in \partial\Lambda$. This site x may be: one of the endpoints of the link, contributing itself or as a part of D, τ or ξ, or the endpoint of some r_{xy} or it may belong to the support of some contour which is a part of some polymer ξ. For all these involved weighted objects we always keep the property that the sum of the absolute values of the statistical weights of all objects passing through given point is less than absolute constant. Multiplying this constant by the factor ℓ_2^d, which depending on the notations is present explicitly or implicitly in front of the sums just discussed, we reproduce (6.24). We need now some additional work to improve on (6.24).

The simplest observation is that using (5.43) and (5.37) one obtains that the sum over the objects containing τ with $L(\tau) > c_{(5.31)}$ or ξ as a constituting element is less than $c\gamma^{4d}$.

Another source of smallness are various expectations $\langle \prod_x \Delta\varrho_x \rangle$ in the corresponding terms. In particular we check that given x and for any $k \geqslant 1$

$$\langle |\Delta\varrho_x|^k \rangle \leqslant c(\ell_2^d)^{-k(5/12)} \tag{6.25}$$

where the expectation means any one of those contributing to (6.22) and (6.23). The power $-\frac{5}{12}$ is taken for the definiteness only and it can be replaced by $-\frac{1}{2}+\varepsilon$. We prove (6.25) adapting to our situation relatively abstract Lemma 8.1 in the spirit of [R2]. The correspondence between current notations and those of Section 8 is given by the following list of analogous objects: ϕ_x and $\Delta\varrho_x$, κ and ℓ_2^d, a and 2ζ, M and $2\zeta\ell_2^d$, R and $c_{(5.31)}\gamma^{-1}$, J_{xy} and $\hat{\varrho}\tilde{I}^{(2)}(x, y \mid \hat{\varrho})$ and so on. The bound (6.25) is an easy consequence of Lemma 8.1 (which deals the finite range interactions) as the infinite radius part, $s \sum_{i=2,3} \Delta H^{(i)}_{\gamma,\lambda}(\varrho^{(\ell_2)} \mid \bar{q}^{(\Lambda^c)})$, of (6.15) is so small that it can not increase the expectation of $|\Delta\varrho_x|$ more than in $1 + c\gamma^{4d}$ times. Note also that in (6.22) and (6.23) we have terms $\langle |\Delta\varrho_x|^k \rangle$ only with $k \geqslant 2$.

In a similar way applying estimate (8.12) to $|\langle \prod_{x \in X_2(D)} \Delta\varrho_x \rangle_{t,0,\Lambda,\bar{q}^{(\Lambda^c)}} - \langle \prod_{x \in X_2(D)} \Delta\varrho_x \rangle_{t,0}|$ we obtain improved (6.20). Indeed, the RHS of (6.18) is nothing but the upper bound for the Vasserstein distance and a square root of this estimate produces an exponential bound just with twice smaller exponent than in (6.20).

These improvements reduce by the same factor, $(\ell_2^d)^{-2(5/12)}$, the estimates of the cross boundary terms coming from lines 2 and 5 in (6.22) and line 7 in (6.23).

As an immediate consequence of these improvements we can see that the contribution, $e(\lambda)$, to the difference $\ell_2^{-d}(f^A_{+,\lambda,\gamma}-f^A_{-,\lambda,\gamma})$ coming from the non energy terms (lines 7, 8, 9 in (6.23)) is a continuous function of λ and $|e(\lambda)|\leqslant c(\ell_2^d)^{-2(5/12)}$. The energy parts (lines 1–6 in (6.23)) of $\ell_2^{-d}f^A_{+,\lambda,\gamma}$ and $\ell_2^{-d}f^A_{-,\lambda,\gamma}$ coincide with each other at $\lambda=\hat\lambda(\beta,\gamma)$ and they contain explicitly terms which are linear in λ. Hence shiftig $\hat\lambda(\beta,\gamma)$ by at most $c(\ell_2^d)^{-2(5/12)}$ one can find the solution, $\lambda=\lambda(\beta,\gamma)$ of (3.42) which proves Statement 3.3. ▮

