Example 12.6 Consider the Grassman integral

$$I = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \int \prod_{j=1,N} d\bar{\alpha}_j d\alpha_j \prod_{k=1,M} d\bar{\beta}_k d\beta_k \exp\left[(\bar{\alpha}, \bar{\beta})\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right], \quad (12.151)$$

where A and B are square matrices of dimension N and M, respectively, and α and β are column Grassman vectors of length N and M, respectively. By integrating out the β variables first, prove the identity

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\left[A - BD^{-1}C\right]\det D. \quad (12.152)$$

Solution

By expanding the argument of the exponential in I, we can rewrite the integral in the form

$$I = \int\int \prod_{j=1,N} d\bar{\alpha}_j d\alpha_j \exp\left[-\bar{\alpha}A\alpha\right] Y[\bar{\alpha}, \alpha], \quad (12.153)$$

where the inner integral is an integral over the β variables:

$$Y[\bar{\alpha}, \alpha] = \int \prod_{k=1,M} d\bar{\beta}_k d\beta_k \exp\left[-\left(\bar{\beta}D\beta + (\bar{\alpha}B)\beta + \bar{\beta}(C\alpha)\right)\right]. \quad (12.154)$$

The inner integral is of the form (12.141) with $\bar{j} = \bar{\alpha}B$ and $j = C\alpha$, so it can be evaluated as

$$Y[\bar{\alpha}, \alpha] = \det[D] \exp\left[\bar{\alpha}BD^{-1}C\alpha\right].$$

Putting this back into I, we then obtain

α moving in field of β

$$I = \det D \int\int \prod_{j=1,N} d\bar{\alpha}_j d\alpha_j \exp\left[-\bar{\alpha}\left(A - BD^{-1}C\right)\alpha\right] = \det D \det\left(A - BD^{-1}C\right), \quad (12.155)$$

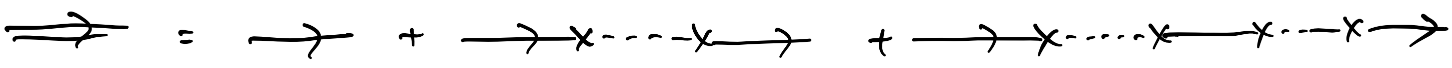
β without α

thus proving the result.

Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} -(G^0_\alpha)^{-1} & \bar{V} \\ V & -(G^0_\beta)^{-1} \end{bmatrix}$ $\quad G^0_\alpha = \longrightarrow$ $\quad G^0_\beta = - - \rightarrow - -$

$$A - BD^{-1}C = -G_\alpha^{-1} = -\left[(G^0_\alpha)^{-1} + \Sigma\right]$$

$$G_\alpha = \frac{1}{G_{\alpha 0}^{-1} - \bar{V}G_{\beta 0}V} \qquad \Sigma = \bar{V}G_{\beta 0}V = \overset{V}{\bullet} - - \rightarrow - - \overset{\bar{V}}{\bullet}$$



$$= G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0$$

$$= G_0 \left(\frac{1}{1 - \Sigma G_0} \right) = \frac{1}{G_0^{-1} - \Sigma}$$

$$\mathcal{H} = \underbrace{E_f f^\dagger f}_{\alpha} + V(c_k^\dagger f + f^\dagger c_k) + \sum \epsilon_k c_{k\sigma}^\dagger c_{k\sigma}$$

e.g. Anderson impurity model

$$-A = G_{0\alpha}^{-1} = (\omega - E_f)$$

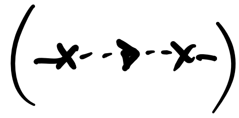
$$-B = G_{0\beta}^{-1} = (\omega - \epsilon_k) \qquad B = C = V$$

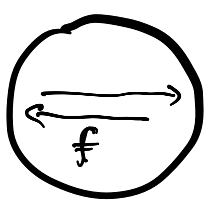
$$\Sigma = V^2 \sum_k \frac{1}{(i\omega_n - \epsilon_k)}$$

