by making the appropriate change in the measure and replacing $\bar{c}c$ in the exponent by the dot product:

$$d\bar{c}dc \to \prod_j d\bar{c}_j dc_j, \tag{12.113}$$
$$\bar{c}c \to \sum_j \bar{c}_j c_j.$$

12.4.2 Path integral for the partition function: fermions

This section very closely parallels the derivation of the bosonic path integral in Section 12.3, but for completeness we include all relevant steps. To begin with, we consider a single fermion, with Hamiltonian

$$H = \epsilon \hat{c}^\dagger \hat{c}. \tag{12.114}$$

Using the trace formula (12.110), the partition function

$$Z = \mathrm{Tr}\, e^{-\beta H} \tag{12.115}$$

can be rewritten in terms of coherent states as

$$Z = -\int d\bar{c}_N dc_0 e^{\bar{c}_N c_0} \langle \bar{c}_N | e^{-\beta H} | c_0 \rangle, \tag{12.116}$$

where the labeling anticipates the next step. Now we expand the exponential into a sequence of time-slices (see Figure 12.5):

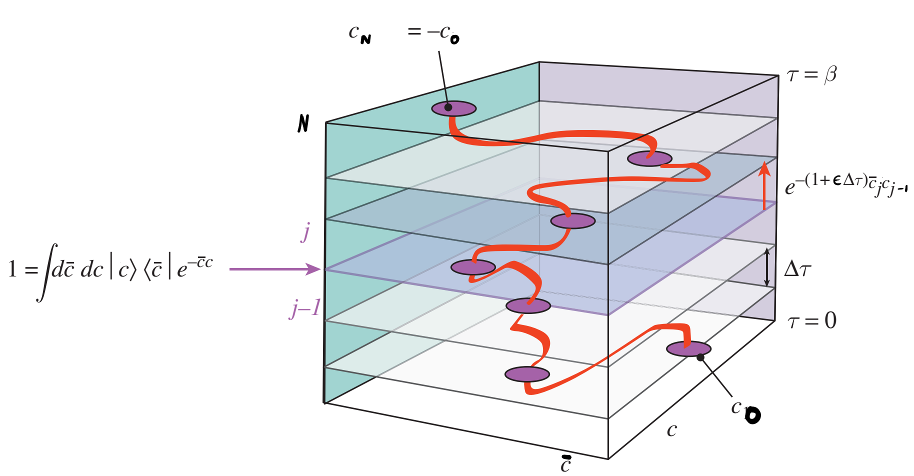
$$e^{-\beta H} = \left(e^{-\Delta\tau H}\right)^N, \qquad \Delta\tau = \beta/N. \tag{12.117}$$

Between each time-slice we introduce the completeness relation

$$\int d\bar{c}_j dc_j |c_j\rangle\langle \bar{c}_j| e^{-\bar{c}_j c_j} = 1, \tag{12.118}$$

so that

$$Z = -\int d\bar{c}_N dc_0 e^{\bar{c}_N c_0} \prod_{j=1}^{N-1} d\bar{c}_j dc_j e^{-\bar{c}_j c_j} \prod_{j=1}^{N} \langle \bar{c}_j | e^{-H\Delta\tau} | c_{j-1} \rangle, \tag{12.119}$$



Division of Grassmanian time evolution into time-slices.

Fig. 12.5

where the first integral is associated with the trace and the subsequent integrals with the $N-1$ completeness relations. Now if we define

$$c_N = -c_0, \tag{12.120}$$

we are able to identify the Nth time-slice with the zero-th time-slice. In this way, the integral associated with the trace

$$-\int d\bar{c}_N dc_0 e^{\bar{c}_N c_0} \langle \bar{c}_N | \cdots | c_0 \rangle = \int d\bar{c}_N dc_N e^{-\bar{c}_N c_N} \langle \bar{c}_N | \cdots | c_0 \rangle \tag{12.121}$$

can be absorbed into the other $N-1$ integrals. Furthermore, we notice that the fields entering into the discrete path integral are *antiperiodic*.

