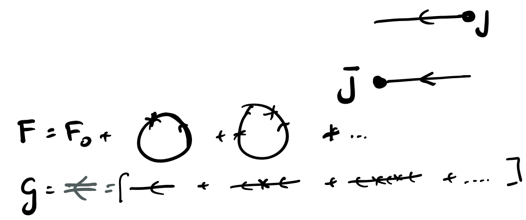
Topics for L5

- Feynman Diagrams

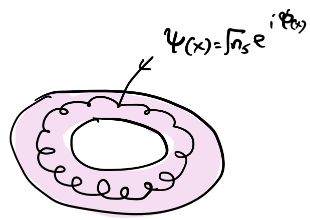
$G_0 = \longrightarrow$



- Quick look at: Superfluids + BECs

$$\boxed{S = \iint \left[\bar\psi\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi + \frac{1}{2}g(\bar\psi\psi)^2\right] d\tau}$$

superfluid / BEC



$$\boxed{\oint \nabla\phi \, dx = 2\pi n}$$

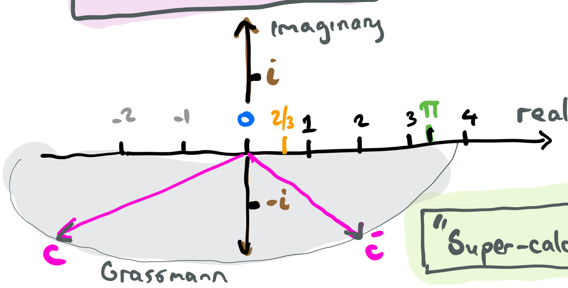
Topological invariant

$$|\psi(x)\rangle = \exp\left[\int d^3x \, \hat\psi^\dagger(x)\psi(x)\right]|0\rangle$$

$$\hat\psi(x)|\psi(x)\rangle = \psi(x)|\psi(x)\rangle$$

- **Fermions!**

$$|c\rangle = e^{\sum_x \hat c_x^\dagger c_x}|0\rangle \qquad \hat c_x|c\rangle = c_x|c\rangle$$



"Super-calculus"

anticommutes

$$\hat c_x \hat c_{x'} = -\hat c_{x'}\hat c_x \Rightarrow c_x c_{x'} = -c_{x'}c_x$$

$$\hat c_x^2 = 0 \Rightarrow c_x^2 = 0$$

$$\partial_x x^n = n x^{n-1} \qquad \int dx \, x^n = \frac{x^{n+1}}{n}$$

$$\partial_x e^{xa} = a e^{ax} \ \ldots \qquad \ldots$$

$$\partial_x \ln x = 1/x \qquad \int dx \, \frac{1}{x} = \ln x$$

$\vdots$

$$\partial_c 1 = 0 \qquad \partial_c = \int dc \qquad \int dc \, 1 = 0$$

$$\partial_c c = 1 \qquad \int dc \, c = 1$$

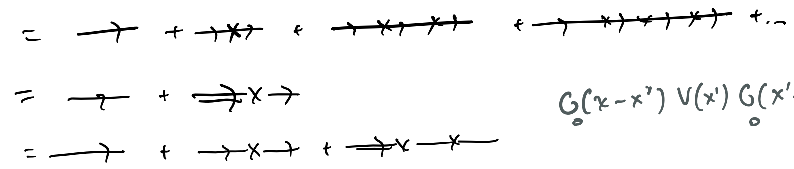
GRASSMANN CALCULUS

$$\partial_c F[c] \stackrel{!}{=} \int dc \, F[c]$$

$\boxed{\text{FEYNMAN DIAGRAMS}}$

$$= G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \dots$$

Artist $\mathcal{G} = -\frac{1}{\partial_\tau + h}$ $\Longrightarrow$



$G_0(x-x')\,V(x')\,G_0(x'-x'')$

Engineer $\mathcal{G}_{\lambda\lambda'} = \frac{\delta_{\lambda\lambda'}}{i\nu_n - \omega_\lambda}$

