Lecture 1

XIX century: deterministic diff'l eq's.

XX century: QM, chaos, <u>fluctuating phenomena</u>

("curve + noise")

<u>Brownian motion</u>

Robert Brown (1827) $\Rightarrow$ particles suspended in water undergo random motion

Einstein's explanation: introduce $\tau \ll t$
& independently, (1905) Smoluchowski

τ: collision time; t: successive observ'n times

Subseq. collisions are <u>indep.</u>; part's also move independently

In a time τ, x changes by Δ:

$\varphi(\Delta)$ is the prob. distr'n: $\int_{-\infty}^{\infty} \varphi(\Delta)\, d\Delta = 1$,

$$dn = n\,\varphi(\Delta)\, d\Delta$$

n: total # part. suspended in liquid

dn: # part. ~~check~~ whose coordinates change in the $(\Delta, \Delta + d\Delta)$ interval

Assume: 1. $\varphi(\Delta) \neq 0$ only for "small" Δ;
2. $\varphi(\Delta) = \varphi(-\Delta)$ [symm.]

Consider $f(x,t)$ – # part. per unit volume @ x, @ t.

Then
$$f(x,t+\tau)\, dx = dx \int_{-\infty}^{\infty} d\Delta\, \varphi(\Delta)\, f(x \mp \Delta, t)$$

τ small:
$$f(x,t+\tau) \approx f(x,t) + \tau \frac{\partial f}{\partial t} + \ldots$$

Δ small:
$$f(x \mp \Delta, t) \approx f(x,t) \mp \Delta \frac{\partial f}{\partial x} + \frac{\Delta^2}{2} \frac{\partial^2 f}{\partial x^2} + \ldots$$

Then $$f + \tau \frac{\partial f}{\partial t} = f \underbrace{\int_{-\infty}^{\infty} \varphi(\Delta)\, d\Delta}_{=1} + \frac{\partial f}{\partial x} \underbrace{\int_{-\infty}^{\infty} d\Delta\, \Delta\, \varphi(\Delta)}_{0} +$$

$$+ \frac{\partial^2 f}{\partial x^2} \underbrace{\int_{-\infty}^{\infty} d\Delta \frac{\Delta^2}{2} \varphi(\Delta)}_{\text{"} D\tau}$$

$$\boxed{\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}} \quad \text{diff'n eq'n}$$

$$f(x,t) = \frac{n}{\sqrt{4\pi D}} \frac{e^{-x^2/4Dt}}{\sqrt{t}}$$

$$\sigma_x = \sqrt{2Dt}$$

<u>Langevin's eq'n</u>

Consider part. of mass m & diam. a suspended in liquid. Two forces:

1. Viscous drag force : from hydrodynamics

$$-6\pi \eta a v$$

η ← viscosity

2. Fluctuating force X from random collisions

stochastic diff'l eq'n

So, $$m \frac{d^2 x}{dt^2} = -6\pi \eta a \frac{dx}{dt} + X \quad | * x$$

$$\underbrace{\frac{m}{2} \frac{d^2}{dt^2}(x^2) - m\left(\frac{dx}{dt}\right)^2}_{} = -3\pi \eta a \frac{d(x^2)}{dt} + Xx$$

$$\frac{m}{2} \frac{d}{dt}\left(2x \frac{dx}{dt}\right) - m\left(\frac{dx}{dt}\right)^2 =$$

$$= m x \frac{d^2 x}{dt^2}$$

$\langle \ldots \rangle$ - average over all particles:

use $\langle \frac{m v^2}{2} \rangle = \frac{k_B T}{2}$ equip. theorem

$$\frac{m}{2}\frac{d^2}{dt^2}\langle x^2\rangle + 3\pi\eta a \frac{d\langle x^2\rangle}{dt} = k_B T + \underbrace{\langle X x\rangle}_{\langle X\rangle\langle x\rangle = 0}$$

Then $$\frac{d\langle x^2\rangle}{dt} = \frac{k_B T}{3\pi\eta a} + C e^{-\frac{6\pi\eta a t}{m}}$$

const

$\frac{6\pi\eta a}{m} \gg 1 \Rightarrow$ neglect 2nd term

Then $$\langle x^2\rangle - \langle x_0^2\rangle = \underbrace{\frac{k_B T}{3\pi\eta a}}_{2D} t$$

"2D", same as Einstein's result (see σ_x above)

Birth-death processes

Consider X : ~~predator~~ prey

Y : predator

A : food for prey

$$\begin{cases} X + A \to 2X \\ X + Y \to 2Y \\ Y \to \emptyset \end{cases}$$

x - # prey

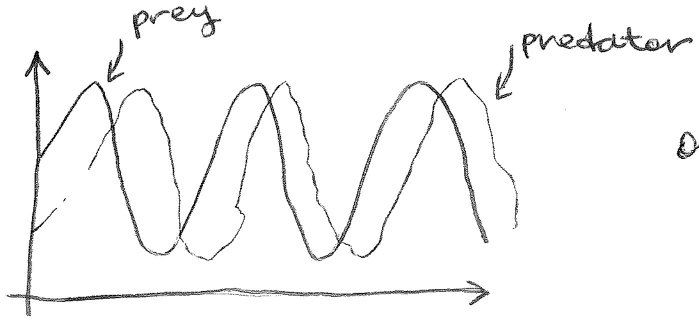
y - # predators

amount of food

Then $$\begin{cases} \frac{dx}{dt} = k_1 a x - k_2 x y, \\ \frac{dy}{dt} = k_2 x y - k_3 y \end{cases}$$

$\{k_i\}$ are rates

Lotka ~~Lotka~~ - Volterra eq'ns



oscillatory Sol's

But: real data has noise ...

How to include fluct's ?

Birth-death master eq'n (ME):

Consider $P(x,y,t)$ ← prob. distr'n

Consider small Δt:

transition probs
$$\begin{cases} P(x\to x+1;\ y\to y) = k_1 a x \Delta t, \\ P(x\to x-1;\ y\to y+1) = k_2 x y \Delta t, \\ P(x\to x;\ y\to y-1) = k_3 y \Delta t, \\ P(x\to x;\ y\to y) = 1-(k_1 a x + k_2 x y + k_3 y)\Delta t \end{cases}$$

no memory (Markov assumption): probs can be constructed simply from (x,y)

Then
$$\overbrace{\frac{P(x,y,t+\Delta t)-P(x,y,t)}{\Delta t}}^{\frac{\partial P}{\partial t}} =$$

$$= k_1 a(x-1)P(x-1,y,t) + k_2(x+1)(y-1)\times$$
$$\times P(x+1,y-1,t) + k_3(y+1)P(x,y+1,t) -$$
$$-(k_1 a x + k_2 x y + k_3 y)P(x,y,t)$$

Is Markov postulate justified? (not clear...)

⇒ ME ; determines both fluct's and average behavior