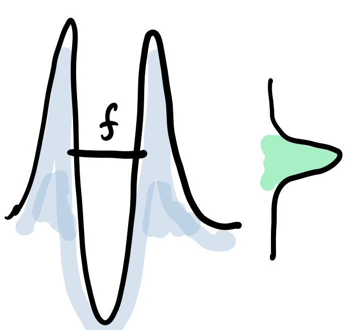
$$A = \begin{pmatrix} E_f - \omega & V & V & V \cdots V \\ V & \ddots & & \\ V & & \epsilon_k - \omega & \\ V & & & \ddots \\ V & & & \end{pmatrix}$$

$$G_f = \frac{1}{i\omega_n - E_f - V^2 \sum_k \frac{1}{(i\omega_n - \epsilon_k)}} = $$ 

$$= \frac{1}{i\omega_n - E_f + i\Delta \,\mathrm{sgn}\,\omega_n}$$



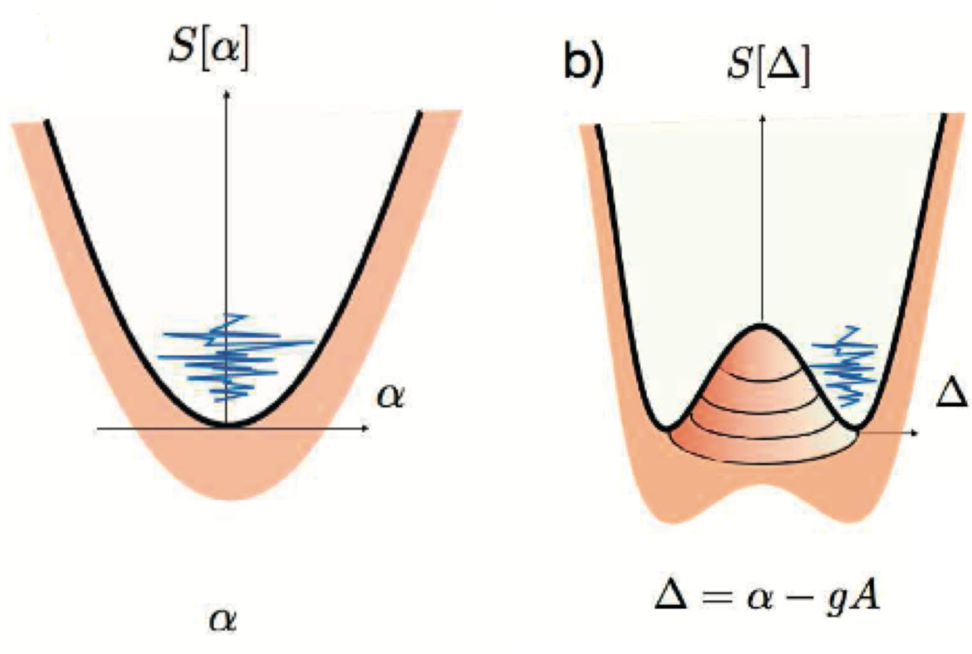




$$\sum \frac{V^2}{i\omega_n - \epsilon_k} = V^2 \int \frac{N(\epsilon)\, d\epsilon}{(i\omega_n - \epsilon)}$$

$$\simeq -i\pi N(0) V^2 \,\mathrm{sgn}\,\omega_n$$

$$= -i\Delta \,\mathrm{sgn}\,\omega_n$$



⟨T φ(1)φ(2)⟩

$= g\delta(\mathbf{1}-\mathbf{2})$.

Exchange Boson with White noise Correlations.

12.5 The Hubbard–Stratonovich transformation

12.5.1 Heuristic derivation

The *Hubbard–Stratonovich transformation*, named after John Hubbard and Ruslan Stratonovich [9, 10, 14], provides a means of representing the interactions between fermions in terms of an exchange boson. It is in essence a way of replacing an instantaneous interaction by a force-carrying boson that describes the fluctuations of an emergent order parameter. Using this method it becomes possible to formally "integrate out" the microscopic fermions, rewriting the problem as an effective field theory describing the thermal and quantum fluctuations of the order parameter as a path integral with a new *effective action*. The method also provides an important formal basis for the order-parameter and mean-field description of broken-symmetry states.

