which we can interpret as the result of hybridization. We will return to use these expressions to calculate the low-energy part of the tunneling spectrum.

17.7 Kondo insulators

The Kondo insulator is the simplest version of the Kondo lattice, in which the formation of Kondo singlets leads to a fully gapped, insulating state. While the term "Kondo insulator" dates back to the early 1990s [5], these are the oldest heavy-fermion materials. The first heavy-fermion or Kondo insulator, SmB_6, was discovered in 1969 by Menth, Buehler, and Geballe at AT&T Bell Laboratories [6], followed closely by SmS under pressure [7]. It was these materials that inspired Neville Mott to propose that Kondo insulators involve a kind of excitonic ordering between localized f-electrons and conduction electrons [8], driving the emergent hybridization that we have been discussing. A predecessor of the large-N path integral approach to Kondo insulators was proposed in 1979 by Claudine Lacroix and Michel Cyrot at the Laboratoire Louis Néel in Grenoble [28]. At the time of writing this book, SmB_6 has once again been thrust into the main-stream of research, with the proposal [39] that this is an example of a topological insulator – a *topological Kondo insulator* with robust conducting surfaces [40, 41]. This is a topic we will return to in Chapter 18 when we consider mixed valence.

17.7.1 Strong-coupling expansion

In many ways, the Kondo insulator is the simplest ground state of the Kondo lattice. Let us begin by returning to the $SU(2)$ Kondo lattice model:

$$H = -t\sum_{(i,j)\sigma}(c^\dagger_{i\sigma}c_{j\sigma} + \text{H.c.}) + J\sum_{j,\alpha\beta}\vec{\sigma}_j\cdot\vec{S}_j \qquad (\vec{\sigma}_j \equiv (c^\dagger_{j\beta}\vec{\sigma}_{\beta\alpha}c_{j\alpha})), \tag{17.145}$$

corresponding to a tight-binding Kondo lattice where the electrons at each site are coupled antiferromagnetically to a local moment. We can gain a lot of insight by examining the strong-coupling limit, in which the dispersion of the conduction sea is much smaller than J, so that $t/J << 1$ is a small parameter. In this limit, the intersite hopping is a perturbation to the onsite Kondo insteraction:

$$H \overset{t/J\to 0}{\longrightarrow} J\sum_{j,\alpha\beta}\vec{\sigma}_j\cdot\vec{S}_j + O(t), \tag{17.146}$$

and the ground state corresponds to the formation of a spin singlet at each site, denoted by the wavefunction

$$|KI\rangle = \prod_j \frac{1}{\sqrt{2}}\Big(\Uparrow_j\downarrow_j - \Downarrow_j\uparrow_j\Big), \tag{17.147}$$

where the double and single arrows denote the localized moment and conduction electron, respectively, as illustrated in Figure 17.13(a).

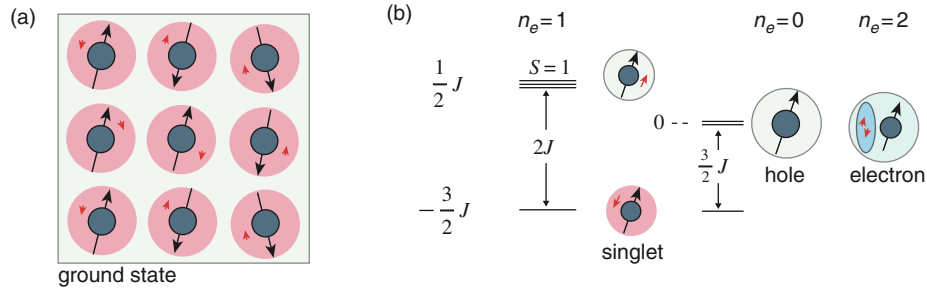


Fig. 17.13 (a) Illustrating the ground state of the Kondo insulator in the strong-coupling limit. (b) Excitations of the Kondo insulator, showing triplet excitation with spin gap $2J$ and $S = \frac{1}{2}$ hole and electron excitations with excitation energy $\frac{3}{2}J$.

Each singlet has a ground-state energy $E = -\frac{3}{2}J$ per site and a singlet–triplet spin gap of magnitude $\Delta E = 2J$. Moreover, if we remove an electron from site i, we break a Kondo singlet and create an unpaired spin with excited energy $\frac{3}{2}J$,

$$|\mathrm{qp}^+, i\uparrow\rangle = \Uparrow_i \prod_{j\neq i} \frac{1}{\sqrt{2}}\left(\Uparrow_j\downarrow_j - \Downarrow_j\uparrow_j\right) = \sqrt{2}c_{i\downarrow}|KI\rangle, \tag{17.148}$$

while if we add an electron, we create an electron quasiparticle, corresponding to an unpaired local moment and a doubly occupied conduction electron orbital,

$$|\mathrm{qp}^-, i\uparrow\rangle = \Uparrow_i \left(\uparrow_i\downarrow_i\right) \prod_{j\neq i} \frac{1}{\sqrt{2}}\left(\Uparrow_j\downarrow_j - \Downarrow_j\uparrow_j\right) = \sqrt{2}c^\dagger_{j\uparrow}|KL\rangle, \tag{17.149}$$

as illustrated in Figure 17.13(b).

If we now reintroduce the hopping $-t$ between sites, then these quasiparticle excitations become mobile, as illustrated in Figure 17.14(a) and (b). From the explicit form of the states, we see that the matrix elements to hop these quasiparticles between nearest-neighbor (n.n.) sites are given by

$$\langle \mathrm{qp}^\pm, i\sigma|H|\mathrm{qp}^\pm, j\sigma\rangle = \pm\frac{t}{2} \qquad \big((i,j) \in \text{n.n.}\big), \tag{17.150}$$

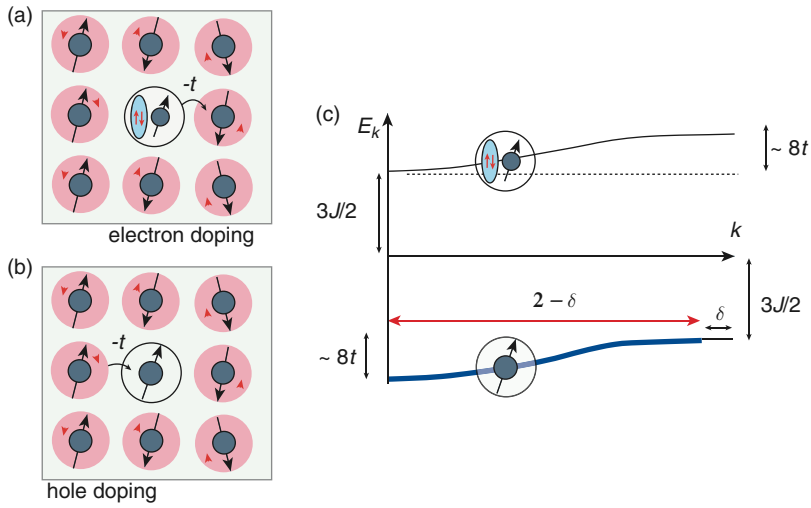
corresponding to one-half the bare hopping, giving quasiparticle energies

$$E_{\mathrm{qp}^\pm}(\mathbf{k}) = \pm t(c_x + c_y + c_z) + \frac{3}{2}J. \tag{17.151}$$

To transform from the quasiparticle to the electron basis, we need to reverse the sign of the hole (qp^+) dispersion to obtain the valence band dispersion, so that the band energies predicted by the strong-coupling limit of the Kondo lattice are

$$E^\pm_{\mathbf{k}} = -t(c_x + c_y + c_z) \pm \frac{3}{2}J, \tag{17.152}$$

separated by an energy $3J$ as shown in Figure 17.14(c). Note that these are hard-core fermions that cannot occupy the same lattice site simultaneously.



Showing (a) electron and (b) hole doping of strong-coupling Kondo insulator. (c) Dispersion of strong-coupling Kondo insulator. A small amount of hold doping δ gives rise to a Fermi surface containing $2-\delta = (1-\delta)+1$ heavy electrons, showing that the Fermi surface counts the number of electrons and spins. **Fig. 17.14**

In this way, the half-filled strong-coupling Kondo lattice forms an insulator with a charge gap of size $3J$ and a spin gap of size $2J$. Notice finally that if we dope the insulator with an amount δ of holes, we form a band of heavy-fermions. We can regard the resulting "hole" Fermi surface as containing $2-\delta = (1-\delta)+1$ heavy electrons, which is one more than we expect based on the density of conduction electrons. In this way, we see once again that the Fermi surface of the Kondo lattice counts both the electrons and the spins.

17.7.2 Large-*N* treatment of the Kondo insulator

Let us now re-examine the Kondo insulator using the large-N expansion. Here, our advantage is that we are not restricted to strong coupling. The Kondo insulator is a special case of the large-N Kondo lattice, in which the chemical potential lies between the upper conduction and lower valence band. We start with the mean-field hybridization model (17.50):

$$H_{MF} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_j \left[\bar{V} \left(c^\dagger_{j\sigma} f_{j\sigma} \right) + \left(f^\dagger_{j\sigma} c_{j\sigma} \right) V_j + \lambda (n_{fj} - Q) + N\mathcal{N}_s \frac{|V|^2}{J} \right]. \tag{17.153}$$

We can simplify the problem by considering the special case of particle–hole symmetry, with $Q = N/2$ and $n_e = N/2$ per site, in which case the mean-field constraint $\langle n_f \rangle = N/2$ is satisfied with $\lambda = 0$ and we only need to optimize the value of the hybridization. Following the steps of (17.111)–(17.115), the mean-field dispersions for the Kondo lattice are

$$E_{\mathbf{k}\pm} = \left[\frac{\epsilon_{\mathbf{k}}}{2}\right] \pm \left[\left(\frac{\epsilon_{\mathbf{k}}}{2}\right)^2 + |V|^2\right], \tag{17.154}$$

where we have set $\lambda = 0$. Following (17.125), the ground-state energy is then given by

$$\frac{E_g}{N} = \sum_{\mathbf{k}} \left(\left[\frac{\epsilon_{\mathbf{k}}}{2}\right] - \left[\left(\frac{\epsilon_{\mathbf{k}}}{2}\right)^2 + |V|^2\right]\right) + \mathcal{N}_s \frac{|V|^2}{J}. \tag{17.155}$$

This expression is strongly reminiscent of BCS theory, and differs from the mean-field theory of the heavy-fermion metal, in that the energy involves an unrestricted sum over momenta in the lower valence band. If we differentiate this expression with respect to the hybridization, we obtain

$$\frac{1}{N\mathcal{N}_s}\frac{\partial E_g}{\partial |V|^2} = 0 = -\int_{\mathbf{k}} \frac{1}{\sqrt{\epsilon_{\mathbf{k}}^2 + 4V^2}} + \frac{1}{J}, \tag{17.156}$$

which is a kind of gap equation. To get an approximate treatment of the problem, let us replace the momentum integral by an energy integration. Assuming the conduction electron density of states $\rho = \frac{1}{2D}$ can be treated as constant, the Kondo insulator gap equation becomes

$$\frac{1}{J} = \rho \int_{-D}^{D} \frac{d\epsilon}{\sqrt{\epsilon^2 + 4V^2}} = 2\rho \sinh^{-1}\left(\frac{D}{2V}\right). \tag{17.157}$$

Putting $\rho = \frac{1}{2D}$, we then obtain

$$V = \frac{D}{2\sinh\left(\frac{D}{J}\right)}. \tag{17.158}$$

As we increase the half-bandwidth D from a value that is small to a value that is large compared with J, we see that V interpolates from $V = J/2$ at strong coupling to $V = D\exp\left[-\frac{1}{2J\rho}\right] = \sqrt{DT_K}$, where $T_K = De^{-\frac{1}{J\rho}}$. We can also calculate the indirect gap of the insulator, determined by the value of the dispersion when the conduction electrons are at the edge of the band, i.e.

$$\begin{aligned}
\Delta_g = E_{\mathbf{k}+}|_{\epsilon_{\mathbf{k}}=-D} - E_{\mathbf{k}-}|_{\epsilon_{\mathbf{k}}=D} &= \sqrt{D^2 + 4V^2} - D \\
&= 2V\left[\sqrt{\left(\frac{D}{2V}\right)^2 + 1} - \left(\frac{D}{2V}\right)\right] \\
&= 2V\left[\cosh\frac{D}{J} - \sinh\frac{D}{J}\right] \\
&= 2V\exp\left[-\frac{D}{J}\right] = \frac{2D}{e^{\frac{2D}{J}} - 1},
\end{aligned} \tag{17.159}$$

where we have used (17.158) to make the substitutions on the third and fourth lines. We see that, in the large-N limit, the gap undergoes a crossover from $\Delta_g = J$ to $\Delta_g = 2T_K$, as shown in Figure 17.15.