A useful consequence of our improvements is the estimate

$$\begin{aligned}|g_\sigma(\Lambda\mid\bar q^{(\Lambda^c)})&+(\tilde H_{\gamma,\lambda}(\hat\varrho_{\beta,\lambda,\gamma}(\Lambda)\mid\hat\varrho_{\beta,\lambda,\gamma}(\Lambda^c))-\tfrac12\tilde U_\gamma(\hat\varrho_{\beta,\lambda,\gamma}(\Lambda^c)\mid\hat\varrho_{\beta,\lambda,\gamma}(\Lambda^c))\\&-\tilde H_{\gamma,\lambda}(\hat\varrho(\Lambda)\mid\bar q^{(\Lambda^c)}))|\leqslant c\gamma^{-1}(\ell_2^d)^{-2(5/12)}\,|\bar\Lambda\cap\bar\Lambda^c|\leqslant\gamma^{1/2}\,|\bar\Lambda\cap\bar\Lambda^c|\qquad(6.26)\end{aligned}$$

where $\tilde U_\gamma(\hat\varrho_{\beta,\lambda,\gamma}(\Lambda)\mid\hat\varrho_{\beta,\lambda,\gamma}(\Lambda^c))$ is defined as in (2.11) but in terms of the Hamiltonian $\tilde H(\cdot)$. (We remind that $|\Lambda\cap\Lambda^c|$ is the $(d-1)$-dimensional volume of the hypersurface separating closed domains Λ and Λ^c.) Observe that $\tilde H_{\gamma,\lambda}(\hat\varrho_{\beta,\lambda,\gamma}(\Lambda)\mid\hat\varrho_{\beta,\lambda,\gamma}(\Lambda^c))-\tilde H_{\gamma,\lambda}(\hat\varrho(\Lambda)\mid\bar q^{(\Lambda^c)})$ is rather small while the compensating term $-\frac12\tilde U_\gamma(\hat\varrho_{\beta,\lambda,\gamma}(\Lambda)\mid\hat\varrho_{\beta,\lambda,\gamma}(\Lambda^c))$ is of order $c\gamma^{-1}\,|\Lambda\cap\Lambda^c|$.

Proof of Statement 3.5. To check estimate (3.45) consider $\mathrm{Int}^{\neq}(\Gamma)=\bigcup_{m:\,\sigma_m(\Gamma)\neq\sigma(\Gamma)}\mathrm{Int}_m(\Gamma)$ and introduce a strip $S(\Gamma)=\partial^{(\ell_3/2)}\,\mathrm{Int}^{\neq}(\Gamma)\setminus\tilde\delta^{\neq}(\Gamma)$, where $\tilde\delta^{\neq}(\Gamma)$ is defined similarly to $\tilde\delta^{=}(\Gamma)$ (see (4.57) and below). First we rewrite the numerator of (3.39) as

$$\int_{\mathcal{Q}(\mathrm{Supp}(\Gamma)\setminus S(\Gamma))}dq\,\mathbb{1}_{\eta(q)=\eta^\Gamma}\,e^{-\beta H_{\gamma,\lambda}(q\mid\bar q^{(\delta^=(\Gamma))})}\,Z^A_{\gamma,\beta,\lambda}(\mathrm{Int}^{\neq}(\Gamma)\cup S(\Gamma)\mid q^{(\tilde\delta^{\neq}(\Gamma))})\tag{6.27}$$

Observe that for any q contributing to (6.27) the restriction of q to $\tilde\delta^{\neq}(\Gamma)$ belongs to the ground state ensemble of the phase $-\sigma(\Gamma)$. Given such $q^{(\tilde\delta^{\neq}(\Gamma))}$ denote by $\hat\varrho(\mathrm{Int}^{\neq}(\Gamma)\cup S(\Gamma))=\hat\varrho^{q^{(\tilde\delta^{\neq}(\Gamma))}}$ the minimizer of (6.2). Then using representation (6.22) and definition (6.23) one can see that