With this observation,

$$Z = \int \prod_{j=1}^{N} d\bar{c}_j dc_j e^{-\bar{c}_j c_j} \langle \bar{c}_j | e^{-H\Delta\tau} | c_{j-1} \rangle. \tag{12.122}$$

Provided each time-slice is of sufficiently brief duration, we can replace $e^{-\Delta\tau H}$ by its normal-ordered form, so that

$$\langle \bar{c}_j | e^{-H\Delta\tau} | c_{j-1} \rangle = e^{\bar{c}_j c_{j-1}} e^{-H[\bar{c}_j, c_{j-1}]\Delta\tau} + O(\Delta\tau^2), \tag{12.123}$$

where $H[\bar{c}, c] = \epsilon \bar{c} c$ is the normal-ordered Hamiltonian, with Grassman numbers replacing operators.

Combining (12.116) and (12.119) we can write

$$\begin{aligned} Z &= \mathrm{Lt}_{N\to\infty} Z_N \\ Z_N &= \int \prod_{j=1}^{N} d\bar{c}_j dc_j \exp\Big[-S\Big] \\ S &= \sum_{j=1}^{N} \Big[\bar{c}_j (c_j - c_{j-1})/\Delta\tau + \epsilon \bar{c}_j c_{j-1}\Big] \Delta\tau, \end{aligned} \tag{12.124}$$

As in the bosonic case, this path integral represents a sum over all possible values "histories" of the fields:

$$c(\tau_j) \equiv \{c_1, c_2, \ldots c_N\}, \tag{12.125}$$

$$\bar{c}(\tau_j) \equiv \{\bar{c}_1, \bar{c}_2, \ldots \bar{c}_N\} \tag{12.126}$$

as illustrated in Figure 12.5.

This kind of integral is also called a *functional integral*, because it involves integrating over all possible values of the functions $c(\tau)$ and $\bar{c}(\tau)$. When we take the thickness of the time-slices to zero, the discrete functions $c(\tau)$ and $\bar{c}(\tau)$ become functions of continuous time. The boundary condition (12.120) implies that the set of complete functions which we sum over must satisfy antiperiodic boundary conditions

$$c(\tau + \beta) = -c(\tau), \qquad \bar{c}(\tau + \beta) = -\bar{c}(\tau).$$

In the continuum limit $N \to \infty$, we now replace

$$\begin{aligned} \bar{c}_j(c_j - c_{j-1})/\Delta\tau &\to \bar{c}\partial_\tau c, \\ \sum_j \Delta\tau &\to \int_0^\beta d\tau. \end{aligned} \tag{12.127}$$

The sense in which c_j becomes “close” to c_{j+1} needs to be carefully understood. Suppose we rewrite the antiperiodic c_j in terms of their frequency components as

$$c_j = \frac{1}{\sqrt{\beta}} \sum_{|n| \leq N/2} c(i\omega_n) e^{-i\omega_n \tau_j}.$$

Then, in this new basis,

$$\sum_j \bar{c}_j(c_j - c_{j-1}) = \sum_{|n| \leq N/2} \bar{c}(i\omega_n) \left[\frac{1 - e^{i\omega_n \Delta\tau}}{\Delta\tau} \right] c(i\omega_n).$$

In practice, the path integral is dominated by functions c_j with a maximum characteristic temporal frequency max $(|\omega_n|) \sim \epsilon$, so that as $\Delta\tau \to 0$ we can replace

$$\left[\frac{1 - e^{i\omega_n \Delta\tau}}{\Delta\tau} \right] \to -i\omega_n,$$

which is the Fourier transform of ∂_τ.

With these provisos, the continuum limit of the action and path integral are then

$$\begin{aligned} S &= \int_0^\infty d\tau \left[\bar{c}(\partial_\tau + \epsilon)c \right] \\ Z &= \int \mathcal{D}[\bar{c}, c] \exp\left[-S \right], \end{aligned} \tag{12.128}$$

where we use the notation

$$\mathcal{D}[\bar{c}, c] = \prod_{\tau_l} d\bar{c}(\tau_l) dc(\tau_l).$$

At first sight, it might seem a horrendous task to carry out the integral over all possible functions $c(\tau)$. How can we possibly do this in a controlled fashion? The clue to this problem lies in the observation that the set of functions $c(\tau)$ and its conjugate $\bar{c}(\tau)$ are spanned by a *discrete* but complete set of antiperiodic functions, as follows:

$$c(\tau) = \frac{1}{\sqrt{\beta}} \sum_n c_n e^{-i\omega_n \tau}.$$

We can integrate over all possible functions $c(\tau)$ by integrating over all possible values of the coefficients c_n, and since the transformation which links these two bases is unitary, the Jacobian which links the two bases is unity, i.e.