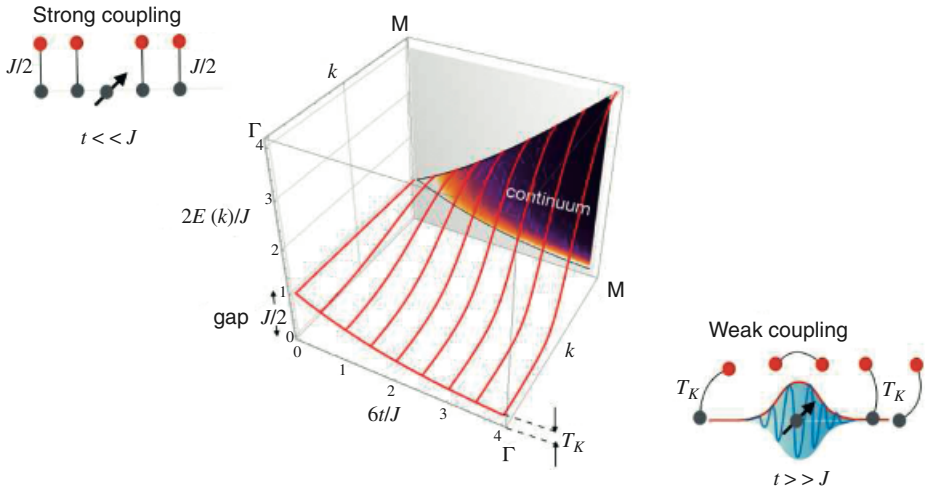


Fig. 17.15 Evolution of Kondo insulator from strong to weak coupling, calculated in the large-N limit. The points $\Gamma \equiv (0,0,0)$ and $M \equiv (\pi,\pi,\pi)$ correspond to the origin and the zone-center of the Brillouin zone. As the bandwidth t of the conduction sea is increased, the quasiparticle excitation energy drops from $O(J)$ to $O(T_K)$. At strong coupling the excitations are immobile unpaired spins, but at weak coupling, the gapped excitations are mobile heavy-fermions. At finite N, we expect that, when T_K becomes too small, the insulator will ultimately become unstable to antiferromagnetism.

17.8 The composite nature of the f-electron

17.8.1 A thought experiment: a Kondo lattice of nuclear spins

In electronic materials the Kondo effect involves localized f- or d-electrons. However, a Kondo effect could occur equally well with a *nuclear* spin. This might seem absurd, yet nuclear spins do couple antiferromagnetically with conduction electrons to produce RKKY interactions that drive nuclear antiferromagnetism. In practice the coupling is far too small to destabilize the nuclear magnetism and produce a nuclear Kondo effect. Nevertheless, we learn something from the thought experiment in which the nuclear spin coupling to electrons is strong enough to overcome the nuclear magnetism. In this case, resonant bound states would form with the nuclear spin lattice, giving rise to *charged* heavy electrons, presumably with an expanded Fermi surface.

From this line of argument we see that, while it's tempting to associate the heavy fermion in the Kondo effect with a physical f- or d-electron localized inside the local moment, from a renormalization group perspective the heavy electron is an emergent excitation: a fermionic bound state formed between the conduction sea and the neutral localized moments. The only memory of the underlying localized electrons is encoded in the spatial symmetry of the Kondo coupling, which of course for rare-earth systems is an f-form factor.

The composite picture of heavy electrons is useful for several reasons:

- As we will see in Section 17.9, it allows us to understand the formation of Fano resonant structures in Kondo lattices.
- It allows us to envisage processes in which the Kondo effect breaks down, leading to the loss of the large Fermi surface. Such *Kondo breakdown* phenomena are thought to be the origin of certain types of non-Fermi-liquid behavior in heavy-electron systems.
- It opens up the possibility of new kinds of composite structures – bosons that might pair-condense – or composite fermions with quantum numbers which are different to electrons, such as neutral, spinless, or integer-spin fermions.

17.8.2 Cooper pair analogy

In a superconductor, electron pairs behave as loose composite bosons described by the relation

$$\overbrace{\psi_\uparrow(x)\psi_\downarrow}(x') = -F(x-x'). \tag{17.160}$$

Here $F(x-x') = -\langle T\psi_\uparrow(1)\psi_\downarrow(2)\rangle$ is the anomalous Gor'kov Green's function which determines the Cooper pair wavefunction, extended over the coherence length $\xi \sim v_F/T_c$. We can treat the pair operator as a c-number because the pairs condense.

A similar phenomenon takes place in the Kondo effect, but here the bound state develops between spins and electrons, forming a fermion, rather than a boson. For an isolated Kondo impurity, the analogue of the coherence length in the superconductor is the *Kondo screening length* $\xi_K \sim v_F/T_K$, but in a lattice the renormalization of the heavy-fermion velocity means that this screening length is of the order of a lattice spacing. In this situation, it is perhaps more useful to think in terms of a *screening time* $\tau_K \sim \hbar/T_K$, rather than a length, governing the electron spin-flip correlations. Both Cooper pairs and heavy electrons involve a binding process that spans decades of energy up to a cut-off, be it the Debye energy ω_D in superconductivity or the (much larger) bandwidth D in the Kondo effect [42, 43].

To follow this analogy in greater depth, recall that in the path integral the Kondo interaction factorizes as

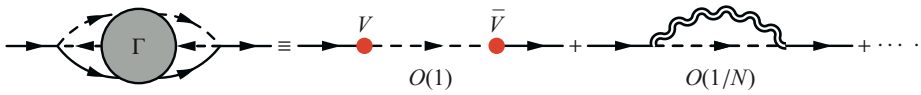
$$\frac{J}{N}c^\dagger_\beta S_{\alpha\beta}c_\alpha \longrightarrow \bar{V}\left(c^\dagger_\alpha f_\alpha\right) + \left(f^\dagger_\alpha c_\alpha\right)V + N\frac{\bar{V}V}{J}, \tag{17.161}$$

so by comparing the right- and left-hand sides, we see that the composite operators $S_{\beta\alpha}c_\beta$ and $c^\dagger_\beta S_{\alpha\beta}$ behave as a single fermion denoted by the contractions

$$\frac{1}{N}\sum_\beta \overbrace{S_{\beta\alpha}c_\beta} = \left(\frac{\bar{V}}{J}\right)f_\alpha, \qquad \frac{1}{N}\sum_\beta \overbrace{c^\dagger_\beta S_{\alpha\beta}} = \left(\frac{V}{J}\right)f^\dagger_\alpha. \tag{17.162}$$

composite fermion

Physically, this means that the spins bind high-energy electrons, transforming themselves into composites which then hybridize with the conduction electrons. The resulting *heavy fermions* can be thought of as moments ionized in the magnetically polar electron fluid to form mobile, negatively charged heavy electrons while leaving behind a positively charged *Kondo singlet*. Microscopically, the many-body amplitude to scatter an electron off a local moment develops a bound-state pole, which for large N we can denote by the diagrams



The leading diagram describes a kind of condensation of the hybridization field; the second and higher terms describe the smaller $O(1/N)$ fluctuations around the mean-field theory.

By analogy with superconductivity, we can associate a wavefunction with the temporal correlations between spin-flips and conduction electrons, as follows:

$$\frac{1}{N}\sum_{\beta}\overbrace{c_\beta(\tau)S_{\beta\alpha}(\tau')} = g(\tau-\tau')\hat{f}_\alpha(\tau'), \tag{17.163}$$

where the spin-flip correlation function $g(\tau-\tau')$ is an analogue of the Gor'kov function, extending over a coherence time $\tau_K \sim \hbar/T_K$. Notice that, in contrast to the Cooper pair, this composite object is a fermion and thus requires a distinct operator $\hat{f}_\alpha$ for its expression. The Fourier (Laplace) decomposition of $g(\tau)$ describes the spectral distribution of electrons and spin-flips inside the composite f-electron, which we may calculate as follows:

$$\begin{aligned}\frac{1}{N}\sum_{\beta}\overbrace{c_\beta(\tau)S_{\beta\alpha}(\tau')} &= \frac{1}{N}\sum_{\beta}\overbrace{c_\beta(\tau)f^\dagger_\beta(\tau')}f_\alpha(\tau') \\ &= \frac{1}{N}\sum_{\beta}\langle Tc_\beta(\tau)f^\dagger_\beta(\tau')\rangle f_\alpha(\tau') \\ &= -G_{cf}(\tau-\tau')f_\alpha(\tau').\end{aligned} \tag{17.164}$$

In this way, we identify

$$g(\tau-\tau') = \langle Tc_\beta(\tau)f^\dagger_\beta(\tau')\rangle = -G_{cf}(\tau-\tau') \tag{17.165}$$

with the anomalous Green's function between the f- and conduction electrons at the same site.

A detailed calculation (see Example 17.5) shows that $g(\tau)$ is logarithimically correlated at short times, but decays as $\frac{1}{\tau}$ at times $|\tau| >> \frac{\hbar}{T_K}$:

$$g(\tau) \sim \begin{cases} \rho V \ln\left(\frac{T_K\tau}{\hbar}\right) & (\hbar/D << \tau << \hbar/T_K) \\ \frac{1}{\tau} & (\tau >> \hbar/T_K). \end{cases} \tag{17.166}$$

The short-time logarithimic correlations between the spin-flip and electron ($\tau << \hbar/T_K$) represent the weak-coupling interior of the composite fermion, whereas the long-time power law correlations reflect the development of the Fermi liquid correlations at long times.

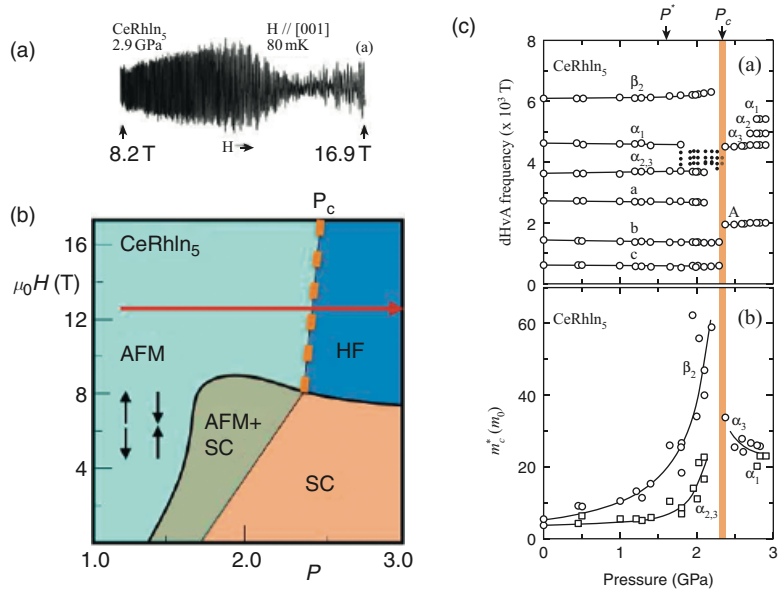


Fig. 17.16 Jump in the Fermi surface area of $CeRhIn_5$ at an antiferromagnetic phase transition, imaged from de Haas–van Alphen (dHvA) oscillations. (a) de Haas–van Alphen signal [44] at 2.9 GPa. (b) Schematic zero-temperature pressure-field phase diagram for $CeRhIn_5$, showing antiferromagnetic, superconducting, and heavy-fermion regions of the phase diagram [45]. The red arrow shows the approximate region of the phase diagram swept by the dHvA measurements. (c) Pressure dependence of dHvA frequencies [44], which are a measure of the Fermi surface area of extremal orbits on the Fermi surface. The data show the jump in Fermi surface area at the critical pressure where antiferromagnetism disappears, and a corresponding divergence in effective mass. Panels (a) and (c) reprinted with permission from H. Shishido, *et al.*, *J. Phys. Soc. Jpn.*, vol. 74, no. 4, p. 1103, 2005. Copyright 2005 by the Physical Society of Japan. Panel (b) reprinted with permission from T. Park, *et al.*, *Nature*, vol. 440, no. 7080, p. 65, 2006. Copyright 2006 Macmillan Publishers.