Free-energy

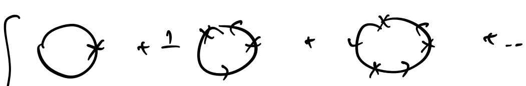
$$F = T \ln\det -G^{-1} = T\,\mathrm{Tr}\ln(-G^{-1}) = T\ln(-\partial_\tau + h)$$

$h = h_0 + V$

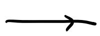
$G_0 = (-\partial_\tau + h_0)^{-1}$

$$F = F_0 + T \sum \ln(1 - G_0 V)$$

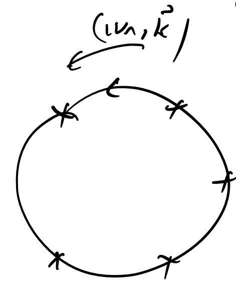
$$= F_0 - \frac{1}{\beta}\sum_{kn} G_0 V + \frac{1}{2}(G_0 V)^2 + \frac{1}{3}(G_0 V)^3$$

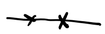
$= F_0 - \frac{1}{\beta}\Big[$  $\Big]$

$$= F_0 - T\sum_{k,n}\left[\frac{1}{i\nu_n - \omega_k} V + \frac{1}{2}\left(\frac{V}{i\nu_n - \epsilon_k}\right)^2 + \frac{1}{3}\left(\frac{V}{i\nu_n - \epsilon_k}\right)^3 + \dots\right.$$

 $\frac{1}{i\nu_n - \omega_k} = G_0$

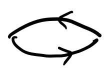
Sum of linked diagrams



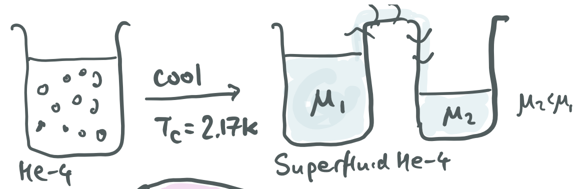
 V

$\frac{1}{n}$ symmetry fct.

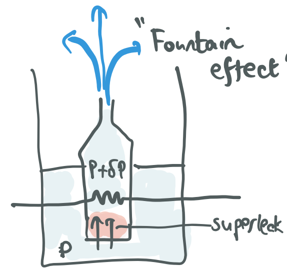
Closed loop: Sum over all intermediate frequencies & momenta. $T\sum_{n,\vec{k}}$

$-\frac{\partial^2 F}{\partial V_{q=0}^2} =$  $= T\sum \frac{1}{(i\nu_n - \omega_k)^2}$

Quick look at BECs / Superfluids



$$\boxed{\vec{v}_s = \frac{\hbar}{m}\nabla\phi}$$



$$\boxed{\frac{d\vec{j}}{dt} = -\frac{\hbar n_s}{m}\nabla\mu}$$

Classical limit of a bosonic fluid = Superfluid

$$\psi = \frac{1}{\sqrt{V}}\sum b_k e^{i\vec{k}\cdot\vec{x}}$$

$$\boxed{S = \int d\tau \int \left[\bar{\psi}(\partial_\tau - \nabla^2 - \mu)\psi + \frac{g}{2}(\bar{\psi}\psi)^2\right]}$$

$$S = \int d\tau \left[\sum \bar{b}_k(\partial_\tau + \omega_k)b_k + \frac{g}{2V}\sum \bar{b}_{\vec{k}_1}\bar{b}_{\vec{k}_2} b_{k_3} b_{k_4}\, \delta_{\vec{k}_1+\vec{k}_2-k_3-k_4}\right]$$

$$b_{\vec{k}=0} = \sqrt{N_c} = b_0 \qquad |b_0|^2/V = N_c/V = n_c$$

$$|\psi\rangle = \exp\left[\int \hat{\psi}^\dagger(x)\psi(x)\, d^3x\right]|0\rangle$$