$$\mathcal{D}[\bar{c}, c] \equiv \prod_n d\bar{c}_n dc_n.$$

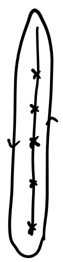
It is much easier to visualize and work with a discrete basis. We can transform to this basis by replacing $\partial_\tau \to -i\omega_n$ in the action, rewriting it as

$$S = \sum_n \bar{c}_n(-i\omega_n + \epsilon)c_n.$$

Now the path integral is just a discrete Gaussian integral,

$$Z = \int \prod_n d\bar{c}_n dc_n \exp\left[-\sum_n \bar{c}_n(-i\omega_n + \epsilon)c_n\right] = \prod_n(-i\omega_n + \epsilon),$$

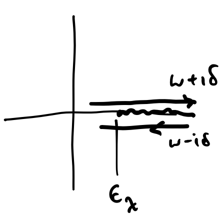
so that the free energy is given by



$$F = -T\ln Z = -T\sum_n \ln(\epsilon - i\omega_n)e^{i\omega_n 0^+}.$$

Here we have added a small convergence factor $e^{i\omega_n 0^+}$ because the time evolution from $\tau = 0$ to $\tau = \beta$ is equivalent to time evolution from $\tau = 0$ to $\tau = 0^-$.

We can show that this reverts to the standard expression for one-particle free energy by replacing the Matsubara sum with a contour integral:



$$F = \oint \frac{dz}{2\pi i} f(z)\ln[\epsilon_\lambda - z]e^{z0^+}, \tag{12.129}$$

where the contour integral passes counterclockwise around the poles of the Fermi function at $z = i\omega_n$, and the choice of $f(z)$ is dictated by the convergence factor. We take the logarithm to have a branch cut which extends from $z = \epsilon_\lambda$ to infinity. By deforming the integral around this branch cut, we obtain

$$\begin{aligned} F &= \int_\epsilon^\infty \frac{d\omega}{2\pi i} f(\omega)\Big[\ln(\epsilon - \omega - i\delta) - (\text{c.c.})\Big] \\ &= \int_\epsilon^\infty d\omega f(\omega) \\ &= -T\ln[1 + e^{-\beta\epsilon}], \end{aligned} \tag{12.130}$$

which is the well-known free energy of a single fermion.

Of course, here we have used a sledgehammer to crack a walnut, but the virtue of the method is the ease with which it can be generalized to more complex problems. Three important points need to be made about this result:

- This result can easily be generalized to an arbitrary number of Fermi fields. In this case,

  $$S = \int_0^\infty d\tau\left[\sum_\lambda \bar{c}_\lambda \partial_\tau c_\lambda + H[\bar{c}, c]\right],$$

  and the measure for the path integral becomes

  $$\mathcal{D}[\bar{c}, c] = \prod_{\tau_l, r} d\bar{c}_\lambda(\tau_l)dc_\lambda(\tau_l).$$

- The derivation did not depend on any details of H, and can thus be simply generalized to interacting Hamiltonians. In both cases, the conversion of the normal-order Hamiltonian occurs by simply replacing operators with the appropriate Grassman variables:

$$: H[\hat{c}^\dagger, \hat{c}] :\rightarrow H[\bar{c}, c].$$

- The amplitude associated with a particular path can be split into the product of two terms, as follows:

$$e^{-S_{PATH}} = \exp\left[i\gamma - \int_0^\beta H[\bar{c}, c]d\tau\right], \tag{12.131}$$

where the quantity

$$\gamma = i\sum_\lambda \int_0^\beta d\tau \bar{c}_\lambda \partial_\tau c_\lambda \tag{12.132}$$

is known as a *Berry phase*, named after the British physicist Michael Berry. The Berry phase can be identified as the phase picked up by the coherent state $|\psi\rangle = \exp\left[\sum_\lambda \hat{c}^\dagger_\lambda c_\lambda\right]|0\rangle$ as it is adiabatically evolved along the path defined by the functions $c_\lambda(\tau)$. For this reason, the time-derivative term in a path integral is often refered to as the *Berry phase term*.