The internal structure of the composite fermion, spread over several decades up to the bandwidth, guarantees that the composite f-state is orthogonal to the low-energy conduction electrons, behaving as an emergent electron field, injected into the low-energy Fermi sea. The physical manifestation of this phenomenon is an expansion of the Fermi surface by the composite fermions. A particularly dramatic example of this expansion is seen in the material $CeRhIn_5$, which is an antiferromagnetic metal at ambient pressures but becomes superconducting as the f-electrons delocalize at higher pressures (Figure 17.16). De Haas–van Alphen experiments on the normal state show that the Fermi surface expands as the mobile f-electrons are formed. Similar effects are also seen in Hall-constant measurements. Most remarkably of all, in cases where the Fermi surface expands to fill the entire Brillouin zone, the resulting system becomes an insulator, a *Kondo insulator*.

Example 17.5 Calculate the internal spin-flip correlation function of the composite f-electron,

$$\frac{1}{N}\sum_{\beta}\overrightarrow{c_\beta(\tau)S}_{\beta\alpha}(\tau') = g(\tau-\tau')\hat{f}_\alpha(\tau'), \tag{17.167}$$

in the large-N expansion. Carry this out using a Fourier decomposition,

$$g(\tau) = -T \sum_{\mathbf{k}, i\omega_n} G_{cf}(\mathbf{k}, i\omega_n) e^{-i\omega_n \tau}, \tag{17.168}$$

where $G_{cf}(\mathbf{k}, \tau) = -\langle c_{\mathbf{k}\sigma}(\tau) f_{\mathbf{k}\sigma}(0) \rangle$ is the anomalous propagator between the conduction and f-state.

Solution

Transforming to Fourier space, we have

$$G_{cf}(\mathbf{k}, i\omega_n) = \quad f \quad V \quad c \quad = \left(\frac{V}{i\omega_n - \epsilon_k}\right) \frac{1}{i\omega_n - \lambda - \frac{V^2}{i\omega_n - \epsilon_{\mathbf{k}}}}$$

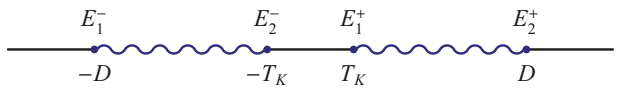
$$= \frac{V}{(i\omega_n - \epsilon_{\mathbf{k}})(i\omega_n - \lambda) - V^2}, \tag{17.169}$$

where the double dashed line is the full f-electron propagator. We can approximate the summation over momentum in (17.168) as an integral over energy:

$$G_{cf}(z) = \sum_{\mathbf{k}} G_{cf}(\mathbf{k}, z) = \rho \int_{-D}^{D} d\epsilon \frac{V}{(z - \epsilon)(z - \lambda) - V^2}$$

$$= \frac{\rho V}{z - \lambda} \ln \left[\frac{(z + D)(z - \lambda) - V^2}{(z - D)(z - \lambda) - V^2} \right]. \tag{17.170}$$

This function contains two branch cuts along the real axis, corresponding to the upper and lower bands, which run from $E_1^{\pm} \longrightarrow E_2^{\pm}$, where $(E_{1,2}^{\pm} \pm D)(E_{1,2}^{\pm} - \lambda) - V^2 = 0$. The low-energy ends of the branch cut $|E_2^+| \sim |E_1^-| \sim V^2/D \sim T_K$ are of the order of the Kondo scale, whereas the high-energy ends $|E_1^-| \sim |E_2^+| \sim D$ are set by the bandwidth:



There are thus two energy scales in this function – the bandwidth D and the Kondo temperature $T_K \sim \lambda$. The internal structure of the composite fermion is thus determined by the spectral function

$$g(\omega) = -\frac{1}{\pi} \mathrm{Im}\, G_{cf}(\omega - i\delta) = -\frac{\rho V}{\omega - \lambda} \sum_{\pm} [\theta(\omega - E_1^{\pm}) - \theta(\omega - E_2^{\pm})], \tag{17.171}$$

as shown in Figure 17.17.

In the time domain,

$$g(\tau) = -\int_{-\infty}^{\infty} d\omega g(\omega) \left[(1 - f(\omega))\theta(\tau) - f(\omega)\theta(-\tau) \right] e^{-\omega\tau}. \tag{17.172}$$

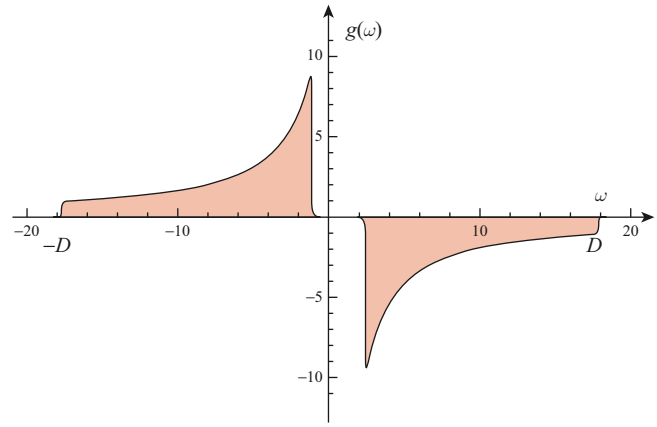


Fig. 17.17 Spectral distribution function $g(\omega)$ (17.171) describing the internal correlations of spin and electron inside a composite f-electron. See Example 17.5.

For simplicity, let's examine the case where the Fermi energy is in the lower band ($\lambda > 0$). Now by (17.167) and (17.162), the bound-state amplitude V is given by the equal-time Green's function,

$$\frac{V}{J} = g(0^-) = -V\rho \int_{-D}^{0} \frac{d\omega}{\omega - \lambda} = \rho V \ln \frac{D}{\lambda}, \tag{17.173}$$

from which we deduce that

$$\frac{V}{J} = \rho V \ln \frac{D}{\lambda} \Rightarrow \lambda = De^{-\frac{1}{J\rho}} = T_K, \tag{17.174}$$

as obtained earlier from the minimization of the energy. Note that the argument in the bound-state integral (17.173) depends on the inverse of the energy, right out to the bandwidth. If we divide the band on a logarithmic scale into n equal parts, where the ratio of the lower and upper energies is $s > 1$, we see that each decade of energy counts equally to the bound-state amplitude:

$$\begin{aligned} \frac{V}{J} &= -\rho V \int_{-D}^{-\lambda} d\epsilon \frac{1}{\epsilon} = -\rho V \left\{ \int_{-D}^{-D/s} + \int_{-D/s}^{-D/s^2} + \cdots + \int_{-D/s^{n-1}}^{-\lambda} \right\} \frac{d\epsilon}{\epsilon} \\ &= \rho V \left\{ \ln s + \ln s + \cdots + \ln \frac{Ds^{-n+1}}{\lambda} \right\}, \end{aligned} \tag{17.175}$$

demonstrating that the low-energy heavy-fermion bound state is formed from electron states that are spread out over decades of energy out to the bandwidth.

Finally, returning to the time dependence,

$$g(\tau) = -\int_{-D}^{0} d\omega \overbrace{\frac{V\rho}{\omega - \lambda}}^{-g(\omega)} e^{-\omega\tau} \qquad (\tau < 0), \tag{17.176}$$

Luttinger sum rules and spin fractionalization in the SU(N) Kondo lattice

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(Received 8 March 2021; accepted 19 July 2021; published 28 September 2021)

We show how Oshikawa's theorem for the Fermi surface volume of the Kondo lattice can be extended to the SU(N) symmetric case. By extending the theorem, we can show that the mechanism of Fermi surface expansion seen in the large N mean-field theory is directly linked to the expansion of the Fermi surface in a spin-$\frac{1}{2}$ Kondo lattice. This linkage enables us to interpret the expansion of the Fermi surface in a Kondo lattice as a fractionalization of the local moments into heavy electrons. Our method allows extension to a pure U(1) spin liquid, where we find the volume of the spinon Fermi surface by applying a spin twist, analogous to Oshikawa's, [Phys. Rev. Lett. **84**, 3370 (2000)] flux insertion. Lastly, we discuss the possibility of interpreting the FL* phase characterized by a small Fermi surface in the absence of symmetry breaking, as a nontopological coexistence of such a U(1) spin liquid and an electronic Fermi liquid.

DOI: 10.1103/PhysRevResearch.3.033284

I. INTRODUCTION

Over two decades ago, Oshikawa [1] applied the Lieb-Schultz-Mattis approach [2] to the Kondo lattice, using its response to a flux insertion to demonstrate that its Fermi surface volume counts the combined density of electrons and local moments. Although the expansion of the Fermi surface in the Kondo lattice had been informally established from arguments of continuity based on the Anderson lattice model [3], from the large N limit of the Kondo lattice [4–7], Oshikawa's [1] result provided a rigorous foundation for the Fermi surface expansion in a strict $S = \frac{1}{2}$ system Kondo lattice.

Curiously, since this hallmark development, Oshikawa's [1] result has not been generalized to higher group symmetries. Here, we show that this generalization is readily established for a family of SU(N) Kondo lattices. The key result is that, for local moments in an antisymmetric representation of the group constructed from Q elementary spinons, a Fermi liquid ground state will have an expanded Fermi surface volume $V_{\rm FS}$ given by

$$N v_c \frac{V_{\rm FS}}{(2\pi)^D} = n_e + N_S Q, \tag{1}$$

where n_e and N_S are, respectively, the number of electrons and number of local moments per unit cell of volume v_c. For all N, the electronic Fermi surface expands to incorporate the number of elementary spinons forming the local moments, and by increasing N to arbitrarily large values, we can link Oshikawa's [1] original result to the basin of attraction of large N field theoretic approaches to the Kondo lattice [4,5,7]. The importance of this link is that the Kondo fractionalization of local moments into *charged* heavy fermions, inferred field theoretically, is rigorously confirmed.

One of the unexpected outcomes of our analysis is the discovery that Oshikawa's [1] flux attachment method can also be applied to spin liquids [8,9]. Previously, it was assumed that, since spin liquids are neutral, they are immune to flux attachment, stimulating an alternative topological interpretation of spin-liquid ground states in coexistence with a Fermi liquid. However, because the unitary transformation that attaches a flux involves both a charge and a spin twist of the wave function, a spin liquid is sensitive to the flux attachment. This enables us to show that a U(1) spin liquid in an SU(N) Heisenberg model will have a Fermi surface volume determined purely by the number of spinons in the representation, i.e.,

$$N v_c \frac{V_{\rm FS}}{(2\pi)^D} = N_S Q. \tag{2}$$

This result suggests that fractionalization in a U(1) spin liquid and the Kondo lattice does not require a topological interpretation, i.e., that fractionalization and topology are not inevitably tied together.

The outline of this paper is as follows. In Sec. II, we derive the Luttinger sum rule for the SU(N) Kondo lattice. In Sec. III, we interpret the result as a signature of spin fractionalization, cementing an intuition derived from the large N mean-field theories as a general feature of the Kondo lattice.

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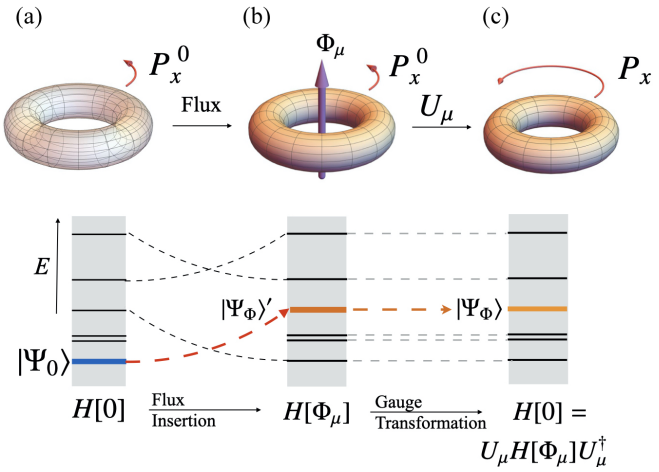

FIG. 1. Flux insertion strategy: (a) Initial state $|\Psi_0\rangle$ with momentum P_x^0. (b) State $|\Psi_\Phi\rangle'$ after flux insertion for electrons with spin component μ has unchanged canonical momentum, (c) After gauge transformation, $|\Psi_\Phi\rangle = U_\mu|\Psi_\Phi\rangle'$ has canonical momentum P_x. The change in momentum $\Delta P_x = P_x - P_x^0$ determines the Fermi surface volume.