$$\psi(x) = \sqrt{n_s}\, e^{-i\phi(x)} \qquad \frac{\langle\bar{\psi}|\hat{\psi}(x)|\psi\rangle}{\langle\bar{\psi}|\psi\rangle} = \sqrt{n_s}\, e^{i\phi(x)} \qquad \vec{v}_s = \frac{\hbar}{m}\nabla_x\phi\, n_s$$

$$\vec{P}|\psi\rangle = (\hbar\vec{\nabla}_x\phi)\,\psi|\psi\rangle$$

$$\frac{\langle\bar{\psi}|\vec{P}/m|\psi\rangle}{\langle\bar{\psi}|\psi\rangle} = \frac{\hbar}{m}(\vec{\nabla}\phi)\, n_s = \vec{J}$$

Superflow

$$\boxed{\vec{V}_s = \frac{\hbar}{m}\nabla\phi}$$

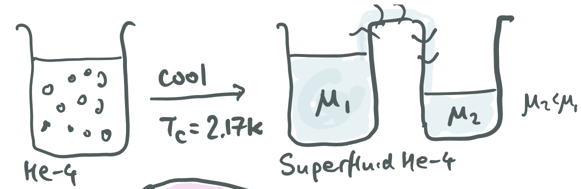
$$\mu \Rightarrow \quad \phi = -\frac{\mu t}{\hbar} \qquad \frac{d\vec{j}}{dt} = -\frac{\hbar}{m} n_s \nabla\mu$$

$$\rho(x,y) = \langle N|\psi(x)\psi^\dagger(y)|N\rangle \approx \langle\psi(x)\psi^\dagger(y)\rangle$$

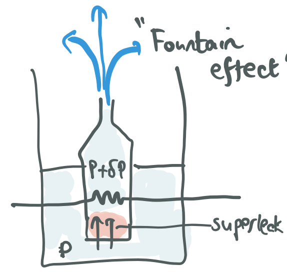
$$|N_0\rangle = \int\frac{d\phi}{2\pi}\, e^{i\phi(\hat{N}-N_0)}\,|\psi\rangle \qquad \xrightarrow{|x-y|\to\infty} \langle\psi(x)\rangle\langle\psi^\dagger(y)\rangle$$

ODLRO

Quick look at BECs / Superfluids



$$\vec{v}_s = \frac{\hbar}{m} \nabla\phi$$



$$\boxed{\frac{d\vec{J}}{dt} = -\frac{\hbar n_s}{m} \nabla\mu}$$

Classical limit of a bosonic fluid = Superfluid

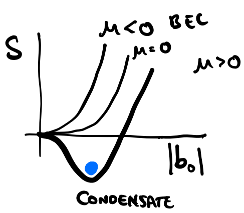
$$\psi = \frac{1}{\sqrt{V}} \sum b_k e^{i\vec{k}\cdot\vec{x}}$$

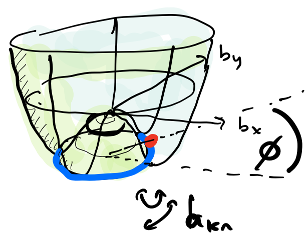
$$\boxed{S = \int d\tau \left[\bar{\psi}(\partial_\tau - \nabla^2 - \mu)\psi + \frac{g}{2}(\bar{\psi}\psi)^2 \right]}$$

$$S = \int d\tau \left[\sum \bar{b}_k (\partial_\tau + \omega_k) b_k + \frac{g}{2V} \sum \bar{b}_{k_1} \bar{b}_{k_2} b_{k_3} b_{k_4} \delta_{\vec{k}_1 + \vec{k}_2 - k_3 - k_4} \right]$$