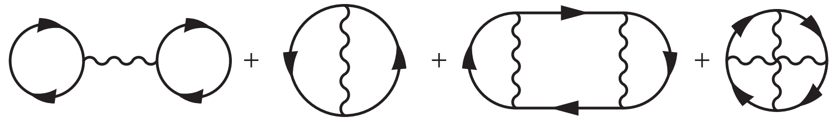
To motivate this approach, we begin with a heuristic derivation. Consider a simple attractive point interaction between particles $V(\mathbf{x}-\mathbf{x}') = -g\delta(\mathbf{x}-\mathbf{x}')$, given by the interaction Hamiltonian

$$H_I = -\frac{g}{2}\int_{\mathbf{x}} \rho(\mathbf{x})^2. \tag{12.156}$$

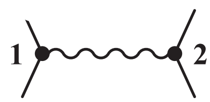
We can write the partition function as a path integral,

$$Z = \int \mathcal{D}[\psi] \exp\left[-\int_{\mathbf{x},\tau} \left(\bar{\psi}(x)(\partial_\tau + h)\psi(x) - \frac{g}{2}\rho(x)^2\right)\right]. \tag{12.157}$$

If we expand the logarithm of the partition function diagrammatically, then we get a series of linked-cluster diagrams,

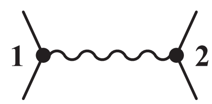
$\ln(Z/Z_0) =$  $+\cdots,$ (12.158)

where the point interaction is represented by the Feynman diagram



$$= g\delta(\mathbf{1}-\mathbf{2}). \tag{12.159}$$

Rather that thinking of an instantaneous contact interaction, we can regard this diagram as the exchange of a force-carrying boson, writing the diagram as



$$= \underbrace{(i)^2}_{\text{vertices}} \times \overbrace{-g\delta(\mathbf{1}-\mathbf{2})}^{-\langle T\phi(1)\phi(2)\rangle}, \tag{12.160}$$

where the vertices $(-i)$ derive from an interaction $S'_I = \int_{\mathbf{x},\tau} \rho(x)\phi(x)$, between the fermions and the boson with imaginary-time Green's function

$$G(\mathbf{1}-\mathbf{2}) = -\langle T\phi(\mathbf{1})\phi(\mathbf{2})\rangle = -g\delta(\mathbf{1}-\mathbf{2}). \tag{12.161}$$

But this implies that the exchange boson has a white-noise correlation function $\langle T\phi(\mathbf{1})\phi(\mathbf{2})\rangle = \delta(\mathbf{1}-\mathbf{2})$: this kind of white-noise correlation is exactly what we expect for a field governed by a simple Gaussian path integral, where

$$\frac{\int D[\phi]\phi(\mathbf{1})\phi(\mathbf{2})e^{-S_\phi}}{\int D[\phi]e^{-S_\phi}} = g\delta(\mathbf{1}-\mathbf{2}), \tag{12.162}$$

with the Gaussian action

$$S_\phi = \int_{\mathbf{x}}\int_0^\beta d\tau\, \frac{\phi(x)^2}{2g}. \tag{12.163}$$

By adding $S_\phi + S'_I$ to the free fermion action, we can thus represent original point interaction by a fluctuating white-noise potential,

$$-\frac{g}{2}\rho(x)^2 \rightarrow \rho(x)\phi(x) + \frac{\phi(x)^2}{2g}. \tag{12.164}$$

If we now insert this transformed interaction into the action, the transformed path integral expression of the partition function becomes

$$Z = \int \mathcal{D}[\psi,\phi] \exp\left[-\int_{\mathbf{x},\tau}\left(\bar{\psi}(x)[\partial_\tau + \underline{h} + \phi(x)]\psi(x) + \frac{1}{2g}\phi(x)^2\right)\right]. \tag{12.165}$$

Note the following:

- Although our derivation is heuristic, we shall shortly see that the Hubbard–Stratonovich transformation is *exact* so long as we allow $\phi(x) = \phi(\mathbf{x}, \tau)$ to describe a fluctuating quantum variable inside the path integral.
- If we replace $\phi(\mathbf{x}, \tau)$ by its average value, $\phi(\mathbf{x}, \tau) \rightarrow \langle\phi(\mathbf{x}, \tau)\rangle = \phi(\mathbf{x})$, we obtain a *mean-field theory*. Suppose, instead of carrying out the Hubbard–Stratonovich transformation, we choose to expand the density in powers of its fluctuations $\delta\rho(x)$ about its average value $\langle\rho(\mathbf{x})\rangle$, writing $\rho(x) = \langle\rho(\mathbf{x})\rangle + \delta\rho(x)$. The interaction can then be written

$$H_I = -\frac{g}{2}\int_{\mathbf{x}}(\langle\rho(\mathbf{x})\rangle + \delta\rho(x))^2$$
$$= -\frac{g}{2}\int_{\mathbf{x}}\Big[\langle\rho(\mathbf{x})\rangle^2 + 2\langle\rho(\mathbf{x})\rangle\delta\rho(x)\Big] + O(\delta\rho(x)^2). \tag{12.166}$$