- Because the Jacobian for a unitary transformation is unity, we can change basis inside the path integral. For example, if we start with the action for a gas of fermions,

$$S = \int_0^\beta d\tau \sum_{\mathbf{k}} \bar{c}_{\mathbf{k}}(\partial_\tau + \epsilon_{\mathbf{k}})c_{\mathbf{k}},$$

where $\epsilon_{\mathbf{k}} = (k^2/2m) - \mu$, we can transform to a completely discrete basis by Fourier transforming in time:

$$\begin{aligned} c_{\mathbf{k}} &= \frac{1}{\sqrt{\beta}}\sum_n c_{\mathbf{k}n}e^{i\omega_n\tau}, \\ \partial_\tau &\rightarrow -i\omega_n \\ \mathcal{D}[\bar{c}, c] &\rightarrow \prod_{\mathbf{k},n} d\bar{c}_{\mathbf{k}n}dc_{\mathbf{k}n}. \end{aligned} \tag{12.133}$$

In this discrete basis, the action becomes

$$S = \sum_{\mathbf{k},n}(\epsilon_{\mathbf{k}} - i\omega_n)\bar{c}_{\mathbf{k}n}c_{\mathbf{k}n}.$$

This basis usually proves very useful for practical calculations.

- We can also transform to a continuum real-space basis, as follows:

$$\begin{aligned} c_{\mathbf{k}} &= \frac{1}{\sqrt{V}}\int d^3x\psi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \\ \epsilon_{\mathbf{k}} &\rightarrow -\frac{\nabla^2}{2m} - \mu \\ \mathcal{D}[\bar{c}, c] &\rightarrow \mathcal{D}[\bar{\psi}, \psi]. \end{aligned} \tag{12.134}$$

In the new basis, the action becomes

$$S = \int_0^\beta d\tau \int d^3x \bar{\psi}(\mathbf{x}) \left[\partial_\tau - \frac{\nabla^2}{2m} - \mu \right] \psi(\mathbf{x}).$$

The discrete and continuous measures, (12.133) and (12.134), respectively, are equivalent:

$$\prod_{\mathbf{k},n} d\bar{c}_{\mathbf{k}n} dc_{\mathbf{k}n} \equiv \mathcal{D}[\bar{\psi}, \psi]$$

because the space of continuous functions $\psi(x)$ is spanned by a complete but discrete set of basis functions:

$$\psi(\mathbf{x}, \tau) = \frac{1}{\sqrt{\beta V}} \sum_{\mathbf{k},n} c_{\mathbf{k}n} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_n \tau)}.$$

We can integrate over all possible functions $\psi(\mathbf{x}, \tau)$ by integrating over all values of the discrete vector $c_{\mathbf{k}n}$.

12.4.3 Gaussian path integral for fermions

For non-interacting fermions the action only involves bilinears of the Fermi fields, so the path integral is of Gaussian form and can always be evaluated. To discuss the most general case, we shall include source terms in the original Hamiltonian, writing

$$H(\tau) = \sum_\lambda \left[\epsilon_\lambda \hat{c}^\dagger_\lambda \hat{c}_\lambda - \bar{j}_\lambda(\tau) \hat{c}_\lambda - \hat{c}^\dagger_\lambda j_\lambda(\tau) \right],$$

where $\hat{c}^\dagger_\lambda$ is Schrödinger field that creates a fermion in the eigenstate with energy ϵ_λ. With source terms, the partition function becomes a *generating functional*,

$$Z[\bar{j}, j] = \mathrm{Tr}\left[\mathrm{T} \exp\left\{ -\int_0^\beta d\tau H(\tau) \right\} \right].$$