In Sec. IV, we show how the method can be extended to a Kondo-Heisenberg model. In Sec. V, we discuss the role of spin-exchange interactions and identify the spinon Fermi surface volume of a U(1) spin liquid. Finally, in Sec. VI, we discuss whether the coexistence of a spin and small Fermi surface conduction fluid to form an FL* requires a topological interpretation.

II. DERIVATION

We consider the SU(N) symmetric Kondo lattice

$$H_{\rm KL} = -\sum_{\mathbf{r}\mathbf{r}'} t_{\mathbf{r},\mathbf{r}'} c^\dagger_{\mathbf{r}\sigma} c_{\mathbf{r}'\sigma} + J_K \sum_{\mathbf{r}} \vec{\lambda}_{\mathbf{r}} \cdot \vec{\Lambda}_{\mathbf{r}}, \tag{3}$$

where $c^\dagger_{\mathbf{r}\sigma}$, $(\sigma = 1, N)$ creates an electron at site $\mathbf{r}$, moving on a D-dimensional toroid with intersite hopping amplitude $t_{\mathbf{r},\mathbf{r}'}$, with dimensions $L_x, L_y \ldots L_D$. Here, $\vec{\lambda}_{\mathbf{r}} = c^\dagger_{\mathbf{r}\sigma}\vec{\lambda}_{\sigma\sigma'}c_{\mathbf{r}\sigma'}$ is the electron spin density at $\mathbf{r}$, where the $\vec{\lambda} = (\lambda^1, \ldots \lambda^{N^2-1})$ are the SU(N) Gell-Mann matrices. The $\vec{\Lambda}_{\mathbf{r}} = (\Lambda^1_{\mathbf{r}}, \ldots \Lambda^{N^2-1}_{\mathbf{r}})$ are the components of the localized moment at site $\mathbf{r}$. We shall consider local moments composed of Q elementary spinons, in an antisymmetric representation of SU(N), $|\sigma_1, \ldots \sigma_Q\rangle = (-1)^P|\sigma_{P_1} \ldots \sigma_{P_Q}\rangle$. The action of the spin operator Λ^a, $a = (1, N^2 - 1)$ on these states is then $\Lambda^a|\sigma_1 \ldots \sigma_Q\rangle = \sum_{n=1}^{Q} |\sigma_1 \ldots \sigma'_n \ldots \sigma_Q\rangle \lambda^a_{\sigma'_n\sigma_n}$.

The SU(N) Kondo lattice has a global U(1)×SU(N) symmetry, associated with the conserved electron number N_e and magnetization $M^a = \sum_{\mathbf{r}} \lambda^a_{\mathbf{r}} + \Lambda^a_{\mathbf{r}}$. Of particular interest are the diagonal components of the magnetization M^μ, $(\mu \in [1, N-1])$, which form the Cartan subalgebra of the SU(N) group, with Gell-Mann matrices $\lambda^\mu_{\sigma\sigma'} = (\delta^{\mu\sigma} - 1/N)\delta_{\sigma\sigma'}$.

Oshikawa's [1] strategy (see Fig. 1) is to introduce a unit magnetic flux quantum $\Phi_\mu = \frac{h}{e}$ that couples to the μth spin component of the Fermi sea, giving rise to an inductive current which increases the mechanical momentum by an $\Delta P_x = 2\pi/L_x \times V/(2\pi)^D \times V^\mu_{\rm FS}$, directly proportional to the Fermi surface volume. Since the flux insertion does not change the many-body energy eigenstates, it is equivalent to a unitary transformation U_μ of the original Hamiltonian $H[\Phi_\mu] = U^\dagger_\mu H[0] U_\mu$. This enables a direct calculation of the change in the mechanical momentum due to flux insertion in terms of microscopic quantities. Equating the direct calculation with the Fermi liquid result determines the Fermi surface volume.

We now apply this strategy to the SU(N) Kondo lattice. Flux insertion is achieved by a Peierls substitution $t_{\mathbf{r},\mathbf{r}'} \to t_{\mathbf{r},\mathbf{r}'} \exp[-i\mathbf{A}^\sigma \cdot (\mathbf{r} - \mathbf{r}')]$, where $\mathbf{A}^\sigma = \delta^{\mu\sigma}(\frac{2\pi}{L_x})\hat{\mathbf{x}}$. (Note, we are using natural units in which $e = \hbar = 1$, and the dimensions of the unit cell are rescaled to be unity, so that the unit cell volume $v_c = 1$.) This additional gauge field is generated by a large gauge transformation of the electron fields $U^\dagger_\mu c^\dagger_{\mathbf{r}\sigma} U_\mu = c^\dagger_{\mathbf{r}\sigma} \exp(-i\mathbf{A}^\sigma \cdot \mathbf{r})$. The obvious guess $U_\mu = \exp(\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} n^\mu_{\mathbf{r}})$ does not leave the Kondo interaction invariant, but a modified transformation

$$U_\mu = \exp\left[\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} \left(n^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}} + \frac{Q}{N}\right)\right], \tag{4}$$

satisfies this requirement. This is a generalization of Oshikawa's [1] original transformation, in which we have replaced the SU(2) generator $S^z_{\mathbf{r}}$ by $\Lambda^\mu_{\mathbf{r}}$. We have also added an additional gauge transformation which multiplies the wave function by a factor $\exp(\frac{2\pi i}{L_x} x_{\mathbf{r}} \frac{Q}{N})$ at each site, which ensures that the unitary transformation preserves the periodic boundary conditions: $U_\mu(x_r) = U_\mu(x_r + L_x)$. Here U_μ is a product of a U(1) and an SU(N) gauge transformation: in other words, to selectively impart momentum to μth Fermi surface we must "twist" the wave function in charge *and* spin space.

To see that U_μ commutes with the Kondo interaction, we write $n^\mu_{\mathbf{r}} = \lambda^\mu_{\mathbf{r}} + n_{\mathbf{r}}/N$, so that

$$U_\mu = \exp\left[\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} \left(\frac{n_{\mathbf{r}} + Q}{N} + M^\mu_{\mathbf{r}}\right)\right], \tag{5}$$

involves the electron density $n_{\mathbf{r}}$ and local magnetization $M^\mu_{\mathbf{r}} = \lambda^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}}$, which both commute with the Kondo interaction. To confirm that the transformation also preserves periodic boundary conditions, we note that, if we shift the x component of the site at $\mathbf{r}_0$ by L_x, i.e., $x_{\mathbf{r}_0} \to x_{\mathbf{r}_0} + L_x$, the unitary transformation picks up an additional factor $\exp[2\pi i(n^\mu_{\mathbf{r}_0} + \Lambda^\mu_{\mathbf{r}_0} + q)] = \exp[2\pi i(\Lambda^\mu_{\mathbf{r}_0} + q)]$, where $q = \frac{Q}{N}$, and we have used the fact that the $n^\mu_{\mathbf{r}}$ are integers. However, under a 2π rotation, an SU(N) spin picks up a phase factor, i.e., $\exp(2\pi i \Lambda^\mu_{\mathbf{r}_0}) = \exp(-2\pi i q)$, so that the factor $\exp[2\pi i(\Lambda^\mu_{\mathbf{r}_0} + q)] = 1$, and the unitary transformation U_μ preserves periodic boundary conditions.

Written in full, the Hamiltonian with flux inserted is

$$H[\Phi_\mu] = -\sum_{\mathbf{r}\mathbf{r}'\sigma} t_{\mathbf{r},\mathbf{r}'} \exp[-i\mathbf{A}^\sigma \cdot (\mathbf{r}-\mathbf{r}')] c^\dagger_{\mathbf{r}\sigma} c_{\mathbf{r}'\sigma} + J_K \sum_{\mathbf{r}} \vec{\lambda}_{\mathbf{r}} \cdot \vec{\Lambda}_{\mathbf{r}},$$
$$\mathbf{A}^\sigma = \delta^{\sigma\mu}\left(\frac{2\pi}{L_x}\right)\hat{\mathbf{x}}. \tag{6}$$

The process of flux insertion involves adiabatically increasing $\mathbf{A}^\sigma(t) = \mathbf{A}^\sigma \exp(-|t|/\tau)$ from zero at $t = -\infty$ to its full value at $t = 0$, taking $\tau \gg (1/T_K)$ to be much longer than the inverse Kondo temperature, so that the initial eigenstate $|\psi^0\rangle$ evolves smoothly into an excited eigenstate $|\psi_\Phi\rangle'$ of $H_{\rm KL}[\Phi_\mu]$ (see Fig. 1).

Since translational symmetry is preserved by flux insertion, and since the exponential of the canonical momentum $\exp(-iP_x)$ is the eigenstate of translation, it follows that the state retains a fixed canonical momentum $P_x(t) = P_x^0$, so that under a translation,

$$T_x|\psi_\Phi\rangle' = \exp\left(-iP_x^0\right)|\psi_\Phi\rangle'. \tag{7}$$

We can obtain the mechanical momentum P_x of the final state $|\psi_\Phi\rangle'$ by noting that, since this quantity is gauge invariant, it is unchanged when we gauge transform back into the original gauge. Now since $H_{\rm KL}[0] = U_\mu H_{\rm KL}[\Phi_\mu] U_\mu^\dagger$, it follows that $|\psi_\Phi\rangle = U_\mu|\psi_\Phi\rangle'$ is the corresponding transform of $|\psi_\Phi\rangle$ back into the original gauge. However, since the vector potential is now absent, the mechanical and canonical momentum coincide and can be determined from a translation:

$$T_x|\psi_\Phi\rangle = \exp(-iP_x)|\psi_\Phi\rangle. \tag{8}$$

Since $T_x|\psi_\Phi\rangle = (T_x U_\mu T_x^{-1}) T_x|\psi_\Phi\rangle'$, it follows that

$$\exp(-iP_x)|\psi_\Phi\rangle = \left(T_x U_\mu T_x^{-1}\right)\exp\left(-iP_x^0\right)|\psi_\Phi\rangle'. \tag{9}$$

Now $T_x U_\mu T_x^{-1}$ describes the effect of translating the operator U_μ by one lattice spacing in the $\hat{x}$ direction, so that

$$\left(T_x U_\mu T_x^{-1}\right) = \exp\left[\frac{2\pi i}{L_x}\sum_{\mathbf{r}} x_{\mathbf{r}}\left(n^\mu_{\mathbf{r}+\hat{x}} + \Lambda^\mu_{\mathbf{r}+\hat{x}} + q\right)\right] = \exp\left[\frac{2\pi i}{L_x}\sum_{\mathbf{r}} x_{\mathbf{r}-\hat{x}}\left(n^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}} + q\right)\right], \tag{10}$$

where inside the sum, we have shifted the x coordinate of the position vectors $\mathbf{r}$, $\mathbf{r} \to \mathbf{r} - \hat{x}$. Now naïvely, we might expect $x_{\mathbf{r}-\hat{x}} = x_{\mathbf{r}} - 1$. However, this is not the case with sites on the first layer of the crystal, for in this case, $x_{1-1} = x_0$, but the periodic boundary conditions mean that $x_0 = x_{L_x} = L_x = x_1 - 1 + L_x$. Thus, in general, $x_{\mathbf{r}-\hat{x}} = x_{\mathbf{r}} - 1 + L_x\delta_{r_1,1}$. Substituting this into Eq. (11), we obtain

$$\begin{aligned}\left(T_x U_\mu T_x^{-1}\right) &= \exp\left[\frac{2\pi i}{L_x}\sum_{\mathbf{r}}\left(x_{\mathbf{r}} - 1 + L_x\delta_{r_1,1}\right)\left(n^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}} + q\right)\right] \\ &= \exp\left[\frac{2\pi i}{L_x}\sum_{\mathbf{r}}\left(L_x\delta_{r_1,1} - 1\right)\left(n^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}} + q\right)\right]U_\mu \\ &= \exp\left[2\pi i\sum_{\mathbf{r}_\perp}\left(n^\mu_{1,\mathbf{r}_\perp} + \Lambda^\mu_{1,\mathbf{r}_\perp} + q\right)\right]\exp\left[-\frac{2\pi i}{L_x}\sum_{\mathbf{r}}\left(n^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}} + q\right)\right]U_\mu.\end{aligned} \tag{11}$$

The first term derives from the crystal boundary at $x_{\mathbf{r}} = 1$, derived from the shift of the x coordinates by L_x. However, since we have chosen a gauge where U_μ is invariant under such coordinate shifts, this prefactor is unity $\{\exp[2\pi i(n^\mu_{1,\mathbf{r}_\perp} + \Lambda^\mu_{1,\mathbf{r}_\perp} + q)] = 1\}$. Our final answer for the translated U_μ is then

$$\left(T_x U_\mu T_x^{-1}\right) = \exp\left[-\frac{2\pi i}{L_x}\sum_{\mathbf{r}}\left(n^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}} + q\right)\right]U_\mu. \tag{12}$$

We note that this answer is also obtained with Oshikawa's [1] original choice of $U_\mu = \exp[\frac{2\pi i}{L_x}\sum_{\mathbf{r}} x_{\mathbf{r}}(n^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}})]$, but in this case, the q dependence derives from the boundary term.