If we neglect the term which is second-order in the fluctuations, then resubstitute $\delta\rho(x) = \rho(x) - \langle\rho(\mathbf{x})\rangle$, we obtain

$$H_I \approx -\frac{g}{2}\int_{\mathbf{x}}\Big[2\langle\rho(\mathbf{x})\rangle\rho(x) - \langle\rho(\mathbf{x})\rangle^2\Big] = \int_{\mathbf{x}}\left[\rho(x)\phi(\mathbf{x}) + \frac{\phi(\mathbf{x})^2}{2g}\right], \tag{12.167}$$

where we have replaced $-g\langle\rho(\mathbf{x})\rangle = \phi(\mathbf{x})$. This approximate mean-field Hamiltonian (12.167) resembles the result of the Hubbard–Stratonovich transformation (12.164).

With care, this kind of reasoning can be extended to a whole host of interactions between various kinds of charge, spin, and current densities, including both non-local interactions and repulsive interactions. For example, in the Hubbard and Anderson models, the interaction can be written as an attractive interaction in the magnetic channel of the form that is factorized as follows:

$$-\frac{U}{2}(n_\uparrow - n_\downarrow)^2 \rightarrow (n_\uparrow - n_\downarrow)M + \frac{M^2}{2U}, \tag{12.168}$$

corresponding to electrons exchanging fluctuations of the magnetic Weiss field M. The coupling between the field M and the electrons can sometimes stabilize a broken–symmetry state where M develops an expectation value – leading to a magnet. The Hubbard–Stratonovich transformation can also be applied to complex fields, permitting the following factorization:

$$H_I = -gA^\dagger A \rightarrow \bar{A}\Delta + \bar{\Delta}A + \frac{\bar{\Delta}\Delta}{g}, \tag{12.169}$$

where Δ is a complex field. Notice how we have switched $A^\dagger \rightarrow \bar{A}$ to emphasize that the replacement is only exact *under the path integral* (or alternatively, if you wish to switch to operators, under the time-ordering operator). This kind of interaction occurs in a BCS superconductor, where the pairing interaction

$$H_I = -g\sum_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}c_{-\mathbf{k}'\downarrow}c_{\mathbf{k}'\uparrow} = -g\overbrace{\sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}}^{A^\dagger}\overbrace{\sum_{\mathbf{k}} c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}}^{A}.$$

In this case, *under the path integral* the interaction can be rewritten in terms of electrons moving in a fluctuating pair field:

$$H_I \to \bar{\Delta} \sum_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + \sum_{\mathbf{k}} \bar{c}_{\mathbf{k}\uparrow} \bar{c}_{-\mathbf{k}\downarrow} \Delta + \frac{\bar{\Delta}\Delta}{g}.$$

Once superconductivity develops, Δ develops an expectation value, playing the role of an order parameter.

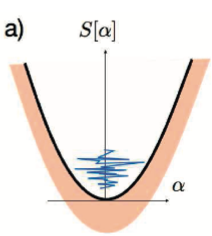
12.5.2 Detailed derivation

Let us examine the above procedure in detail. To be concrete, consider an attractive interaction of the form $H_I = -g \sum_j A_j^\dagger A_j$, where A_j represents an electron bilinear (such as the pair density or spin density of an x-y spin). Consider a fermion path integral on a lattice with interactions $H_I = -g \sum_j A_j^\dagger A_j$,

$$Z = \int \mathcal{D}\Big[\bar{c}, c\Big] \exp\Big[- \int_0^\beta d\tau \Big(\bar{c}\Big(\partial_\tau + \underline{h}\Big)c - g \sum_j \bar{A}_j A_j \Big)\Big], \qquad (12.170)$$

where, inside the path integral, we have replaced $A^\dagger \to \bar{A}$. The next step is to introduce a *white-noise variable* α_j, described by the path integral

$$Z_\alpha = \int \mathcal{D}[\bar{\alpha}, \alpha] \exp\Big[- \sum_j \int_0^\beta d\tau \frac{\bar{\alpha}_j \alpha_j}{g} \Big]. \qquad (12.171)$$