$$b_{\vec{k}=0} = \sqrt{N_c} = b_0 \qquad |b_0|^2/V = N_c/V = n_c$$

$$\frac{\delta S}{\delta \bar{b}_{k=0}} = \underset{-\mu}{\omega_k} b_0 + \frac{g}{V} \bar{b}_0 b_0^2 = 0 \Rightarrow \mu = g\frac{|b_0|^2}{V} \qquad \frac{|b_0|^2}{V} = \frac{\mu}{g} = n_c$$





$$\psi = \sqrt{n_c}\, e^{i\phi}$$

Spontaneously Broken gauge symmetry

$$|\psi\rangle = \exp\left[\int \hat{\psi}^\dagger(x) \psi(x)\, d^3x \right] |0\rangle$$

$$\psi(x) = \sqrt{n_s}\, e^{i\phi(x)} \qquad \frac{\langle \bar{\psi} | \hat{\psi}(x) | \psi \rangle}{\langle \bar{\psi} | \psi \rangle} = \sqrt{n_s}\, e^{i\phi(x)} \qquad \vec{v}_s = \frac{\hbar}{m} \nabla_x \phi\, n_s$$

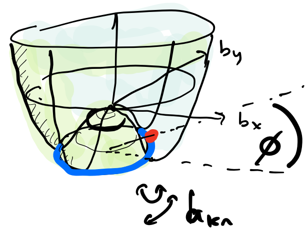
$$\vec{P}|\psi\rangle = (\hbar \vec{\nabla}_x \phi)\, \psi |\psi\rangle$$

$$\frac{\langle \bar{\psi} | \frac{\vec{P}}{m} | \psi \rangle}{\langle \bar{\psi} | \psi \rangle} = \frac{\hbar}{m} (\vec{\nabla}\phi)\, n_s = \vec{J}$$

Superflow

$$\mu \Rightarrow \quad \phi = -\frac{\mu t}{\hbar} \qquad \frac{d\vec{J}}{dt} = -\frac{\hbar n_s}{m} \nabla\mu$$

FLUCTUATIONS



Fluctuations

$$b_k(\tau) = \sqrt{V}\,\tilde{b}_0\,\delta_{\vec{k},0} + \frac{1}{\sqrt{\beta}}\sum b_{kn}\, e^{-i\nu_n \tau}$$

$$S = S_0 + O\left(\bar{a}_k a_k,\ \bar{a}_k \bar{a}_{-k},\ a_{-k} a_k,\ a_{-k}\bar{a}_k\right)$$

$$\frac{\delta S}{\delta b_{kn}} = 0 \quad \text{— saddle point}$$

$$S = S_0 + \sum_{kn} \bar{a}_{kn}\left(-i\nu_n + \omega_k\right) a_{kn} + \frac{g}{2V}\sum \left(\sqrt{V}\,\bar{\tilde{b}}_0\,\delta_{k_1,0} + \bar{a}_{k_1}\right)\left(\sqrt{V}\,\bar{\tilde{b}}_0\,\delta_{k_2,0} + \bar{a}_{k_2}\right)\left(\sqrt{V}\,\tilde{b}_0\,\delta_{k_3,0} + a_{k_3}\right)\left(\sqrt{3}\,\tilde{b}_0\,\delta_{k_4,0} + a_{k_4}\right)\delta_{k_1+k_2-k_3-k_4} + \ldots$$

$$= S_0 + \sum_{kn} \bar{a}_{kn}\left(-i\nu_n + \omega_k\right) a_{kn} + \sum_k \frac{g}{2}\int \left(\bar{a}_k,\ a_{-k}\right)\begin{pmatrix} 2|\tilde{b}_0|^2 & \tilde{b}_0^2 \\ (\bar{\tilde{b}}_0)^2 & 2|\tilde{b}_0|^2 \end{pmatrix}\begin{pmatrix} a_k \\ \bar{a}_{-k} \end{pmatrix} d\tau + \ldots$$

Feedback effect of interactions.