From Eq. (10), it then follows that

$$\exp(-iP_x)|\psi_\Phi\rangle = \exp\left(-iP_x^0\right)\exp\left[-\frac{2\pi i}{L_x}\sum_{\mathbf{r}}\left(n^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}} + q\right)\right]|\psi_\Phi\rangle, \tag{13}$$

i.e., flux insertion changes the mechanical momentum by

$$\Delta P_x = \frac{2\pi}{L_x}\sum_{r}\left(n^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}} + q\right). \tag{14}$$

For N_s spins per unit cell,

$$\Delta P_x = \frac{2\pi}{L_x} V[\nu^\mu + N_s(m^\mu + q)] \mod 2\pi, \tag{15}$$

where $V = L_x L_y \ldots L_D$ is the system volume, while $\nu^\mu = (1/V)\sum_{\mathbf{r}} n^\mu_{\mathbf{r}}$ and $m^\mu = (1/V)\sum_{\mathbf{r}} \Lambda^\mu_{\mathbf{r}}$ are the μth filling fraction and magnetization, respectively.

Alternatively, if we assume a Fermi liquid ground state, we can compute the change in momentum by observing that coupling to the gauge potential shifts the momentum of each μ-quasiparticle by $2\pi/L_x$, so that $\Delta P_x = \frac{2\pi}{L_x} N^\mu_F$ where N^μ_F is the number of μ-quasiparticles. The quasiparticle number operator $\tilde{n}_{\mathbf{k}\mu}$ is conserved in a Fermi liquid and jumps from 1 to 0 across the Fermi surface. This allows us to relate the shift in momentum to the volume of the μ-Fermi surface $V^\mu_{\rm FS} = N^\mu_F (2\pi)^D/V$:

$$\Delta P_x = \frac{2\pi}{L_x} V \left[\frac{V^\mu_{\rm FS}}{(2\pi)^D}\right]. \tag{16}$$

Comparing Eqs. (18) and (19), we find

$$VX \frac{V^\mu_{\rm FS}}{(2\pi)^D} = V[\nu^\mu + N_s(m^\mu + q)] + n_x L_x, \tag{17}$$

with $n_x \in \mathbb{Z}$. Now since the remainder term $n_x L_x$ can be calculated for a flux threading in any of the D directions, the remainder is also equal to $n_y L_y, \ldots n_D L_D$, where the n_i $(i = 1, D)$ are distinct integers for each direction. However, since the integer remainder is independent of direction, $n_x L_x = n_y L_y = \ldots n_D L_D$. If we choose the $L_x, L_y \ldots L_D$ to be coprime (no common denominators), it follows that n_x is proportional to each of the $n_y L_y, \ldots n_D L_D$, so that the remainder is a multiple of the full product, i.e., the volume $V = L_x \ldots L_D$. Factoring out the volume V, we obtain

$$\frac{V^\mu_{\rm FS}}{(2\pi)^D} = \nu^\mu + N_s(m^\mu + q) + n. \tag{18}$$

Since the Fermi surface volume is an intensive quantity, the remainder n is independent of the convenient choice of mutually coprime boundary lengths, and Eq. (19) is valid in the thermodynamic limit.

Finally, if we trace over all N Fermi surfaces, since the members of the Cartan subalgebra are traceless, it follows that $\sum_\mu m^\mu = 0$, so that

$$N v_c \frac{V_{\rm FS}}{(2\pi)^D} = n_e + N_s Q, \tag{19}$$

where $n_e = \sum_\mu \nu_\mu$, and we have restored the unit cell volume v_c and have dropped the integer remainder $p = nN$, with the understanding that the Fermi surface volume is only defined mod $(2\pi)^D$.

III. THE LINK WITH FRACTIONALIZATION

Traditionally, the localized spins of a Kondo lattice are written in terms of an Abrikosov pseudofermion representation:

$$\Lambda^a_{\mathbf{r}} = f^\dagger_{\mathbf{r}\sigma} \lambda^a_{\sigma\sigma'} f_{\mathbf{r}\sigma'}, \qquad (a = 1, N^2 - 1), \tag{20}$$

with a constraint on the local f-fermion (spinon) density $n^{(f)}_{\mathbf{r}} = \sum_\mu f^\dagger_{\mathbf{r}\mu} f_{\mathbf{r}\mu} = Q$, which determines the number of spinons contained in the Qth antisymmetric representation of SU(N). With hindsight, we now see that, since the constraint commutes with every operator involved in the proof, we could have used this representation from the outset, but by tacitly avoiding doing so, we avoided any lingering concerns about the constraint.

In the Abrikosov representation, the Kondo lattice Hamiltonian takes the form [4]

$$H_{\rm KL} = -\sum_{\mathbf{r}\mathbf{r}'} t_{\mathbf{r},\mathbf{r}'} c^\dagger_{\mathbf{r}\sigma} c_{\mathbf{r}'\sigma} - \frac{\tilde{J}_K}{N} \sum_{\mathbf{r}} c^\dagger_{\mathbf{r}\sigma} f_{\mathbf{r}\sigma} f^\dagger_{\mathbf{r}\sigma'} c_{\mathbf{r}\sigma'}, \tag{21}$$

which explicitly commutes with the constraint $n_{f\mathbf{r}} = Q$ and the number of conduction electrons $n_{c\mathbf{r}}$ at site $\mathbf{r}$. With the normalization $\mathrm{Tr}[\lambda^a \lambda^b] = (1 - \frac{1}{N})\delta^{ab}$ set by the Cartan subalgebra, the coupling constants of the Read-Newns form and the original model in Eq. (3) are related by $\tilde{J}_K = J_K(N - 1)$.

The Cartan elements are now represented by $\Lambda^\mu_{\mathbf{r}} = n^\mu_{f\mathbf{r}} - Q/N$, so that the gauge transformation in Eq. (4) that imposes the flux insertion is given by

$$U^\mu = \exp\left[\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} \left(n^\mu_{c\mathbf{r}} + n^\mu_{f\mathbf{r}}\right)\right]. \tag{22}$$

Equation (22) is effectively a large-gauge transformation that counts the f-spinons as quasiparticles. The conduction electrons and spinons transform identically under the flux insertion:

$$U^\dagger_\mu \begin{pmatrix} c^\dagger_{\mathbf{r}\sigma} \\ f^\dagger_{\mathbf{r}\sigma} \end{pmatrix} U_\mu = \exp(-i\mathbf{A}^\sigma \cdot \mathbf{r}) \begin{pmatrix} c^\dagger_{\mathbf{r}\sigma} \\ f^\dagger_{\mathbf{r}\sigma} \end{pmatrix}. \tag{23}$$

In other words, the structure of the unitary transformation, forced upon us by the Kondo coupling, means that the spinons behave exactly as charged particles under the flux attachment, consistent with a fractionalization of spins into heavy electrons in the Fermi liquid phase. Remarkably then, the seeds of fractionalization are present in the original Oshikawa [1] gauge transformation.

The final form of the Luttinger sum rule

$$\frac{V^\mu_{\rm FS}}{(2\pi)^D} = \nu^\mu + \nu^\mu_f, \qquad \mod(1), \tag{24}$$

where $\nu^\mu_f = \frac{N_s}{V}\sum_{\mathbf{r}} n^\mu_{f\mathbf{r}} = N_s(m^\mu + q)$ is the number of spinons with spin index μ per unit cell, is not a surprise because the U(1) $\times$ SU(N) gauge transformation in Eq. (22) audits every spinon entangled into the Fermi sea.

Traditionally, the Kondo-Fermi surface expansion is interpreted by identifying the Kondo-Hamiltonian as the strong coupling renormalization of a periodic Anderson model with the same filling [3]. However, a Kondo lattice Hamiltonian has no knowledge of its high energy origins. From a renormalization group perspective, the Kondo lattice lies on the common scaling trajectory of many high-energy "microscopic" Hamiltonians. Indeed, the model is entirely agnostic as to the origin of the local moments, and they need not have an electronic origin at all; for instance, they equally could be nuclear spins, with a Kondo interaction derived from hyperfine interactions. The main point is that, since the Kondo lattice has no knowledge of its high-energy origins, fractionalization in the Kondo lattice is an emergent property. This alternate interpretation allows us to contemplate the possibility that different kinds

of spin fractionalization may develop in the approach to magnetism or spin liquid behavior.

IV. KONDO-HEISENBERG MODEL

We now consider an extension of our results to a Kondo-Heisenberg model: a Kondo lattice with additional Heisenberg interactions $H_{\rm KH} = H_{\rm KL} + H_H$, where now

$$H_H = \sum_{\langle \mathbf{r}\mathbf{r}'\rangle} J_{\mathbf{r},\mathbf{r}'}\, \vec{\Lambda}_{\mathbf{r}} \cdot \vec{\Lambda}_{\mathbf{r}'}. \tag{25}$$

From Doniach's original arguments [10], we know that, for large enough T_K, the Kondo interaction will stabilize a Fermi liquid, in which case, we expect Oshikawa's [1] result to generalize to the Kondo-Heisenberg model. We are particularly interested in the case of frustrated Kondo lattices, where in the limit of small T_K, rather than forming a state of long-range magnetic order, the system develops into a spin liquid, preserving the Fermi surface of the underlying spinons. We shall show that Oshikawa's [1] theorem can be extended to this case.

Naïvely, one might expect flux insertion to only affect charge particles, leaving the Heisenberg term alone. However, the unitary transformation that accomplishes flux insertion in Eq. (4), $U_\mu = \exp[\frac{2\pi i}{L_x}\sum_{\mathbf{r}} x_{\mathbf{r}}(n^\mu_{\mathbf{r}} + \Lambda^\mu_{\mathbf{r}} + q)]$, adds a charge and a spin flux to the system, thus affecting the Heisenberg interaction terms. Under the gauge transformation, the local moments transform under the adjoint representation of SU(N). To keep track of these transformations, its simpler to switch to a Coqblin-Schrieffer representation of the local moments $\Lambda^{\sigma\sigma'}_{\mathbf{r}} = f^\dagger_{\mathbf{r}\sigma} f_{\mathbf{r}\sigma'} - \frac{Q}{N}\delta_{\sigma\sigma'}$, so that the Heisenberg interaction takes the form

$$H_H = \frac{1}{N}\sum_{\langle \mathbf{r}\mathbf{r}'\rangle} \tilde{J}_{\mathbf{r},\mathbf{r}'} \Lambda^{\sigma\sigma'}_{\mathbf{r}} \Lambda^{\sigma'\sigma}_{\mathbf{r}'}, \tag{26}$$

where $\tilde{J}_{\mathbf{r},\mathbf{r}'} = J_{\mathbf{r},\mathbf{r}'}(N-1)$. Under the flux insertion, $f_{\mathbf{r}\sigma} \to \exp(i\mathbf{A}^\sigma \cdot \mathbf{r}) f_{\mathbf{r}\sigma}$, so that under the gauge transformation in Eq. (26),

$$\Lambda^{\sigma\sigma'}_r \to U^\dagger_\mu \Lambda^{\sigma\sigma'}_r U_\mu = \exp[-i(\mathbf{A}^\sigma - \mathbf{A}^{\sigma\prime}) \cdot \mathbf{r}]\Lambda^{\sigma\sigma\prime}_r, \tag{27}$$

which describes the transformation of the spin operator under the adjoint representation of SU(N), corresponding to a slow twist of the local moments about the μ axis, created by the spin component of U_μ, through an angle $2\pi(x/L_x)$ that increases from 0 to 2π across the sample.