$$\begin{aligned}&Z^A_{\gamma,\beta,\lambda}(\mathrm{Int}^{\neq}(\Gamma)\cup S(\Gamma)\mid q^{(\tilde\delta^{\neq}(\Gamma))})\\&\quad=Z^A_{\gamma,\beta,\lambda}(S(\Gamma)\mid q^{(\tilde\delta^{\neq}(\Gamma))},\hat\varrho^{q^{(\tilde\delta^{\neq}(\Gamma))}}(\mathrm{Int}^{\neq}(\Gamma)))\exp(\ell_2^{-d}f^A_{-\sigma(\Gamma),\lambda,\gamma}\,|\mathrm{Int}^{\neq}(\Gamma)|\\&\qquad-\tfrac12 U(\hat\varrho^{q^{(\tilde\delta^{\neq}(\Gamma))}}(\mathrm{Int}^{\neq}(\Gamma))\mid\hat\varrho^{q^{(\tilde\delta^{\neq}(\Gamma))}}(\partial\,\mathrm{Int}^{\neq}(\Gamma)))\\&\qquad-(\tilde H_{\gamma,\lambda}(\hat\varrho^{q^{(\tilde\delta^{\neq}(\Gamma))}}(\mathrm{Int}^{\neq}(\Gamma))\mid\hat\varrho^{q^{(\tilde\delta^{\neq}(\Gamma))}}(\partial\,\mathrm{Int}^{\neq}(\Gamma)))\\&\qquad-\tilde H_{\gamma,\lambda}(\hat\varrho_{\beta,\lambda,\sigma}(\mathrm{Int}^{\neq}(\Gamma))\mid\hat\varrho_{\beta,\lambda,\sigma}(\partial\,\mathrm{Int}^{\neq}(\Gamma))))\\&\qquad+\tilde g_{-\sigma(\Gamma)}(\mathrm{Int}^{\neq}(\Gamma)\mid\hat\varrho^{q^{(\tilde\delta^{\neq}(\Gamma))}}(\partial\,\mathrm{Int}^{\neq}(\Gamma))))\end{aligned}\tag{6.28}$$

The term $\tilde{g}_{-\sigma(\Gamma)}(\mathrm{Int}^{\neq}(\Gamma) \mid \hat{\varrho}^{q^{(\tilde{\delta}^{\neq}(\Gamma))}}(\partial\, \mathrm{Int}^{\neq}(\Gamma)))$ is the sum over "polymer type" terms cross the boundary of $\mathrm{Int}^{\neq}(\Gamma)$ and it can be estimated (see (6.26)) by

$$|\tilde{g}_{-\sigma(\Gamma)}(\mathrm{Int}^{\neq}(\Gamma) \mid \hat{\varrho}^{q^{(\tilde{\delta}^{\neq}(\Gamma))}}(\partial\, \mathrm{Int}^{\neq}(\Gamma)))| \leqslant c\gamma^{1/2}\, |\overline{\mathrm{Int}^{\neq}(\Gamma)} \cap \overline{\mathrm{Int}^{\neq}(\Gamma)^c}| \tag{6.29}$$

On the other hand the strip $S(\Gamma)$ is so wide that in the γ^{-1} neighborhood of $\mathrm{Int}^{\neq}(\Gamma) \cap \mathrm{Int}^{\neq}(\Gamma)^c$ the difference $|\hat{\varrho}^{q^{(\tilde{\delta}^{\neq}(\Gamma))}} - \hat{\varrho}_{\beta,\lambda,\sigma}|$ is exponentially small as follows from a straightforward analogue of (4.36). Thus

$$\begin{aligned} &Z^{A}_{\gamma,\beta,\lambda}(\mathrm{Int}^{\neq}(\Gamma) \cup S(\Gamma) \mid q^{(\tilde{\delta}^{\neq}(\Gamma))}) \\ &\quad = Z^{A}_{\gamma,\beta,\lambda}(S(\Gamma) \mid q^{(\tilde{\delta}^{\neq}(\Gamma))}, \hat{\varrho}_{\beta,\lambda,\sigma}(\mathrm{Int}^{\neq}(\Gamma)))\exp(\ell_2^{-d} f^{A}_{-\sigma(\Gamma),\lambda,\gamma}\, |\mathrm{Int}^{\neq}(\Gamma)| \\ &\qquad - \tfrac{1}{2} U(\hat{\varrho}_{\beta,\lambda,\sigma}(\mathrm{Int}^{\neq}(\Gamma)) \mid \hat{\varrho}_{\beta,\lambda,\sigma}(\partial\, \mathrm{Int}^{\neq}(\Gamma))) + \bar{g}_{-\sigma(\Gamma)}(\mathrm{Int}^{\neq}(\Gamma))) \end{aligned} \tag{6.30}$$

where

$$|\bar{g}_{-\sigma(\Gamma)}(\mathrm{Int}^{\neq}(\Gamma))| \leqslant c\gamma^{1/2}\, |\overline{\mathrm{Int}^{\neq}(\Gamma)} \cap \overline{\mathrm{Int}^{\neq}(\Gamma)^c}| \tag{6.31}$$