EFFECT OF CONDENSATE

$$S = S_0 + \frac{1}{2}\sum_{k,n} (\bar{a}_{kn}, a_{-k,-n}) \underbrace{\begin{pmatrix} -i\nu_n + \omega_k + 2g|\bar{b}_0|^2 & g\bar{b}_0^2 \\ g(\bar{b}_0)^2 & i\nu_n + \omega_k + 2g|\bar{b}_0|^2 \end{pmatrix}}_{-g^{-1}} \begin{pmatrix} a_{kn} \\ \bar{a}_{-k-n} \end{pmatrix}$$

$$- \frac{1}{2}\left[(\bar{a}_{kn}, a_{-kn}) \begin{pmatrix} J_{kn} \\ \bar{J}_{-k-n} \end{pmatrix} + (\bar{J}_{kn}, J_{-k,-n}) \begin{pmatrix} b_{kn} \\ \bar{b}_{-k-n} \end{pmatrix} \right]$$

$$\int \mathcal{D}[\bar{a},a]\, e^{-S} = e^{-S_0} \int \mathcal{D}[\bar{b},b] \exp^{-\frac{1}{2}(\bar{a}Ma - \bar{j}b - \bar{b}j)}$$

$$= e^{-S_0} \frac{e^{\frac{1}{2}\bar{J}M^{-1}J}}{\sqrt{\det M}} = e^{-S_0} e^{\frac{1}{2}\bar{J}M^{-1}J - \frac{1}{2}\mathrm{Tr}\ln M}$$

$$\boxed{H = \sum \omega_k \hat{b}_k^\dagger \hat{b}_k + 2gn_c \sum b_k^\dagger b_k + gn_c \sum b_k^\dagger b_{-k}^\dagger + gn_c \sum (b_{-k} b_k)}$$

$$a_k = u b_k + v b_{-k}^\dagger$$ Bogoliubov Transformation.

$U^\dagger \tau_3 U = \tau_3$ Preserves structure of Action

$$\Rightarrow [a_k, a_k^\dagger] = u_k^2 - v_k^2 = 1$$

$$\mathcal{H} = \frac{1}{2}\sum_k (b_k^\dagger, b_{-k}) \begin{pmatrix} \omega + 2gn_c & gn_c \\ gn_c & \omega + 2gn_c \end{pmatrix} \begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix}$$

$$= \frac{1}{2}\sum (a_k^\dagger, a_{-k}^\dagger) \begin{pmatrix} \tilde{\omega}_k & 0 \\ 0 & \tilde{\omega}_k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}$$

$$\begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} b \\ b_{-k}^\dagger \end{pmatrix}$$

$$\begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} \tilde{\omega}_k & 0 \\ 0 & \tilde{\omega}_k \end{pmatrix} \begin{pmatrix} u & v \\ v & u \end{pmatrix} = \begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} \tilde{\omega} u & \tilde{\omega} v \\ \tilde{\omega} v & \tilde{\omega} u \end{pmatrix} = \tilde{\omega}_k \begin{pmatrix} u^2+v^2 & 2uv \\ 2uv & u^2+v^2 \end{pmatrix}$$

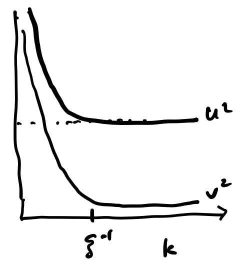
$$\Rightarrow \begin{array}{l} \tilde{\omega}_k (u^2+v^2) = \omega_k + 2gn_c \\ \tilde{\omega}_k 2uv = gn_c \end{array} \Rightarrow \begin{array}{l} \tilde{\omega}_k^2 \overbrace{(u^2-v^2)^2}^{1} = (\omega_k + 2gn_c)^2 - (gn_c)^2 \checkmark \\ \tilde{\omega}_k (2u^2 - 1) = \omega_k + 2gn_c \end{array}$$

$$\Rightarrow \quad u_k^2 = \frac{1}{2}\left[1 + \frac{\omega_k + 2gn_c}{\tilde{\omega}_k}\right]$$

$$v_k^2 = \frac{1}{2}\left[-1 + \frac{\omega_k + 2gn_c}{\tilde{\omega}_k}\right]$$