Derivatives of the generating functional generate the irreducible Green's functions of the fermions; for instance,

$$\frac{\delta \ln Z[\bar{j}, j]}{\delta \bar{j}(1)} = \langle c(1) \rangle \tag{12.135}$$

$$\frac{\delta^2 \ln Z[\bar{j}, j]}{\delta j(2) \delta \bar{j}(1)} = \langle T[c(1) c^\dagger(2)] \rangle - \langle c(2) \rangle \langle c^\dagger(1) \rangle, \tag{12.136}$$

where

$$\langle \ldots \rangle = \frac{1}{Z[\bar{j}, j]} \mathrm{Tr}\left[T \exp\left\{ -\int_0^\beta d\tau H(\tau) \right\} \ldots \right].$$

Transforming to a path integral representation,

$$Z[\bar{j}, j] = \int \mathcal{D}[\bar{c}, c] e^{-S} \tag{12.137}$$

$$S = \int d\tau \left[\bar{c}(\tau)(\partial_\tau + \underline{h}) c(\tau) - \bar{j}(\tau) c(\tau) - \bar{c}(\tau) j(\tau) \right], \tag{12.138}$$

where $\underline{h}_{\alpha\beta} = \epsilon_\alpha \delta_{\alpha\beta}$ is the one-particle Hamiltonian. One can carry out functional derivatives on this integral without actually evaluating it. For example, we find that

$$\langle c(1)\rangle = \frac{1}{Z[\bar{j},j]}\int \mathcal{D}[\bar{c},c]c(1)e^{-S} \quad (12.139)$$

$$\langle T[c(1)c^\dagger(2)]\rangle = \frac{1}{Z[\bar{j},j]}\int \mathcal{D}[\bar{c},c]c(1)\bar{c}(2)e^{-S}. \quad (12.140)$$

Notice how the path integral automatically furnishes us with time-ordered expectation values.

Fortunately, the path integral is Gaussian, allowing us to use the general result obtained in Appendix 12D,

$$\int \prod_j d\bar{\xi}_j d\xi_j \exp[-\bar{\xi}\cdot A\cdot \xi + \bar{j}\cdot\xi + \bar{\xi}\cdot j] = \det A \exp[\bar{j}\cdot A^{-1}\cdot j]. \quad (12.141)$$

In the case considered here, $A = \partial_\tau + \underline{h}$, so we can do the integral to obtain

$$Z[\bar{j},j] = \int \mathcal{D}[\bar{c},c]\exp\left[-\int_0^\beta d\tau\left[\bar{c}(\tau)(\partial_\tau+\underline{h})c(\tau) - \bar{j}(\tau)c(\tau) - \bar{c}(\tau)j(\tau)\right]\right]$$
$$= \det[\partial_\tau + \underline{h}\,]\exp\left[-\int_0^\beta d\tau d\tau' \bar{j}(\tau)\underline{G}[\tau-\tau']\,j(\tau')\right], \quad (12.142)$$

where

$$\underline{G}[\tau-\tau'] = -(\partial_\tau + \underline{h})^{-1}. \quad (12.143)$$

By differentiating (12.142) with respect to j and $\bar{j}$, we are able to identify

$$\left.\frac{\delta^2 \ln Z}{\delta j(\tau')\delta\bar{j}(\tau)}\right|_{\bar{j},j=0} = (\partial_\tau + \underline{h})^{-1} = \langle c(\tau)c^\dagger(\tau')\rangle = -\underline{G}[\tau-\tau'], \quad (12.144)$$

so the inverse of the Gaussian coefficient in the action $-[\partial_\tau + \underline{h}]^{-1}$ directly determines the imaginary-time Green's function of these non-interacting fermions. Higher-order moments of the generating functional provide a derivation of Wick's theorem.

From the partition function in (12.142), the free energy is then given by

$$F = -T\ln Z = -T\ln\det[\partial_\tau + \underline{h}] = -T\mathrm{Tr}\ln[\partial_\tau + \underline{h}] = T\mathrm{Tr}\ln[-G^{-1}],$$

where we have used the result $\ln\det[A] = \mathrm{Tr}\ln[\partial_\tau + \underline{h}]$.