α

The weight function

$$\exp\left[- \sum_j \int_0^\beta d\tau \frac{\bar{\alpha}_j \alpha_j}{g} \right]$$

is a Gaussian distribution function for a white-noise field with correlation function [2]

$$\langle \bar{\alpha}_i(\tau) \alpha_j(\tau') \rangle = g \delta_{ij} \delta(\tau - \tau'). \qquad (12.172)$$

[2] To show this, it is helpful to consider the generating functional

$$\Lambda[\bar{j}, j] = \int \mathcal{D}[\bar{\alpha}, \alpha] \exp\Big[- \sum_r \int_0^\beta d\tau \left(\frac{\bar{\alpha}_r \alpha_r}{g} - \bar{j}_r \alpha_r - \bar{\alpha}_r j_r \right) \Big].$$

By changing variables, $\alpha_r \to \alpha_r + g j_r$ we can absorb the terms linear in j to obtain

$$\Lambda[\bar{j}, j] = \exp\Big[g \sum_r \int_0^\beta d\tau (\bar{j}_r(\tau) j_r(\tau)) \Big]. \quad \Lambda_0$$

Differentiating this with respect to $j_r(\tau)$, we find that

$$\left. \frac{\partial^2 \ln \Lambda[j, \bar{j}]}{\partial \bar{j}_r(\tau) \partial j_{r'}(\tau')} \right|_{j, \bar{j} = 0} = \langle \alpha_r(\tau) \bar{\alpha}_{r'}(\tau') \rangle = g \delta_{rr'} \delta(\tau - \tau').$$

Now the product of these two path integrals,

$$Z \times Z_\alpha = \int \mathcal{D}[\bar{c}, c] \int \mathcal{D}[\bar{\alpha}, \alpha] \exp\left[- \int_0^\beta d\tau \left(\bar{c}(\partial_\tau + \underline{h})c - \sum_j \overbrace{\left(-g\bar{A}_j A_j + \frac{\bar{\alpha}_j \alpha_j}{g} \right)}^{H_I'(j)} \right) \right], \tag{12.173}$$

describes two independent systems. As written, the "α" integrals are on the inside of the path so that, for all configurations of the $\alpha_j(\tau)$ field explored in the inner α integral, the space–time configuration of the $A_j(\tau)$ set by the outer integral are frozen and can hence be regarded as "constants," fixed at each point in space–time. This permits us to define a new variable,

$$\Delta_j(\tau) = \alpha_j(\tau) - gA_j(\tau),$$

and its corresponding conjugate, $\bar{\Delta}_j = \bar{\alpha}_j - g\bar{A}_j$. Formally this is just a shift in the integration variable, so the measure is unchanged and we can write $\mathcal{D}[\bar{\Delta}, \Delta] = \mathcal{D}[\bar{\alpha}, \alpha]$. The transformed interaction becomes

$$\begin{aligned} H_I' &= \sum_j \left\{ -g\bar{A}_j A_j + \frac{(\bar{\Delta}_j + g\bar{A}_j)(\Delta_j + gA_j)}{g} \right\} \\ &= \sum_j \left\{ \bar{A}_j\, \Delta_j + \bar{\Delta}_j A_j + \frac{\bar{\Delta}_j \Delta_j}{g} \right\}. \end{aligned} \tag{12.174}$$

In this way, we arrive at a transformed interaction in which the new variable Δ_j is linearly coupled to the electron operator A_j. If we now re-invert the order of integration inside the path integral (12.173), we obtain

$$\begin{aligned} Z &= \int \mathcal{D}[\bar{\Delta}, \Delta] \exp\left[-\sum_j \int_0^\beta d\tau \frac{\bar{\Delta}_j \Delta_j}{g} \right] \int \mathcal{D}[\bar{c}, c] e^{-\tilde{S}} \\ \tilde{S} &= \int_0^\beta d\tau \left(\bar{c}\partial_\tau c + H_E[\bar{\Delta}, \Delta] \right), \end{aligned} \tag{12.175}$$

where

$$H_E[\bar{\Delta}, \Delta] = \bar{c}\underline{h}c + \sum_j \left\{ \bar{A}_j \Delta_j + \bar{\Delta}_j A_j \right\} \tag{12.176}$$

represents the action for electrons moving in the fluctuating field Δ_j. Notice that, since A and $\bar{A}$ represent fermion bilinear terms, H_E *is itself a bilinear Hamiltonian*.

