Backward & Forward Kolmogorov Equations

FP eq'n:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\Big[\overbrace{\mu(x,t)}^{\frac{\langle \Delta x\rangle}{\Delta t}}\, p(x,t)\Big] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[\underbrace{\sigma^2(x,t)}_{\frac{\langle \Delta x^2\rangle}{\Delta t} + \mathcal{O}(\Delta t)}\, p(x,t)\Big]$$

Prob. density: $p(x,t\,|\,\underbrace{x_0,t_0}_{IC}) = p(x,t)$

Now, consider

Prob. (not density): $\mathbb{P}(A,t\,|\,x_0,t_0) = \mathbb{P}(X_t \in A\,|\,X_{t_0} = x_0)$

$$= \int_A dy\, p(y,t\,|\,x_0,t_0)$$

In particular, $\mathbb{P}(X_t \leq x\,|\,X_{t_0} = x_0) \equiv$

$$\equiv \mathbb{P}(x,t\,|\,x_0,t_0) =$$

$$= \int_{-\infty}^{x} p(z,t\,|\,x_0,t_0)\,dz$$

$$\frac{\partial \mathbb{P}(x,t\,|\,x_0,t_0)}{\partial x} = p(x,t\,|\,x_0,t_0)$$

Now, use $\mathbf{p}(x,t|x_0,t_0) = \int dy\, \mathbf{p}(x,t|y,t_1) \times$

$\times\, p(y,t_1|x_0,t_0)$

Consider

$$\frac{\mathbf{p}(x,t|x_0,t_0-\Delta t) - \mathbf{p}(x,t|x_0,t_0)}{\Delta t} \quad =$$

(Δt small)

$$\mathbf{p}(x,t|x_0,t_0-\Delta t) = \int dy\, \mathbf{p}(x,t|y,t_0)\, p(y,t_0|x_0,t_0-\Delta t)$$

$$= \frac{1}{\Delta t}\Big[\int dy\, \mathbf{p}(x,t|y,t_0)\, p(y,t_0|x_0,t_0-\Delta t) -$$

$$- \mathbf{p}(x,t|x_0,t_0) \underbrace{\int dy\, p(y,t_0|x_0,t_0-\Delta t)}_{=1}\Big] =$$

$$= \frac{1}{\Delta t}\int dy\,[\mathbf{p}(x,t|y,t_0) - \mathbf{p}(x,t|x_0,t_0)] \times$$

$$\times\, p(y,t_0|x_0,t_0-\Delta t) \simeq$$

$$\simeq \frac{\partial \mathbf{p}}{\partial x_0} \underbrace{\frac{1}{\Delta t}\int dy\,(y-x_0)\, p(y,t_0|x_0,t_0-\Delta t)}_{\mu(x_0,t_0)} +$$

$$+ \frac{1}{2}\frac{\partial^2 \mathbf{p}}{\partial x_0^2} \underbrace{\frac{1}{\Delta t}\int dy\,(y-x_0)^2\, p(y,t_0|x_0,t_0-\Delta t)}_{\sigma^2(x_0,t_0)}$$

Finally,

$$-\frac{\partial p(x,t|x_0,t_0)}{\partial t_0} = \mu(x_0,t_0)\frac{\partial p}{\partial x_0} + \frac{1}{2}\sigma^2(x_0,t_0)\frac{\partial^2 p}{\partial x_0^2}. \qquad (*)$$

~~apply $\frac{\partial}{\partial x}^*$ to Eq. (*):~~

$$\cancel{-\frac{\partial p(x,t|x_0,t_0)}{\partial t_0} = \mu(x_0,t_0)\frac{\partial p}{\partial x_0} + \frac{1}{2}\sigma^2(x_0,t_0)\frac{\partial^2 p}{\partial x_0^2}}$$

eq'n for prob. density

Connection with FPT:

$$\underbrace{P_V(x_0, \underbrace{t-t_0}_{\text{homog.}})}_{\substack{\text{survival probability}\\ \text{(no absorption) @ time } t}} = \int_V dy\, p(y,t|x_0,t_0) =$$

$$= 1 - \underbrace{\varphi(x_0, t-t_0)}_{\text{prob. that FPT is } < t-t_0}.$$

Then Eq. (*) gives: [apply $\int_V dy \times \ldots$]

$$-\frac{\partial \varphi(x_0,t-t_0)}{\partial t_0} = \mu(x_0,t_0)\frac{\partial \varphi}{\partial x_0} + \frac{1}{2}\sigma^2(x_0,t_0)\frac{\partial^2 \varphi}{\partial x_0^2}.$$

Finally, $\frac{\partial}{\partial t_0} f(t-t_0) = -\frac{\partial}{\partial t} f(t-t_0)$, yielding

$\left(\begin{array}{l} t_0 \to 0, \\ x_0 \to x \end{array}\right)$

$$\frac{\partial \varphi(x,t)}{\partial t} = \mu(x) \frac{\partial \varphi(x,t)}{\partial x} + \frac{\sigma^2(x)}{2} \frac{\partial^2 \varphi(x,t)}{\partial x^2}.$$

QED