To explicitly compute the free energy, it is useful to transform to Fourier components:

$$c_\lambda(\tau) = \frac{1}{\sqrt{\beta}}\sum_n c_{\lambda n}e^{-i\omega_n\tau}$$

$$j_\lambda(\tau) = \frac{1}{\sqrt{\beta}}\sum_n j_{\lambda n}e^{-i\omega_n\tau}. \quad (12.145)$$

In this basis,

$$(\partial_\tau + \epsilon_\lambda) \longrightarrow (-i\omega_n + \epsilon_\lambda)$$
$$\underline{G} = -(\partial_\tau + \epsilon_\lambda)^{-1} \longrightarrow (i\omega_n - \epsilon_\lambda)^{-1}, \tag{12.146}$$

so that

$$S = \sum_{\lambda,n}\Big[[-i\omega_n + \epsilon_\lambda]\bar{c}_{\lambda n} c_{\lambda n} - \bar{j}_{\lambda n} c_{\lambda n} - \ \bar{c}_{\lambda n} j_{\lambda n}\Big], \tag{12.147}$$

whereupon

$$\det[\partial_\tau + \underline{h}] = = \prod_{\lambda,n}(-i\omega_n + \epsilon_\lambda)$$
$$Z[\bar{j}, j] = \prod_{\lambda,n}(-i\omega_n + \epsilon_\lambda)\exp\Big[\sum_{\lambda,n}(-i\omega_n + \epsilon_\lambda)^{-1}\bar{j}_{\lambda n} j_{\lambda n}\Big]. \tag{12.148}$$

If we set $j = 0$ in Z, we obtain the free energy in terms of the fermionic Green's function:

$$F = -T\sum_{\lambda,n}\ln[-i\omega_n + \epsilon_\lambda].$$

As in the case of a single field, by replacing the Matsubara sum with a contour integral we obtain

$$F = \sum_\lambda \oint \frac{dz}{2\pi i} f(z)\ln[\epsilon_\lambda - z] \tag{12.149}$$
$$= -T\sum_\lambda \ln[1 + e^{-\beta\epsilon_\lambda}]. \tag{12.150}$$

If we differentiate Z with respect to its source terms, we obtain the Green's function:

$$-\frac{\delta^2 \ln Z}{\delta\bar{j}_{\lambda n}\delta j_{\lambda' n'}} = [\underline{G}]_{\lambda n, \lambda' n'} = \delta_{\lambda\lambda'}\delta_{nn'}\frac{1}{i\omega_n - \epsilon_\lambda}.$$

Example 12.6 Consider the Grassman integral

$$I = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \int \prod_{j=1,N} d\bar{\alpha}_j d\alpha_j \prod_{k=1,M} d\bar{\beta}_k d\beta_k \exp\left[(\bar{\alpha},\bar{\beta})\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right], \quad (12.151)$$

where A and B are square matrices of dimension N and M, respectively, and α and β are column Grassman vectors of length N and M, respectively. By integrating out the β variables first, prove the identity

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\left[A - BD^{-1}C\right]\det D. \quad (12.152)$$

Solution

By expanding the argument of the exponential in I, we can rewrite the integral in the form

$$I = \int\int \prod_{j=1,N} d\bar{\alpha}_j d\alpha_j \exp\left[-\bar{\alpha}A\alpha\right] Y[\bar{\alpha},\alpha], \quad (12.153)$$

where the inner integral is an integral over the β variables:

$$Y[\bar{\alpha},\alpha] = \int \prod_{k=1,M} d\bar{\beta}_k d\beta_k \exp\left[-\left(\bar{\beta}D\beta + (\bar{\alpha}B)\beta + \bar{\beta}(C\alpha)\right)\right]. \quad (12.154)$$

The inner integral is of the form (12.141) with $\bar{j} = \bar{\alpha}B$ and $j = C\alpha$, so it can be evaluated as

$$Y[\bar{\alpha},\alpha] = \det[D]\exp\left[\bar{\alpha}BD^{-1}C\alpha\right].$$

Putting this back into I, we then obtain

α moving in field of β

$$I = \det D \int\int \prod_{j=1,N} d\bar{\alpha}_j d\alpha_j \exp\left[-\bar{\alpha}\left(A - BD^{-1}C\right)\alpha\right] = \det D \det\left(A - BD^{-1}C\right), \quad (12.155)$$