Using these results, we can write Heisenberg-Kondo model with a flux insertion in the μ spin channel as

$$H_{\rm KH}[\Phi_\mu] = -\sum_{\mathbf{r}\mathbf{r}'} t_{\mathbf{r},\mathbf{r}'} \exp[-i\mathbf{A}^\sigma \cdot (\mathbf{r}-\mathbf{r}')] c^\dagger_{\mathbf{r}\sigma} c_{\mathbf{r}'\sigma} + \frac{\tilde{J}_K}{N}\sum_{\mathbf{r}} c^\dagger_{\mathbf{r}\sigma} c_{\mathbf{r}\sigma'} \Lambda^{\sigma\sigma}_{\mathbf{r}} \tag{28}$$

$$+\frac{1}{N}\sum_{\langle \mathbf{r}\mathbf{r}'\rangle} \tilde{J}_{\mathbf{r},\mathbf{r}'} \exp[-i(\mathbf{A}^\sigma - \mathbf{A}^{\sigma'}) \cdot (\mathbf{r}-\mathbf{r}')] \Lambda^{\sigma\sigma'}_{\mathbf{r}} \Lambda^{\sigma'\sigma}_{\mathbf{r}'}. \tag{29}$$

The gauge field inside the Heisenberg term

$$\exp[-i\mathbf{A}\cdot(\mathbf{r}-\mathbf{r}')(\delta_{\sigma\mu} - \delta_{\sigma'\mu})] = \exp[-i\mathbf{A}\cdot(\mathbf{r}-\mathbf{r}')\delta_{\sigma\mu}]\exp[i\mathbf{A}\cdot(\mathbf{r}-\mathbf{r}')\delta_{\sigma'\mu}], \qquad \left(\mathbf{r}' \underset{\sigma'}{\overset{\sigma}{\rightleftharpoons}} \mathbf{r}\right), \tag{30}$$

can be interpreted as the product of two Peierls' insertions associated with a spinon exchange: a σ spinon moving from $\mathbf{r}'$ to $\mathbf{r}$, and a σ' spinon moving in the opposite direction. The derivation and final form of the Luttinger sum rule for the Fermi liquid now follows precisely the same route as in the Kondo model. The key identity in Eq. (15) still holds, allowing us to generalize the Oshikawa [1] result in Eq. (1) to the Fermi liquid phases of the SU(N) Kondo-Heisenberg model.

Since our flux insertion works for arbitrary N, it allows us to explicitly examine how the wave function $|\Psi_0\rangle$ evolves at large N, allowing us to the explicit evolution under the flux attachment and subsequent gauge transformation:

$$|\Psi_0\rangle \xrightarrow{\Phi_0} |\Psi_\Phi\rangle' \xrightarrow{U_\mu} |\Psi\rangle. \tag{31}$$

At large N, the ground state wave function is accurately determined by a Gutzwiller wave function:

$$|\Psi_0\rangle = P_G \prod_{\mathbf{k}\in {\rm FS},\sigma} (\alpha_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma} + \beta_{\mathbf{k}} f^\dagger_{\mathbf{k}\sigma})|0\rangle, \tag{32}$$

where the product runs over all wave vectors enclosed by the Fermi surface, and $P_G = \prod_{\mathbf{r}} \delta_{n_f(\mathbf{r}),Q}$ projects out the component of the wave function with $n_f(\mathbf{r}) = Q$ at each site, while the hybridized operators $\alpha_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma} + \beta_{\mathbf{k}} f^\dagger_{\mathbf{k}\sigma}$ define the quasiparticles of the mean-field Hamiltonian. In fact, the Gutzwiller projection P_G can be replaced by an average constraint in the large N limit, but here, we shall keep it for greater generality.

In the large N limit, the dynamics of the wave function are determined by evolution under a time-dependent, translationally invariant mean-field Hamiltonian which preserves the momenta of the quasiparticle states, leaving the Fermi surface unchanged. After the flux insertion, the mean-field wave function then has the form

$$|\Psi_\Phi\rangle' = P_G \prod_{\mathbf{k}\in {\rm FS},\sigma} (\alpha_{\mathbf{k}\sigma}[\Phi] c^\dagger_{\mathbf{k}\sigma} + \beta_{\mathbf{k}\sigma}[\Phi] f^\dagger_{\mathbf{k}\sigma})|0\rangle, \tag{33}$$

where the coefficients $\alpha_{\mathbf{k}\sigma}[\Phi]$ and $\beta_{\mathbf{k}\sigma}[\Phi]$ differ from their zero field value by terms of order $O(1/L)$. Now if we Fourier transform Eq. (26), the transformation of the electron and spinon fields under U_μ in momentum space is given by

$$U_\mu \begin{pmatrix} c^\dagger_{\mathbf{k}\sigma} \\ f^\dagger_{\mathbf{k}\sigma} \end{pmatrix} U^\dagger_\mu = \begin{pmatrix} c^\dagger_{\mathbf{k}+\mathbf{A}^\sigma\sigma} \\ f^\dagger_{\mathbf{k}+\mathbf{A}^\sigma\sigma} \end{pmatrix}, \tag{34}$$

so that under the unitary transformation U_μ, $U_\mu|\psi_\Phi\rangle' = |\psi_\Phi\rangle$ is given by

$$|\psi_\Phi\rangle = P_G \prod_{\mathbf{k}\in FS,\sigma} \left(\alpha_{\mathbf{k}\sigma}[\Phi]c^\dagger_{\mathbf{k}+\mathbf{A}^\sigma\sigma} + \beta_{\mathbf{k}\sigma}[\Phi]f^\dagger_{\mathbf{k}+\mathbf{A}^\sigma\sigma}\right)|0\rangle. \tag{35}$$

If we translate this state in the x direction, then since

$$T_x\begin{pmatrix} c^\dagger_{\mathbf{k}\sigma} \\ f^\dagger_{\mathbf{k}\sigma}\end{pmatrix}T_x^{-1} = \exp(-ik_x)\begin{pmatrix} c^\dagger_{\mathbf{k}\sigma} \\ f^\dagger_{\mathbf{k}\sigma}\end{pmatrix}, \tag{36}$$

it follows that the momentum of the final state P_x is given by

$$T_x|\psi_\Phi\rangle = \exp(-iP_x)|\psi_\Phi\rangle, \tag{37}$$

where

$$P_x = \sum_{\mathbf{k}\in FS,\sigma}\left(k_x + A^\sigma_x\right) = P^{(0)}_x + \frac{2\pi}{L_x}\frac{V}{(2\pi)^D}V^\mu_{\rm FS}, \tag{38}$$

so we see that the shift in momentum per quasiparticle is precisely $A^\sigma_x = \frac{2\pi}{L_x}$ in the μ band.

V. U(1) SPIN LIQUID

The fascinating aspect of this result is that it also allows us to apply the flux attachment idea to a pure Heisenberg model H_H. The Heisenberg model with a spin twist $H_H[\Phi_\mu] = U^\dagger_\mu H_H U_\mu$ is written

$$H_H[\Phi_\mu] = \frac{1}{N}\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}\tilde{J}_{\mathbf{r},\mathbf{r}'}\exp[-i(\mathbf{A}^\sigma - \mathbf{A}^{\sigma'})\cdot(\mathbf{r}-\mathbf{r}')]\Lambda^{\sigma\sigma'}_{\mathbf{r}}\Lambda^{\sigma'\sigma}_{\mathbf{r}'},$$
$$\mathbf{A}^\sigma = \delta^{\sigma\mu}\frac{2\pi}{L_x}\hat{x} \tag{39}$$

and the corresponding gauge transformation is then

$$U^s_\mu = \exp\left[\frac{2\pi i}{L_x}\sum_{\mathbf{r}} x_{\mathbf{r}}\left(\Lambda^\mu_{\mathbf{r}} + q\right)\right]. \tag{40}$$

In this case, the translated gauge transformation takes the form

$$\left(T_x U^s_\mu T_x^{-1}\right) = \exp\left[-\frac{2\pi i}{L_x}\sum_{\mathbf{r}}\left(\Lambda^\mu_{\mathbf{r}} + q\right)\right]U^s_\mu, \tag{41}$$

so the change in momentum associated with the flux insertion is then

$$\Delta P_x = \frac{2\pi}{L_x}\sum_r\left(\Lambda^\mu_{\mathbf{r}} + q\right)$$
$$= \frac{2\pi}{L_x}V(m^\mu + q) \mod 2\pi, \tag{42}$$

where $V = L_xL_y\ldots L_D$ is the volume, and $m^\mu = \frac{1}{V}\sum_{\mathbf{r}}\Lambda^\mu_{\mathbf{r}}$ is the magnetization, and we have assumed $n_s = 1$ local moment per unit cell. Using Abrikosov fermions, $\Lambda^\mu_{\mathbf{r}} + q = n^\mu_{f\mathbf{r}}$ is the number of μ-spinons at site $\mathbf{r}$, so we can interprete $V(m^\mu + q)$ as the number of spinons with spin component μ. In other words, under a flux attachment, each spinon with spin component μ in the ground state acquires a momentum $\frac{2\pi}{L}$.

A U(1) spin liquid can be thought of as an incompressible neutral Fermi liquid. In Appendix A, we demonstrate that such a state is energetically favored in the large N limit over the dimer phase, and the π-flux phase on a square lattice over a range of q. To see how its momentum changes under a flux attachment, consider the model ground state provided by a Gutzwiller wave function

$$|\Psi_0\rangle = P_G\prod_{\mathbf{k}\in\rm FS,\sigma} f^\dagger_{\mathbf{k}\sigma}|0\rangle, \tag{43}$$

where, as in the Kondo lattice, $P_G = \prod_{\mathbf{r}}\delta_{n_f(\mathbf{r}),Q}$ is a Gutzwiller projection onto states with Q elementary spinons at each site. Now the translation operator commutes with P_G, and since $T_x f^\dagger_{\mathbf{k}\sigma}T_x^{-1} = \exp(-ik_x)f^\dagger_{\mathbf{k}\sigma}$, it follows that this state has the initial momentum $P^{(0)}_x = \sum_{\mathbf{k}\in\rm FS,\sigma}k_x$. In the large N limit (see Appendix B), the time evolution of the state is given by a time-dependent mean-field Hamiltonian that is explicitly translationally invariant, so that under a flux attachment, the canonical momenta of the spinons are entirely unchanged. In a one-band fluid of spinons, the corresponding Gutziller ground state is then unchanged after the flux attachment $|\Psi_\Phi\rangle' = |\Psi_0\rangle$. If we now revert back to the original gauge, since $U_\mu f^\dagger_{\mathbf{k}\sigma}U^{-1}_\mu = f^\dagger_{\mathbf{k}+\mathbf{A}^\sigma,\sigma}$, it follows that

$$|\Psi_\Phi\rangle = U_\mu|\Psi_\Phi\rangle' = P_G\prod_{\mathbf{k}\in\rm FS,\sigma} f^\dagger_{\mathbf{k}+\mathbf{A}^\sigma,\sigma}|0\rangle, \tag{44}$$

corresponding to a Fermi sea in which the spinon momenta are shifted by $\mathbf{A}^\sigma = 2\pi/L_x\delta^{\sigma\mu}\hat{x}$, i.e.,

$$\Delta P_x = \frac{2\pi}{L_x}V\frac{V^\mu_{\rm FS}}{(2\pi)^D}. \tag{45}$$

By comparing this result with Eq. (46), we then obtain

$$\frac{V^\mu_{\rm FS}}{(2\pi)^D} = (m_\mu + q) \mod 1. \tag{46}$$

We emphasize that this result remains valid at arbitrary N if the ground state is smoothly connected to the U(1) spin liquid state in Eq. (44). For SU(2), a similar result was also obtained in Ref. [11].