$u^2 - v^2 = 1$

$v^2 = u^2 - 1$



$$M = -\mathcal{G}^{-1} \qquad \mathcal{G} = \begin{pmatrix} i\nu_n - \omega_k - 2g|b_0|^2 & g b_0^2 \\ g(\bar{b}_0)^2 & -i\nu_n - \omega_k - 2g|b_0|^2 \end{pmatrix}^{-1} = \frac{1}{(i\nu_n)^2 - \tilde{\omega}_k^2} \begin{bmatrix} i\nu_n + \omega_k + 2g|b_0|^2 & g b_0^2 \\ g \bar{b}_0^2 & -i\nu_n + \omega_k + 2g|b_0|^2 \end{bmatrix}$$

$$\tilde{\omega}_k^2 = (\omega_k - 2g|b_0|^2)^2 - (g b_0^2)^2$$

$$\tilde{\omega}_k = \sqrt{(\omega_k + g|b_0|^2)^2 - (g|b_0|^2)^2} = \sqrt{(\omega_k + 2gn_c)^2 - (gn_c)^2}$$

$$= \sqrt{(E_k \underbrace{-\mu + 2gn_c}_{gn_c})^2 - (gn_c)^2}$$

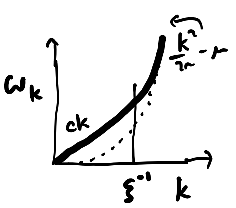
$$2gn_c - \mu = gn_c \qquad \mu = gn_c.$$

Small k

$\boxed{\text{PHONONS}}$

$$\omega_k = \sqrt{(E_k + gn_c)^2 - (gn_c)^2} = \sqrt{E_k^2 + 2E_k g n_c} = \sqrt{\frac{gn_c k^2}{m} + \left(\frac{k^2}{2m}\right)^2}$$

$$\sim c_s k \qquad c_s = \sqrt{\frac{n_c g}{m}}$$



$$E_k \sim gn_c \Rightarrow k^2 \sim 2mgn_c$$

$$\xi^{-1} \sim \sqrt{2mgn_c}$$

$$\xi \sim \frac{1}{\sqrt{2mgn_c}}$$ coherence length.

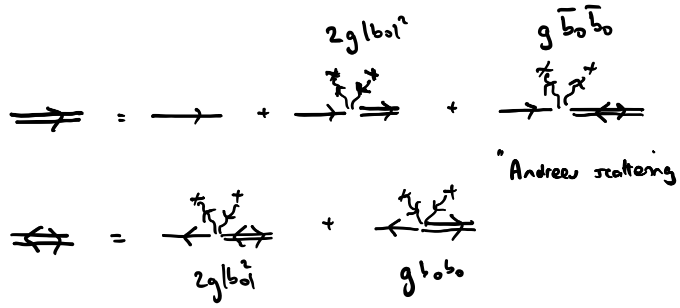
$$-\langle T\, B_k(\tau)\, B_k^+(0) \rangle = -\left\langle T \begin{pmatrix} b_k(\tau) \\ b_{-k}^+(\tau) \end{pmatrix} \otimes \begin{pmatrix} b_k^+(0), & b_{-k}(0) \end{pmatrix} \right\rangle$$

$$= T \sum \mathcal{G}(k, i\nu_n)\, e^{-i\nu_n \tau}$$

$$\boxed{- \begin{pmatrix} \langle T\, b_k(\tau)\, b_k^{\dagger}(0) \rangle & \langle T\, b_k(\tau)\, b_{-k}(0) \rangle \\ \langle T\, b_{-k}^{\dagger}(\tau)\, b_k^{\dagger}(0) \rangle & \langle T\, b_{-k}^{\dagger}(\tau)\, b_{-k}(0) \rangle \end{pmatrix} = T \sum \frac{1}{(i\nu_n)^2 - \tilde{\omega}_k^2} \begin{bmatrix} i\nu_n + \omega_k + 2g|b_0|^2 & g\, b_0^2 \\ g\, \bar{b}_0^2 & -i\nu_n + \omega_k + 2g|b_0|^2 \end{bmatrix} e^{-i\nu_n \tau}}$$