Here $\bar{g}_{-\sigma(\Gamma)}(\mathrm{Int}^{\neq}(\Gamma))$ collects the contribution of $\tilde{g}_{-\sigma(\Gamma)}(\mathrm{Int}^{\neq}(\Gamma) \mid \hat{\varrho}^{q^{(\tilde{\delta}^{\neq}(\Gamma))}}(\partial\, \mathrm{Int}^{\neq}(\Gamma)))$ together with corrections due to replacement of $U(\hat{\varrho}^{q^{(\tilde{\delta}^{\neq}(\Gamma))}}(\mathrm{Int}^{\neq}(\Gamma)) \mid \hat{\varrho}^{q^{(\tilde{\delta}^{\neq}(\Gamma))}}(\partial\, \mathrm{Int}^{\neq}(\Gamma)))$ by $U(\hat{\varrho}_{\beta,\lambda,\sigma}(\mathrm{Int}^{\neq}(\Gamma)) \mid \hat{\varrho}_{\beta,\lambda,\sigma}(\partial\, \mathrm{Int}^{\neq}(\Gamma)))$ and the estimate of $\tilde{H}_{\gamma,\lambda}(\hat{\varrho}^{q^{(\tilde{\delta}^{\neq}(\Gamma))}}(\mathrm{Int}^{\neq}(\Gamma)) \mid \hat{\varrho}^{q^{(\tilde{\delta}^{\neq}(\Gamma))}}(\partial\, \mathrm{Int}^{\neq}(\Gamma))) - \tilde{H}_{\gamma,\lambda}(\hat{\varrho}_{\beta,\lambda,\sigma}\mathrm{Int}^{\neq}(\Gamma)) \mid \hat{\varrho}_{\beta,\lambda,\sigma}(\partial\, \mathrm{Int}^{\neq}(\Gamma)))$.

Using (6.10) we obtain from (6.30) our last estimate

$$\begin{aligned} &Z^{A}_{\gamma,\beta,\lambda}(\mathrm{Int}^{\neq}(\Gamma) \cup S(\Gamma) \mid q^{\tilde{\delta}^{\neq}(\Gamma))}) \\ &\quad = Z^{A}_{\gamma,\beta,\lambda}(S(\Gamma) \mid q^{(\tilde{\delta}^{\neq}(\Gamma))}, \varrho_{\beta,\lambda,\sigma}(\mathrm{Int}^{\neq}(\Gamma)))\exp(\ell_2^{-d} f^{A}_{-\sigma(\Gamma),\lambda,\gamma}\, |\mathrm{Int}^{\neq}(\Gamma)| \\ &\qquad - \tfrac{1}{2} U(\varrho_{\beta,\lambda,\sigma}(\mathrm{Int}^{\neq}(\Gamma)) \mid \varrho_{\beta,\lambda,\sigma}(\partial\, \mathrm{Int}^{\neq}(\Gamma))) + \bar{g}_{-\sigma(\Gamma)}(\mathrm{Int}^{\neq}(\Gamma)) + \Delta \end{aligned} \tag{6.32}$$

where $|\Delta| \leqslant c\gamma^{1/4}\, |S(\Gamma)$.