VI. DISCUSSION

It is interesting to consider the implications of our results for the FL* phase of the Kondo lattice model, in which decoupled spin liquid and conduction electrons co-exist in a state of unbroken symmetry. Earlier work on $S = \frac{1}{2}$ Kondo systems [12,13] has interpreted this phase as a Z_2 spin liquid coexisting with a Fermi liquid. Flux insertion then drives a transition between two topologically degenerate ground states characterized by the presence or absence of vizon states that carry Z_2 flux. However, is the FL* phase necessarily topologically ordered?

Our result on the Kondo-Heisenberg model suggests an alternate interpretation of the FL* phase as the coexistence of a U(1) spin liquid with an electronic Fermi liquid [14]. There are in principle two phases:

(1) the heavy Fermi liquid, a Higgs phase in which the U(1) gauge field of the spinons is locked to the electromagnetic U(1) fields of the conduction electrons, giving rise to a single unified Fermi surface of heavy electrons; and

(2) the FL* in which the U(1) gauge fields of the conduction electrons and spinons are decoupled, so that one is neutral, the other charged.

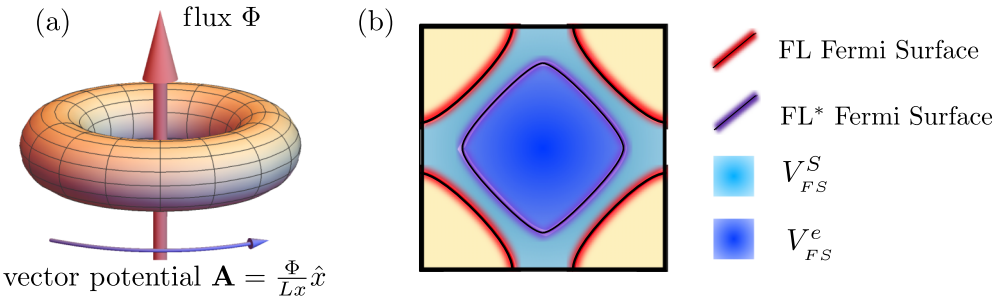

FIG. 2. (a) Flux attachment in the Kondo-Heisenberg model. Threading a flux results in a twist in the U(1) gauge potential and a twist in the spin orientations, imparting momentum to the electrons and the spinons. (b) The total momentum is proportional to the combined Fermi surface volume of the electrons and spinons (red). In the FL* phase, the spinons decouple from the electrons to form a U(1) spin liquid, resulting in a smaller Fermi surface (purple) that only counts the electrons.

Oshikawa's [1] theorem, extended to the Kondo-Heisenberg model makes no judgment on which phase one is in, simply predicting that the combined volume of the Fermi surfaces (see Fig. 2)

$$\frac{V^{\mu}_{\rm FS}}{(2\pi)^D} = \frac{V^{\mu,S}_{\rm FS}}{(2\pi)^D} + \frac{V^{\mu,e}_{\rm FS}}{(2\pi)^D} = N_s(m_\mu + q) + \nu^\mu. \tag{47}$$

If the spin liquid decouples from the electronic fluid, then assuming that the U(1) spin liquid is isomorphic to that of the pure Heisenberg model in Sec. V, the volume of the spinon Fermi surface is given by $\frac{V^{\mu,S}_{\rm FS}}{(2\pi)^D} = N_s(m_\mu + q)$. In this case, the remaining electronic fluid has a Fermi surface volume

$$\frac{V^{\mu,e}_{\rm FS}}{(2\pi)^D} = \nu_\mu. \tag{48}$$

From this perspective, the FL* is understood simply as two decoupled fluids, both of which respond to the flux attachment. One of the interesting aspects of this line of reasoning is that it goes against a commonly held view-point that fractionalization in higher dimensional systems is intimately associated with a topological ground state. It suggests instead that fractionalization does not require such inevitable linkage, and it opens the way for an interpretation of the Kondo effect as a nontopological fractionalization of local moments.

Such U(1) spin liquids are expected from large N treatments [9,15–18] and found in variational studies of Heisenberg-related models [19]. The breakdown of the Kondo effect and the the resulting decoupling between between spinons and electrons in these spin liquids would have sharp experimental signatures owing to the coexistence of a charge-neutral spinon Fermi surface. Such spin liquids are among the prime candidates to account for the anomalous signatures in thermal conductivity [20], spin susceptibility [19], and anomalous quantum oscillations [21–23] that have been observed in various experiments. While this paper makes no claim on the resolution of these experimental puzzles, an exact result on the volume of the spinon Fermi surface in a U(1) spin liquid is a valuable benchmark to compare against. Indeed, in a two-fluid picture [24], Fermi surface volumes intermediate between the large and small Fermi surface may be fit to a heuristic form $V_{\rm FS} = N_s q + f\nu$ to estimate f, the fraction of the spins that are gauge coupled to the electrons.

One of the unsolved questions is whether Oshikawa's [1] approach can be extended to other models. Central to the current derivation of the Luttinger sum rule is the identification of a U(1) gauge symmetry associated with *each* of the N spin components and the presence of translational symmetry. There are two models that fail these requirements:

(1) the Kondo impurity model, where fractionalization and the large N limit tell us that the scattering phase shift is given by $\delta = \pi Q/N$ [7]; and

(2) the family of symplectic SP($2N$) symmetric Kondo lattices, important for extending the notion of pairing to the large N limit [25,26].

At first sight, the absence of a conserved momentum would seem to preclude using flux attachment on the impurity Kondo model; however, by representing the impurities as left-moving particles in a fluid of right-moving electrons, as in Bethe-Ansatz solutions of this problem [15], it may be possible to restore translational invariance required for flux attachment.

Likewise, the absence of many U(1) subgroups in SP($2N$) appears to sabotage the application of Oshikawa's [1] theorem to this case. However, here, too, there may be a way out, for the total number of "up" electrons and spinons is still a conserved U(1) invariant, so that if we attach a flux to all the up electrons and spinons, a Fermi surface sum rule may still be possible. These topics can be considered in future work.

ACKNOWLEDGMENTS

This paper was supported by National Science Foundation Grant No. DMR-1830707 (PC, TH).

APPENDIX A: STABILITY OF THE U(1) SPIN LIQUID IN THE LARGE N LIMIT

The nearest-neighbor Heisenberg model is described by the path integral

$$\mathcal{Z} = \int \mathcal{D}[f^\dagger, f\ , \lambda] \exp\left[-\int_0^\beta d\tau\, \mathcal{L}(\tau) \right],$$
$$\mathcal{L} = \sum_{\mathbf{r}} [f^\dagger_{\mathbf{r}\sigma}(\partial_\tau + \lambda_{\mathbf{r}}) f_{\mathbf{r}\sigma} - \lambda_{\mathbf{r}} Q] + H_H, \tag{A1}$$

TABLE I. Summary of large N mean-field results.

State	Fermionic excitations	Gap equation	Ground state energy
Dimer	$\epsilon_{\pm} = \lambda \mp \chi$	$\chi = \tilde{J}_H q$	$\frac{2E}{\tilde{J}_H NL^2} = -q^2$
π-Flux	$\epsilon_{\mathbf{k}\pm} = \lambda \pm 2\chi\sqrt{\cos^2 k_x + \cos^2 k_y}$	$\frac{2\lvert\chi\rvert}{\tilde{J}_H} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} n_{\mathbf{k}-}\sqrt{\cos^2 k_x + \cos^2 k_y}$	$\frac{2E}{\tilde{J}_H NL^2} = -[\int \frac{d^2\mathbf{k}}{(2\pi)^2} n_{\mathbf{k}-}\sqrt{\cos^2 k_x + \cos^2 k_y}]^2$
U(1) SL	$\epsilon_{\mathbf{k}} = \lambda - 2\chi(\cos k_x + \cos k_y)$	$\frac{2\lvert\chi\rvert}{\tilde{J}_H} = \int \frac{d^2\mathbf{k}}{(2\pi)^2}(\cos k_x + \cos k_y)\, n^f_{\mathbf{k}}$	$\frac{2E}{\tilde{J}_H NL^2} = -\big[\int \frac{d^2\mathbf{k}}{(2\pi)^2}(\cos k_x + \cos k_y)\, n^f_{\mathbf{k}}\big]^2$

with a summation convention over spin indices $\sigma = (1, N)$. The Heisenberg-Hamiltonian H_H is represented in terms of the Abrikosov-Fermi fields $f^\dagger, f$ as

$$H_H = -\frac{\tilde{J}_H}{N}\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}(f^\dagger_{\mathbf{r}\sigma} f_{\mathbf{r}'\sigma})(f^\dagger_{\mathbf{r}'\sigma'} f_{\mathbf{r}'\sigma'}), \tag{A2}$$

while the constraint on the local fermion number is implemented by an integral over the chemical potential $\lambda_{\mathbf{r}}$ [7]. Decoupling the four-fermion term by a Hubbard-Stratonovich transformation to the resonating valence bond fields $\chi_{\mathbf{r}\mathbf{r}'}$ and approximating the integral by the saddle point action leads to the mean-field Hamiltonian

$$H_{\rm MF} = -\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}\chi_{\mathbf{r}\mathbf{r}'} f^\dagger_{\mathbf{r}\sigma} f_{\mathbf{r}'\sigma} + \frac{N}{\tilde{J}_H}\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}|\chi_{\mathbf{r}\mathbf{r}'}|^2 + \sum_{\mathbf{r}}\lambda_{\mathbf{r}}(f^\dagger_{\mathbf{r}\sigma} f_{\mathbf{r}\sigma} - Q), \tag{A3}$$

which becomes exact in the limit of large N. We compare the energies of the spin liquid (SL), dimer, and π-flux phases of this Hamiltonian on a square lattice in two dimensions with linear dimension L in units of the lattice constant. Each of these phases has $\lambda_{\mathbf{r}} = \lambda$.

For the dimer or Peierls phase [27], $\chi_{\mathbf{r}\mathbf{r}'} = 0$ on all but one of the nearest-neighbor bonds to each site, as shown in Fig. 3(a). For the π-flux phase [9], $\chi_{\mathbf{r}\mathbf{r}'} = |\chi|\exp(i\pi/4)$ if $\mathbf{r} \to \mathbf{r}'$ is oriented along the arrows in Fig. 3(b). For the uniform U(1) spin liquid, $\chi_{\mathbf{r}\mathbf{r}'} = \chi \in \mathbb{R}$ for all bonds [Fig. 3(c)]. Table I summarizes the results of the large N mean-field analysis for these states when $q = Q/N \leqslant \frac{1}{2}$, and the ground state energies are compared in Fig. 3(d). Near half-filling, the Peierls phase has the lowest energy. However, for low filling up to $q \sim 0.3$, the lowest energy state is the uniform U(1) spin liquid. For intermediate filling $0.3 < q < 0.48$, the flux phase is most stable.

When $q \ll 1$, the dispersion of the filled states is approximately quadratic, and we obtain the following analytical expressions for the ground state energy

$$\frac{E}{NV\tilde{J}_H} = \begin{cases} -\big[\frac{1-(1-2\pi q)^{3/2}}{3\pi}\big]^2 \simeq -q^2, & \pi\text{-flux} \\ -\frac{q^2}{2}, & \text{dimer} \\ -2q^2\big(1-\frac{\pi q}{2}\big)^2 \simeq -2q^2, & \text{SL} \end{cases}, \tag{A4}$$

so for small q, the uniform spin liquid is the most energetically favorable state. We note that, while the dimer phase is stable only near $q = \frac{1}{2}$, similar phases may be present and favorable at other rational fillings. For instance, r-site ring polymer states have energy $E/(NV\tilde{J}_H) = -q^2$ at $q = 1/r$. At $q = \frac{1}{4}$, the four-site plaquette states have lower energy than the uniform U(1) spin liquid. As q becomes smaller, the likely ground state involves larger and larger decoupled clusters with vanishing energy differences ΔE from the U(1) spin liquid. Above temperatures of the order of ΔE, the system behaves like a spin liquid. Additionally, on finite-sized systems, incommensuration between the cluster size and the system size may frustrate the valence bond crystal and favor the spin liquid.