These noisy fluctuations mediate the interaction between the fermions, much as an exchange boson mediates interactions in the vacuum. More schematically,

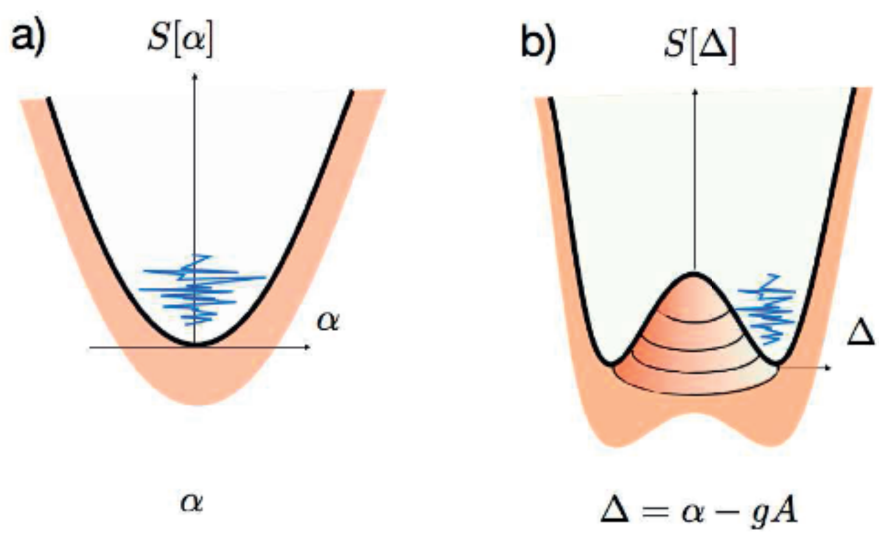


Fig. 12.6 (a) Action for initial white-noise variable α. (b) Action for shifted variable Δ is shifted off-center when the related quantity A has a predisposition towards developing an expectation value.

$$Z = \sum_{\{\Delta\}} \exp\left(-\sum_j \int d\tau \frac{|\Delta_j|^2}{g}\right) \times \Big[\text{path integral of fermions moving in field } \Delta \Big], \tag{12.177}$$

where the summation represents a sum over all possible configurations $\{\Delta\}$ of the auxiliary field Δ. The transformed field

$$\Delta_j = \alpha_j - gA_j$$

is a combination of a white-noise field α_j and the physical field $-gA_j$, so its fluctuations now acquire the correlations associated with the electron fluid. Indeed, when the associated variable A is prone to the development of a broken-symmetry expectation value, the distribution function for Δ becomes concentrated around a non-zero value (Figure 12.6). We call Δ_j a *Weiss field* after Weiss, who first introduced such a field in the context of magnetism.

12.5.3 Effective action

Since the fermionic action inside the path integral is actually Gaussian, we can formally integrate out the fermions as follows:

$$e^{-S_\psi[\bar{\Delta},\Delta]} = \int \mathcal{D}[\bar{c}, c] e^{-\tilde{S}} = \det[\partial_\tau + \underline{h}_E[\bar{\Delta}, \Delta]], \tag{12.178}$$

where $\underline{h}_E$ is the matrix representation of H_E. The full path integral may thus be written

$$Z = \int \mathcal{D}[\bar{\Delta}, \Delta] e^{-S_E[\bar{\Delta},\Delta]},$$

where

$$S_E[\bar{\Delta}, \Delta] = \sum_j \int d\tau \frac{\bar{\Delta}_j \Delta_j}{g} - \ln \det[\partial_\tau + \underline{h}_E[\bar{\Delta}, \Delta]]$$
$$= \sum_j \int d\tau \frac{\bar{\Delta}_j \Delta_j}{g} - \mathrm{Tr} \ln[\partial_\tau + \underline{h}_E[\bar{\Delta}, \Delta]]. \quad (12.179)$$

effective action

Here we have made the replacement $\ln \det \to \mathrm{Tr} \ln$. This quantity is called the *effective action* of the field Δ. The additional fermionic contribution to this action can profoundly change the distribution of the field Δ. For example, if S_E develops a minima around $\Delta = \Delta_o \neq 0$, then $\Delta = -A/g$ will acquire a vacuum expectation value. This makes the Hubbard–Stratonovich transformation an invaluable tool for studying the development of broken symmetry in interacting Fermi systems.