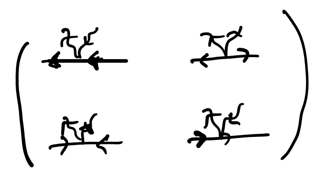
FEYNMAN DIAGRAM PERSPECTIVE

$$\overset{0}{\Rightarrow}\overset{\tau}{} = -\langle b_k(\tau) b_k^{+} \rangle \qquad = = -\langle b_k(\tau) b_{-k}(0) \rangle$$



v = critical velocity of superfluid.
= "sound velocity" of superfluid.

$$\mathcal{G}^{-1} = \mathcal{G}_0^{-1} - V \qquad V = \begin{pmatrix} 2g|b_0|^2 & g b_0^2 \\ g(\bar{b}_0)^2 & 2g|b_0|^2 \end{pmatrix} =$$

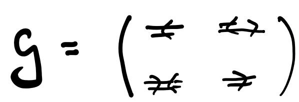


$$\mathcal{G} = \frac{1}{\mathcal{G}_0^{-1} - V}$$

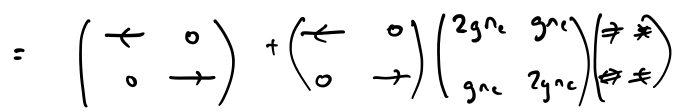
$$= \mathcal{G}_0 \frac{1}{1 - V\mathcal{G}_0} = \mathcal{G}_0 + \mathcal{G}_0 V \mathcal{G}_0 + \dots$$

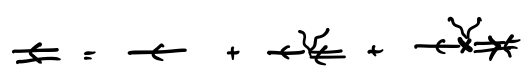
$$\mathcal{G}_0 \mathcal{G}^{-1} = 1 - \mathcal{G}_0 V$$

$$\mathcal{G}_0 = \mathcal{G} - \mathcal{G}_0 V \mathcal{G} \qquad \mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0 V \mathcal{G}$$



$$\sim -\begin{pmatrix} \langle \psi\psi^{+} \rangle & \langle \psi\psi \rangle \\ \langle \psi^{+}\psi^{+} \rangle & \langle \psi^{+}\psi \rangle \end{pmatrix}$$





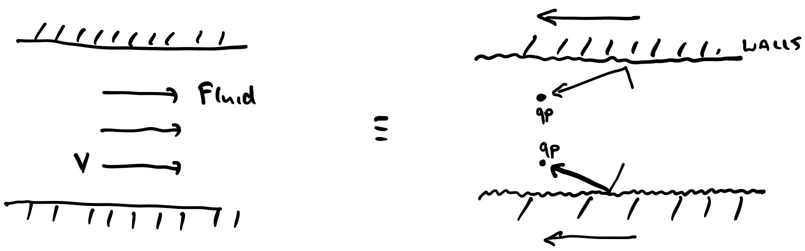
$$\frac{g}{2}\,\psi^{+}\psi^{+}\psi\psi \rightarrow \frac{g}{2}\langle\psi^{+}\rangle^2\psi\psi + \frac{g}{2}\langle\psi\rangle^2\psi^{+}\psi^{+} + 2g\langle\psi^{+}\psi\rangle\psi^{+}\psi$$

$$\mathcal{H} = \sum \epsilon_k \hat{b}_k^{+}\hat{b}_k + 2g n_c \sum b_k^{+} b_k + g n_c \sum \hat{b}_k^{+} b_{-k}^{+} + g n_c \sum (b_{-k} b_k)$$