APPENDIX B: FLUX INSERTION IN THE LARGE N LIMIT OF HEISENBERG MODEL

In this section, we explicitly demonstrate the flux insertion and concomitant change in momentum in the Heisenberg

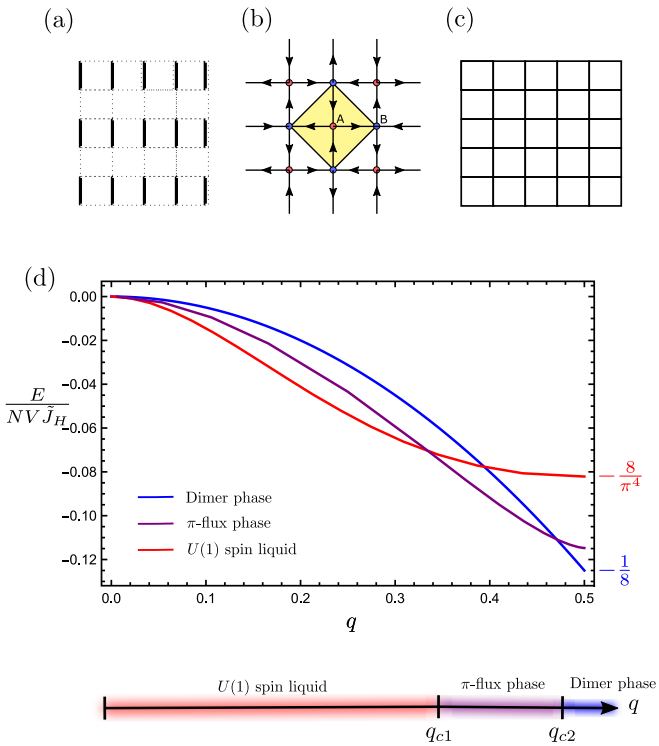

FIG. 3. (a) π-flux phase: the phase of the bond order parameter χ is positive along the direction of the arrows. The unit cell (yellow) is expanded to include two inequivalent sites A and B, corresponding to a reduced Brillouin zone. (b) Peierls phase in which the spin on each site forms a dimer with its nearest neighbor and decouples from the lattice. (c) Comparison of ground state energies for the spin liquid (red), flux phase (purple), and dimer phase (blue). The dashed curve is an analytical approximation for the spin liquid ground state energy valid at small q. For a range of filling $q < q_{c1} \sim 0.3$, the uniform U(1) spin liquid is stable with respect to the flux and dimer phases.

model on a square lattice in the limit of large N, in terms of the mean-field Hamiltonian in Eq. (A3). We discuss the response of the global U(1) gauge corresponding to the phase of the bond order parameters to the insertion of the flux and explicitly show that the volume of the spinon Fermi surface is given by Eq. (2). As opposed to the main text, we consider a flux that couples to P of the N spin degrees of freedom, so that the effect of the flux threading on the relative change in the ground state energy, for instance, is nonvanishing in the large N limit.

The Heisenberg model in the presence of such a flux is given by

$$H_H[\Phi_{\{\mu\}}] = -\frac{\tilde{J}_H}{N}\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}(f^\dagger_{\mathbf{r}\sigma}f_{\mathbf{r}'\sigma})(f^\dagger_{\mathbf{r}'\sigma'}f_{\mathbf{r}'\sigma'}) \times \exp[-i(\mathbf{A}^\sigma - \mathbf{A}^{\sigma'})\cdot(\mathbf{r}-\mathbf{r}')], \tag{B1}$$

where $\mathbf{A}^\sigma = (2\pi/L_x)\hat{x}\sum_{\{\mu\}}\delta_{\sigma\mu}$, with the sum over P spin channels to which the flux is coupled, where $\{\mu\} = \{\mu_1\ldots,\mu_P\}$. In the large N limit, this is exactly captured by the mean-field Hamiltonian

$$H_{\rm MF}[\Phi_{\{\mu\}}] = -\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}\chi_{\mathbf{r}\mathbf{r}'}\exp[-i\mathbf{A}^\sigma\cdot(\mathbf{r}-\mathbf{r}')]f^\dagger_{\mathbf{r}\sigma}f_{\mathbf{r}'\sigma} + \frac{N}{\tilde{J}_H}\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}|\chi_{\mathbf{r}\mathbf{r}'}|^2 + \sum_{\mathbf{r}}\lambda_{\mathbf{r}}(f^\dagger_{\mathbf{r}\sigma}f_{\mathbf{r}\sigma} - Q). \tag{B2}$$

The saddle point condition for a uniform order parameter leads to the self-consistency equation

$$\chi_{\mathbf{r}\mathbf{r}'} = |\chi|\exp[-i\mathcal{A}\cdot(\mathbf{r}-\mathbf{r}')] = \frac{\tilde{J}_H}{2NV}\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}\langle f^\dagger_{\mathbf{r}'\sigma}f_{\mathbf{r}\sigma}\rangle\exp[i\mathbf{A}^\sigma\cdot(\mathbf{r}-\mathbf{r}')], \tag{B3}$$

where $\mathcal{A}$ is the global (spin-independent) U(1) gauge potential, and V is the volume of the system (with the unit cell volume set to unity). With this, the mean-field Hamiltonian is diagonal in momentum space

$$H_{\rm MF}[\Phi_{\{\mu\}}] = \sum_{\mathbf{k}}\epsilon_{\mathbf{k}+\mathcal{A}+\mathbf{A}^\sigma}f^\dagger_{\mathbf{k}\sigma}f_{\mathbf{k}\sigma} + \frac{2NV|\chi|^2}{\tilde{J}_H} - \lambda QV, \tag{B4}$$

where $\epsilon_{\mathbf{k}} = -2\chi(\cos k_x + \cos k_y) + \lambda$ is the dispersion of the f-fermions. Recall that the ground state is the same as before the flux insertion since momentum is conserved throughout the process. The final state $|\psi_\Phi\rangle' = \Pi_{\mathbf{k}\in{\rm FS},\sigma}f^\dagger_{\mathbf{k}\sigma}|0\rangle$ and the canonical x momentum $P^0_x = \sum_{\mathbf{k}\in{\rm FS},\sigma}\mathbf{k} = 0$ as before the flux insertion. The gauge transformation that removes the flux is now given by

$$U^s_{\{\mu\}} = \exp\left[\frac{2\pi i}{L_x}\sum_{\mathbf{r},\{\mu\}}x_{\mathbf{r}}\left(\Lambda^\mu_{\mathbf{r}} + q\right)\right]. \tag{B5}$$

To transform back to the original gauge, we note that

$$U^s_{\{\mu\}}f^\dagger_{\mathbf{k}\sigma}U^{s\dagger}_{\{\mu\}} = \frac{1}{\sqrt{V}}\sum_{\mathbf{r}}U^s_{\{\mu\}}f^\dagger_{\mathbf{r}\sigma}U^{s\dagger}_{\{\mu\}}\exp(i\mathbf{k}\cdot\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{r}}f^\dagger_{\mathbf{r}\sigma}\exp[i(\mathbf{k}+\mathbf{A}^\sigma)\cdot\mathbf{r}] = f^\dagger_{\mathbf{k}+\mathbf{A}^\sigma\sigma}. \tag{B6}$$

We transform the ground state to this gauge $|\psi_\Phi\rangle \equiv U^s_{\{\mu\}}|\psi_\Phi\rangle' = \Pi_{\mathbf{k}\in{\rm FS},\sigma}f^\dagger_{\mathbf{k}+\mathbf{A}^\sigma,\sigma}|0\rangle$ and evaluate the physical momentum

$$P_x = \sum_{\mathbf{k}\in{\rm FS},\sigma}(\mathbf{k}+\mathbf{A}^\sigma) = \sum_{\mathbf{k}\in{\rm FS},\sigma}\mathbf{A}^\sigma = PV\frac{V_{\rm FS}}{(2\pi)^2}\frac{2\pi}{L_x}, \tag{B7}$$

when P out of N spin components are coupled to the flux. In this case, the change in momentum on flux insertion can be independently computed following the arguments in the main text [Eqs. (42)–(46)] to yield

$$\Delta P_x = V\left(\frac{2\pi}{L_x}\right)Pq, \tag{B8}$$

when the ground state is unpolarized. Comparing with Eq. (B7), we find the volume of the Fermi surface to be $V_{\rm FS} = (2\pi)^2 q$, consistent with Eq. (46).

As the flux is inserted, the global U(1) gauge potential $\mathcal{A}$ adjusts in response to preserve a zero total spinon current. Symmetry dictates that $\mathcal{A}\parallel\hat{x}$, and the new self-consistent value of $\mathcal{A}$ is determined by the saddle point condition $\partial_{\mathcal{A}}E = 0$ leading to

$$\sum_{\mathbf{k}\sigma}(\partial_{\mathcal{A}}\epsilon_{\mathbf{k}+\mathcal{A}+\mathbf{A}^\sigma})n^f_{\mathbf{k}} = -2|\chi|\frac{\partial}{\partial_{\mathcal{A}}}\sum_{\mathbf{k}\sigma}\cos(\mathcal{A}+\mathbf{A}^\sigma)\cos k_x n^f_{\mathbf{k}} = 2|\chi|\left(\sum_{\mathbf{k}}\cos k_x n^f_{\mathbf{k}}\right)\left[\sum_\sigma\sin(\mathcal{A}+\mathbf{A}^\sigma)\right] = 0 \Rightarrow P\sin\left(\mathcal{A}-\frac{2\pi}{L_x}\right) + (N-P)\sin\mathcal{A} = 0. \tag{B9}$$

Since $L_x \gg 1$, we find that the saddle point value of $\mathcal{A}$ is

$$\mathcal{A} = -\frac{2\pi}{L_x}\rho, \tag{B10}$$

where $\rho = \frac{P}{N}$. The global U(1) gauge potential adjusts to oppose the inserted flux and is proportional to the fraction of spin components coupled to the flux. In fact, this keeps the net charge current fixed at zero, as expected for a response to a spin twist. The flux imparts momentum to the μ-fermions and elicits a diamagnetic response from all the fermions. This can be seen explicitly by calculating the ground state energy in the presence of the flux:

$$E - \frac{2NV|\chi|^2}{\tilde{J}_H} = -2|\chi|\sum_{\mathbf{k}\sigma}\left[\cos\left(k_x+\mathcal{A}+A^\sigma_x\right)+\cos k_y\right]n^f_{\mathbf{k}} = -2|\chi|I_0V\sum_\sigma\left[\cos\left(\mathcal{A}+A^\sigma_x\right)+1\right], \tag{B11}$$

where $I_0 \equiv (1/V)\sum_{\mathbf{k}} \cos k_x n^f_{\mathbf{k}}$. When P of the N spin components couple to the flux,

$$\frac{E}{V} = -2|\chi|I_0[P\cos(\mathcal{A}+A_x) + (N-P)\cos\mathcal{A} - N] + \frac{2N|\chi|^2}{\tilde{J}_H}. \tag{B12}$$

With $\mathcal{A} \approx -\rho A_x$, the saddle point condition $\partial_{|\chi|}E = 0$ yields $|\chi| = (\frac{1}{2})I_0\tilde{J}_H\{\rho\cos[(1-\rho)A_x] + (1-\rho)\cos(\rho A_x) + 1\}$. As a result, the ground state energy is

$$\begin{aligned}\frac{E}{NV} &= -\frac{2|\chi|^2}{\tilde{J}_H}\\ &= -\frac{1}{2}\tilde{J}_H I_0^2\{\rho\cos[(1-\rho)A_x] + (1-\rho)\cos[\rho A_x] + 1\}^2\\ &\approx -2\tilde{J}_H I_0^2\left\{1 - \frac{1}{8}[\rho(1-\rho)^2 + (1-\rho)\rho^2]A_x^2\right\}^2\\ &= \frac{E_0}{NV} + \frac{1}{2}\rho\widetilde{D}_{\{\mu\}}A_x^2 + O\big(A_x^4\big),\end{aligned} \tag{B13}$$

where $E_0 = -2NV\tilde{J}_H I_0^2$ is the energy in absence of the flux, and $\widetilde{D}_{\{\mu\}} = 2\tilde{J}_H I_0^2(1-\rho)$. The quantity $j_x = -(1/V)\partial E/\partial A_x = -P\tilde{D}_{\{\mu\}}A_x$ can be interpreted as the diamagnetic spin current response of the P spin channels to the flux insertion, while $\widetilde{D}_{\{\mu\}}$ is their spin stiffness.

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