A similar estimate is true for the denominator of (3.39) which after integration over $q^{(\tilde{\delta}^{\neq}(\Gamma))}$ implies (3.45). ∎

7. APPENDIX. POLYMER EXPANSION THEOREM

Consider a finite or countable set Θ the elements of which are called (abstract) *diagrams* and denoted θ, θ', etc. Fix some reflexive and symmetric relation on $\Theta \times \Theta$. A pair $\theta, \theta' \in \Theta \times \Theta$ is called incompatible ($\theta \nsim \theta'$)

if it satisfies given relation and compatible ($\theta \sim \theta'$) in the opposite case. A collection $\{\theta_j\}$ is called a *compatible collection of diagrams* if any two its elements are compatible. Every diagram θ is assigned a complex-valued *statistical weight* denoted by $w(\theta)$, and for any finite $\Lambda \subseteq \Theta$ an (abstract) *partition function* is defined as

$$Z(\Lambda) = \sum_{\{\theta_j\} \subseteq \Lambda} \prod_j w(\theta_j) \tag{7.1}$$

where the sum is extended to all compatible collections of diagrams $\theta_i \in \Lambda$. The empty collection is compatible by definition, and it is included in $Z(\Lambda)$ with statistical weight 1.

A polymer $\pi = [\theta_i^{\varepsilon_i}]$ is an (unordered) finite collection of different diagrams $\theta_i \in \Theta$ taken with positive integer multiplicities ε_i, such that for every pair θ', $\theta'' \in \pi$ there exists a sequence $\theta' = \theta_{i_1}, \theta_{i_2},..., \theta_{i_s} = \theta'' \in \pi$ with $\theta_{i_j} \nsim \theta_{i_{j+1}}$, $j = 1, 2,..., s-1$. The notation $\pi \subseteq \Lambda$ means that $\theta_i \in \Lambda$ for every $\theta_i \in \pi$.

With every polymer π we associate an (abstract) graph $G(\pi)$ which consists of $\sum_i \varepsilon_i$ vertices labeled by the diagrams from π and edges joining every two vertices labeled by incompatible diagrams. It follows from the definition of $G(\pi)$ that it is connected and we denote by $r(\pi)$ the quantity

$$r(\pi) = \prod_i (\varepsilon_i!)^{-1} \sum_{G' \subset G(\pi)} (-1)^{|G'|} \tag{7.2}$$

where the sum is taken over all connected subgraphs G' of $G(\pi)$ containing all of $\sum_i \varepsilon_i$ vertices and $|G'|$ denotes the number of edges in G'. For any $\theta \in \pi$ we denote by $\varepsilon(\theta, \pi)$ the multiplicity of θ in the polymer π.

The polymer expansion theorem below is a modification of results of [Se] and [KP] proven in [MSu]. See also [D3] for similar results.

Theorem 7.1. Suppose that there exists a function $a(\theta)\colon \Theta \mapsto \mathbf{R}^+$ such that for any diagram θ

$$\sum_{\theta' :\, \theta' \nsim \theta} |w(\theta')|\, e^{a(\theta')} \leqslant a(\theta) \tag{7.3}$$

Then for any finite Λ,

$$\log Z(\Lambda) = \sum_{\pi \subseteq \Lambda} w(\pi) \tag{7.4}$$

where the statistical weight of a polymer $\pi = [\theta_i^{\varepsilon_i}]$ equals

$$w(\pi) = r(\pi) \prod_i w(\theta_i)^{\varepsilon_i} \tag{7.5}$$

Moreover, the series (7.4) for $\log Z(\Lambda)$ is absolutely convergent in view of the estimate

$$\sum_{\pi:\, \pi \ni \theta} \varepsilon(\theta, \pi)\, |w(\pi)| \leqslant |w(\theta)|\, e^{a(\theta)} \tag{7.6}$$

which is true for any diagram θ.

Corollary 7.2. For any function $b(\theta)\colon \Theta \mapsto \mathbb{R}^+$ consider modified statistical weights of diagrams

$$\tilde{w}(\theta) = w(\theta)\, e^{b(\theta)} \tag{7.7}$$

and suppose that still

$$\sum_{\theta':\, \theta' \nsim \theta} |\tilde{w}(\theta')|\, e^{a(\theta')} \leqslant a(\theta) \tag{7.8}$$

Then for any family, Π, of polymers such that any $\pi \in \Pi$ contains given diagram θ

$$\sum_{\pi \in \Pi} \varepsilon(\theta, \pi)\, |w(\pi)| \leqslant |\tilde{w}(\theta)|\, e^{a(\theta)} \left[\min_{\pi \in \Pi} \left(\prod_{\theta' \in \pi} e^{b(\theta')} \right) \right]^{-1} \tag{7.9}$$