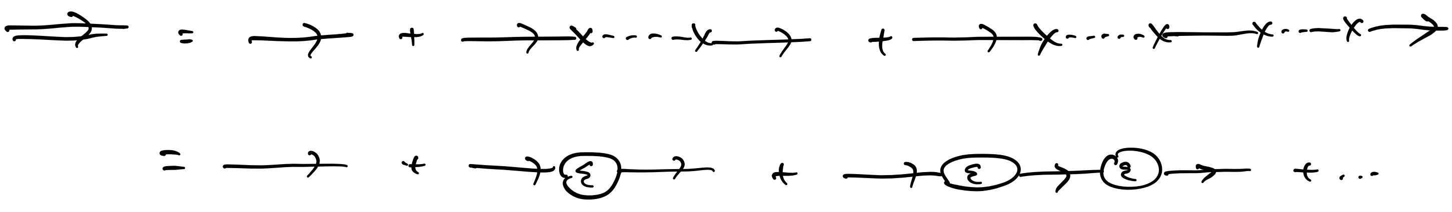
β without α

thus proving the result.

Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} -(G^0_\alpha)^{-1} & \bar{V} \\ V & -(G^0_\beta)^{-1} \end{bmatrix}$ $\quad G^0_\alpha = \longrightarrow$ $\quad G^0_\beta = \text{- - ->- - -}$

$$A - BD^{-1}C = -G_\alpha^{-1} = -\left[(G^0_\alpha)^{-1} + \Sigma\right]$$

$$G_\alpha = \frac{1}{G_{\alpha 0}^{-1} - \bar{V}G_{\beta 0}V} \qquad \Sigma = \bar{V}G_{\beta 0}V = \overset{V}{\bullet}\text{- ->-}\overset{\bar{V}}{\bullet}$$



$$= G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0$$

$$= G_0 \left(\frac{1}{1 - \Sigma G_0} \right) = \frac{1}{G_0^{-1} - \Sigma}$$

e.g. Anderson impurity model

$$\mathcal{H} = \underbrace{E_f f^\dagger f}_{\alpha} + V(c_k^\dagger f + f^\dagger c_k) + \sum \epsilon_k c_{k\sigma}^\dagger c_{k\sigma}$$

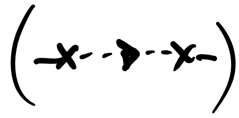
$$-A = G_{0\alpha}^{-1} = (\omega - E_f)$$

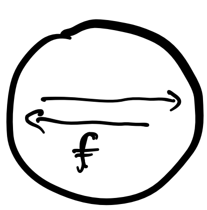
$$-B = G_{0\beta}^{-1} = (\omega - \epsilon_k) \qquad B = C = V$$

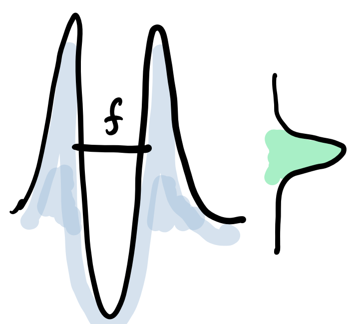
$$\Sigma = V^2 \sum_k \frac{1}{(i\omega_n - \epsilon_k)}$$

$$A = \begin{pmatrix} E_f - \omega & V & V & V \cdots V \\ V & \ddots & & \\ V & & \epsilon_k - \omega & \\ V & & & \ddots \\ V & & & \end{pmatrix}$$

$$G_f = \frac{1}{i\omega_n - E_f - V^2 \sum_k \frac{1}{(i\omega_n - \epsilon_k)}} = \frac{1}{i\omega_n - E_f + i\Delta \,\mathrm{sgn}\,\omega_n}$$







$$\sum \frac{V^2}{i\omega_n - \epsilon_k} = V^2 \int \frac{N(\epsilon)\, d\epsilon}{(i\omega_n - \epsilon)}$$

$$\simeq -i\pi N(0) V^2 \,\mathrm{sgn}\,\omega_n$$

$$= -i\Delta \,\mathrm{sgn}\,\omega_n$$