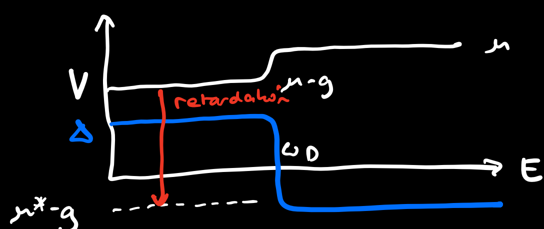
Migdal Eliashberg Theory

$\mu = N(0)\, V^{\text{Coulomb}}$ Coulomb Pseudopotential

$$V_{k,k'} = \frac{1}{N(0)} \begin{cases} \mu - g & \text{Low energies} \quad |\epsilon_k|, |\epsilon_{k'}| < \omega_D \\ \mu & \text{High energies} \quad \text{otherwise} \end{cases}$$



$$\mu^* = \frac{\mu}{1 + \mu \ln \frac{D}{\omega_D}}$$

Coulomb Pseudopotential

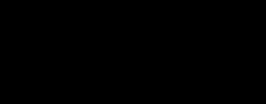
$$\Delta = 2\omega_D \exp\left[- \frac{1}{g-\mu^*} \right]$$

$g-\mu^* > 0$

But says nothing about $g-\mu$!

More generally we expect gap to be frequency dependent.

Migdal Self Energy



Self-consistency.

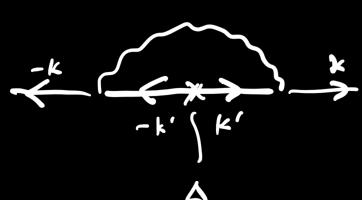
- Vertex corrections $\sim \frac{\omega_D}{\epsilon_F} \sim 5\%$ neglected
- $\Sigma(k,\omega) \sim \overline{\Sigma(k,\omega)} = \Sigma(\omega)$

$$\alpha^2 F(\nu) = \frac{1}{\underset{\uparrow\, d^2k}{\int dS}} \sum_{\lambda} \int \frac{dS\, dS'}{v_F'} \delta(\nu - \omega_{k-k'\lambda}) \left|\alpha_{k,k'\lambda}\right|^2$$

$$\boxed{\Sigma(z) = \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\epsilon\ \alpha^2(\nu) F(\nu) (\mathrm{sgn}\, \nu) \left(\frac{1 + n(\nu) - f(\epsilon)}{z - (\epsilon + \nu)} \right)}$$

$$\lambda = 2 \int_0^{\infty} d\nu\, \frac{\alpha^2(\nu) F(\nu)}{\nu}. \qquad \frac{m^*}{m} = (1 + \lambda)$$

Eliashberg : Combine BCS Theory + Migdal approach.



$$\mathcal{G}_{\mathrm{Local}}(z) = \frac{1}{N(0)} \sum_{k} \frac{1}{z - \epsilon_k} \sim \int d\epsilon\, \frac{1}{z - \epsilon} \qquad \text{Local Propagator}$$

NAMBU BCS

$$\mathcal{G}_{\mathrm{local}}(z) = \frac{1}{N(0)} \sum_{k} \frac{1}{z - \epsilon_k \tau_3 - \Delta_1 \tau_1 - \Delta_2 \tau_2}$$

$$= \int d\epsilon \frac{1}{z - \epsilon\tau_3 - \Delta_1\tau_1 - \Delta_2\tau_2}$$

$$= -\int d\epsilon \left\{ \frac{z + \epsilon\tau_3 + \vec{\Delta}\cdot\vec{\tau}}{-z^2 + \epsilon^2 + \Delta_1^2 + \Delta_2^2} \right\}$$ Lorentzian Integral

$$\mathcal{G}(z) = -\frac{\pi}{\sqrt{|\Delta|^2 - z^2}} \left(z + \vec{\Delta}\cdot\vec{\tau} \right)$$

$$\int d\epsilon \frac{1}{\epsilon^2 + \Gamma^2} = \frac{\pi}{\Gamma}$$

$$\Gamma^2 = \Delta_1^2 + \Delta_2^2 - z^2$$

$$\underline{A}(\omega)\ \frac{1}{\pi}\ \mathrm{Im}\ \mathcal{G}(\omega - i\delta) = \mathrm{Im}\left(\frac{-\pi}{\sqrt{\Delta^2 - (\omega - i\delta)^2}} \left(\omega + \vec{\Delta}\cdot\tau \right) \right)$$