8. APPENDIX. A TECHNICAL LEMMA

Consider a spin model on the lattice $\mathbb{Z}^d$ given by the formal Hamiltonian

$$H(\phi) = \sum_x \phi_x^2 + \tfrac{1}{2} \sum_{x \neq y} J_{xy} \phi_x \phi_y + \sum_{A \in \mathscr{A}} K_A \phi_A \tag{8.1}$$

Here the spin variable ϕ_x takes M discrete values, including 0, from the bounded interval $[-a, a]$. The last sum runs over some family $\mathscr{A}$ of set $A = \{x_1,..., x_{|A|} \in \mathbb{Z}^d\}$ containing $|A|$ not necessarily different sites x and $\phi_A = \prod_{x \in A} \phi_x$. The interaction, J_{xy} and K_A, is of finite range $R < \infty$, i.e., $J_{xy} = 0$ if $\operatorname{dist}(x, y) \geqslant R$ and $K_A = 0$ if $\operatorname{diam}(A) \geqslant R$. Suppose that

$$\sum_{y \neq x} |J_{xy}| = 1 - \alpha, \qquad 0 < \alpha < 1 \tag{8.2}$$

and

$$\sum_{A \ni x} a^{|A|}\, |K_A| \ll \alpha \tag{8.3}$$

and denote by $\mu(\phi(\Lambda))$ the corresponding Gibbs distribution in the finite domain $\Lambda \subset \mathbb{Z}^d$ with zero ($\equiv$empty) boundary condition in Λ^c.

Lemma 8.1. For any $c > 1$ and sufficiently large inverse temperature

$$\kappa > \frac{M+R}{c} \tag{8.4}$$

one has

$$\mu(|\phi_x| \geqslant \kappa^{-5/12}) \leqslant e^{-c\kappa^{1/6}}, \qquad \forall x \in \Lambda \tag{8.5}$$

Proof. for an arbitrary configuration $\phi(\Lambda)$ define *spots* $S(\phi(\Lambda))$ as R-connected components of site $x \in \Lambda$ with $|\phi_x| \geqslant \kappa^{-5/12}$. Taking $x \in S$ set

$$h(x, \phi(\Lambda)) = \sum_{y \neq x} J_{xy} \phi_x \phi_y \mathbb{1}_{|\phi_x| \geqslant |\phi_y|} + \sum_{A \in \mathscr{A}:\, A \ni x} K_A \phi_A \mathbb{1}_{|\phi_x| \geqslant \max_{y \in A, y \neq x} |\phi_y|} \tag{8.6}$$

Then

$$H(\phi(S) \mid \phi(\Lambda \backslash S)) = \sum_{x \in S} h(x, \phi(\Lambda)) \tag{8.7}$$

and

$$h(x, \phi(\Lambda)) \geqslant \frac{\alpha}{2} \phi_x^2 \tag{8.8}$$

Here in (8.7)–(8.8) we used (8.2), (8.3) and the fact that $|\phi_x| \geqslant |\phi_y|$ for any $x \in S$ and $y \in \partial^{(R)} S$.

Now the probability that $\phi(\Lambda)$ contains a spot S with the fixed value of $\phi(S)$ can be estimated as

$$\begin{aligned}
\mu(\phi(S)) &= \frac{\sum_{\phi(\Lambda \backslash S)} \exp(-\kappa H(\phi(S) \mid \phi(\Lambda \backslash S)) - \kappa H(\phi(\Lambda \backslash S)))}{\sum_{\phi(\Lambda)} \exp(-\kappa H(\phi(\Lambda)))} \\
&\leqslant \exp\left(-\kappa \frac{\alpha}{2} \sum_{x \in S} \phi_x^2\right) \frac{\sum_{\phi(\Lambda \backslash S)} \exp(-\kappa H(\phi(\Lambda \backslash S)))}{\sum_{\phi(\Lambda):\, \phi(S) \equiv 0} \exp(-\kappa H(\phi(\Lambda)))} \\
&\leqslant \exp\left(-\frac{\alpha}{2} \kappa^{1/6} |S|\right) \frac{\sum_{\phi(\Lambda \backslash S)} \exp(-\kappa H(\phi(\Lambda \backslash S)))}{\sum_{\phi(\Lambda):\, \phi(S) \equiv 0} \exp(-\kappa H(\phi(\Lambda)))} \\
&= \exp\left(-\frac{\alpha}{2} \kappa^{1/6} |S|\right)
\end{aligned} \tag{8.9}$$