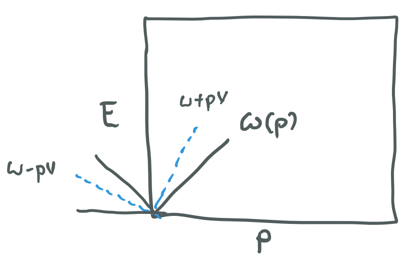
$$= \frac{1}{2}\sum (b^{+}\ b) \begin{pmatrix} & \end{pmatrix} \begin{pmatrix} b \\ b^{+} \end{pmatrix}$$

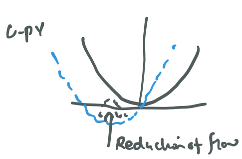
Gallilean Invarance + Superfluidity

In a conventional fluid, set in motion, the walls produce excitations that change the momentum of the fluid, giving rise to a viscosity. Why doesn't this happen in a superfluid?



We can use ideas of Gallilean relativity to address this problem.



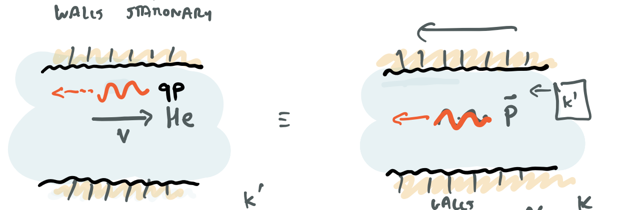


Gallilean Invariance. $u = -\vec{v}$

Excitation $\Delta E = \omega_p + \vec{p}\cdot\vec{v}$

$= \omega_p - pv$

Stationary frame | "Moving" frame



$$\boxed{\begin{aligned} E &= E_0 + \vec{P}_0\cdot\vec{v} + \tfrac{1}{2}Mv^2 \\ &= \omega_p + \vec{p}\cdot\vec{v} + \tfrac{1}{2}Mv^2 \end{aligned}}$$

$E_0 = \omega(p)$

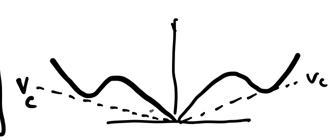
$\vec{P}_0 = \vec{p}$

choose p antiparallel to $\vec{v}$

If $v > \left(\frac{\omega}{p}\right)$ then $\omega_p < 0$

If $\omega = c_s p$

$$\boxed{v_c = c_s = \sqrt{\frac{n_c g}{m}}}$$



$$\boxed{|\Omega\rangle = \underbrace{e^{\sqrt{n_c}\, b^\dagger_{k=0}}}_{\text{One particle}} \; \underbrace{e^{-\sum_{k>0} \frac{v_k}{u_k} b^\dagger_{k} b^\dagger_{-k}}}_{\text{Two particle Condensate.}} |\phi\rangle}$$

$$a_k|\Omega\rangle = 0 = (u b_k + v b^\dagger_{-k})|\Omega\rangle = 0$$

$$[a_k, b^\dagger_k b^\dagger_{-k}] = u b^\dagger_{-k} \qquad a\, b^\dagger_k b^\dagger_{-k} = u b^\dagger_{-k} + b^\dagger_k b^\dagger_{-k} a_k$$

$$a \frac{(b^\dagger_k b^\dagger_{-k})^n}{n!} = u_k b^\dagger_{-k} \frac{(b^\dagger_k b^\dagger_{-k})^{n-1}}{(n-1)!} + \frac{(b^\dagger b^\dagger)^n}{n!} a_k$$

$$a_k \exp\left[-\frac{v_k}{u_k} b^\dagger_k b^\dagger_{-k}\right]|0\rangle = e^{-\frac{v}{u}(b^\dagger b^\dagger)} (\overset{ub+vb^\dagger_{-k}}{a_k} - v_k b^\dagger_{-k})|0\rangle$$

$$= e^{-v/u\, b^\dagger_k b^\dagger_{k}} (u_k b_k)|0\rangle = 0$$