Hence the probability of the spot S

$$\mu(S)=\sum_{\phi(S)}\mu(\phi(S))\leqslant(c_{(8.4)}\kappa)^{|S|}\exp\left(-\frac{\alpha}{2}\kappa^{1/6}|S|\right)$$
$$\leqslant\exp\left(-\frac{\alpha}{4}\kappa^{1/6}|S|\right) \tag{8.10}$$

where the last inequality is true for κ large enough.

Finally for any $x\in\Lambda$

$$\mu(|\phi_x|\geqslant\kappa^{-5/12})=\sum_{S\ni x}\mu(S)$$
$$\leqslant\sum_{|S|=1}^{\infty}(c_{(8.4)}\kappa)^{|S|}\exp\left(-\frac{\alpha}{4}\kappa^{1/6}|S|\right)$$
$$\leqslant\sum_{|S|=1}^{\infty}\exp\left(-\frac{\alpha}{8}\kappa^{1/6}|S|\right)$$
$$\leqslant e^{-c\kappa^{1/6}} \tag{8.11}$$

where again κ is large enough. ∎

Let now $\mu^{(0)}(\phi_x,\phi_y)$ and $\mu^{(1)}(\phi_x,\phi_y)$ be a distribution at sites $x, y\in\Lambda$ of conditional Gibbs measures in Λ with two arbitrary boundary conditions $\bar\phi^{(0)}(\Lambda^c)$ and $\bar\phi^{(1)}(\Lambda^c)$ respectively. The Vasserstein distance $R(\mu^{(0)},\mu^{(1)})$ between $\mu^{(0)}$ and $\mu^{(1)}$ is given by some coupling $\mu(\phi_x^{(0)},\phi_y^{(0)};\phi_x^{(1)},\phi_y^{(1)})$. Denote by $\langle\cdot\rangle^{(0)}$, $\langle\cdot\rangle^{(1)}$ and $\langle\cdot\rangle$ the expectation with respect to $\mu^{(0)}$, $\mu^{(1)}$ and μ and suppose that $2a<1$. Then

$$|\langle\phi_x,\phi_y\rangle^{(0)}-\langle\phi_x,\phi_y\rangle^{(1)}|$$
$$=\langle(\phi_x^{(0)}-\phi_x^{(1)})\,\phi_y^{(0)}+(\phi_y^{(0)}-\phi_y^{(1)})\,\phi_x^{(1)}\rangle$$
$$\leqslant\langle(\phi_x^{(0)}-\phi_x^{(1)})^2\rangle^{1/2}\langle(\phi_y^{(0)})^2\rangle^{1/2}+\langle(\phi_y^{(0)}-\phi_y^{(1)})^2\rangle^{1/2}\langle(\phi_x^{(1)})^2\rangle^{1/2}$$
$$\leqslant\langle|\phi_x^{(0)}-\phi_x^{(1)}|\rangle^{1/2}\langle(\phi_y^{(0)})^2\rangle^{1/2}+\langle|\phi_y^{(0)}-\phi_y^{(1)}|\rangle^{1/2}\langle(\phi_x^{(1)})^2\rangle^{1/2}$$
$$\leqslant\max((\langle\phi_x^2\rangle^{(0)})^{1/2},(\langle\phi_y^2\rangle^{(1)})^{1/2})$$
$$\times\max(R(\mu^{(0)}(\phi_x),\mu^{(1)}(\phi_x))^{1/2},R(\mu^{(0)}(\phi_y),\mu^{(1)}(\phi_y))^{1/2}) \tag{8.12}$$

Similarly one can estimate via the Vasserstein distance and the second moment the difference $|\langle\prod_{x\in X}\phi_x\rangle^{(0)}-\langle\prod_{x\in X}\phi_x\rangle^{(1)}|$ for any $X\subset\Lambda$.

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