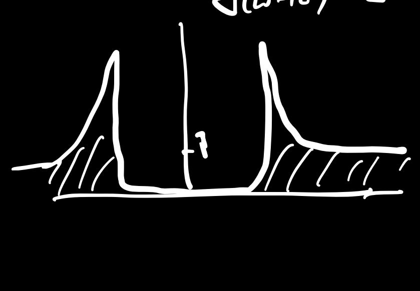
$$A(\omega) = \mathrm{Re}\left(\frac{\mathrm{sgn}\,\omega}{\sqrt{\omega^2 - \Delta^2}} \left(\omega + \vec{\Delta}\cdot\vec{\tau} \right) \right)$$

$$= \left(\frac{|\omega| + \vec{\Delta}\cdot\vec{\tau}(\mathrm{sgn}\,\omega)}{\sqrt{\omega^2 - \Delta^2}} \right)$$

Density of states

$|\omega| > \Delta$.

$$\mathrm{Re}\frac{|\omega|}{\sqrt{(\omega - i\delta)^2 - \Delta^2}}$$



S.c density of states

Local Greens Fn

$$= \underline{\Sigma}(\omega) = \Sigma_\mu(\omega)\tau^\mu$$

Local Nambu Propagator

unit matrix

$$\tau^\mu = (\tau_0, \tau_1, \tau_2, \tau_3)$$

$$= \left[\left(1 - Z(\omega)\right)\omega\, \tau_0 + \varphi_1(\omega)\tau_1 + \varphi_2(\omega)\tau_2 + \Sigma_3(\omega)\tau_3 \right]$$

renormalization of wave function

ω-dependent gap

Normal comp of self energy

Choose a gauge where φ_2 vanishes.

$$\mathcal{G}(\vec{k},\omega) = \frac{1}{\omega - \epsilon_k\tau_3 - \Sigma}$$

$$= \frac{1}{Z\omega - (\epsilon_k + \Sigma_3)\tau_3 - \varphi(\omega)\tau_1}$$

$$\mathcal{G}(k,\omega) = \frac{Z^{-1}(\omega)}{\omega - \dfrac{(\epsilon_k + \Sigma_3(\omega))}{Z(\omega)} - \Delta(\omega)\tau_1}$$

$$\Delta(\omega) = \frac{\varphi(\omega)}{Z(\omega)}$$

$$\mathcal{G}(\epsilon,\omega) = \frac{Z\omega + (\epsilon + \Sigma)\tau_3 + \varphi\tau_1}{(Z\omega)^2 - (\epsilon+\Sigma)^2 - \varphi^2}$$

Local propagator

$$G(z) = \int d\epsilon \; G(\epsilon, z)$$

$$= \int d\epsilon \; \frac{Z\omega + (\epsilon + \Sigma)\tau_3 + \varphi\tau_1}{(Z\omega)^2 - (\epsilon+\Sigma)^2 - \varphi^2}$$

$$\begin{cases} g(\omega) = -\dfrac{\pi}{\sqrt{\varphi^2(\omega) - (Z\omega)^2}} \left(Z\omega + \varphi\tau_1 \right) \\[2em] g(i\omega_n) = \dfrac{-\pi}{\sqrt{\varphi^2(i\omega_n) + (Z\omega_n)^2}} \left(Z i\omega_n + \varphi(i\omega_n)\tau_1 \right) \end{cases}$$

$$\Delta = \varphi / Z$$

$$\boxed{g_L(\omega) = \frac{-\pi}{\sqrt{\Delta(\omega)^2 - \omega^2}} \left(\omega + \Delta(\omega)\tau_1 \right)}$$

$$g(\omega - i\delta) = \int \frac{d\omega'}{\pi} \frac{1}{(\omega - \omega' - i\delta)} \; \mathrm{Im}\left[g(\omega' - i\delta) \right]$$

$$\Sigma_M(z) = \int d\epsilon \, d\nu \; \alpha^2(\nu) F(\nu) \operatorname{sgn}\nu \; \frac{[1 + n(\nu) - f(\epsilon)]}{[z - (\epsilon + \nu)]}$$

$$= \int \frac{d\omega'}{\pi} \int d\nu \; \alpha^2(\nu) F(\nu) \operatorname{sgn}\nu \; \frac{1 + n(\nu) - f(\omega')}{(z - \nu - \omega')} [\operatorname{Im} G_L(\omega')]$$

Local electron propagator

Reduced everything to local propagators.

$$D_{Local}(z) = \int \frac{d\nu}{\pi} \frac{1}{(z - \nu)} \alpha^2(\nu) F(\nu) \operatorname{sgn}\nu$$

$$G_{Local}(z) = \int \frac{d\omega'}{\pi} \frac{1}{z - \omega'} \operatorname{Im} G_L(\omega' - i\delta)$$

$$\boxed{\Sigma(i\omega_n) = -T \sum D_{Local}(i\nu_n) \, G_{Local}(i\omega_n - i\nu_n)}$$

$$\Sigma(\tau) = - G_{Local}(\tau) \, D_{Local}(\tau)$$

Generalize to Nambu $\quad G^{N}_{Local}(i\omega_n) \xrightarrow{?} G^{S.C.}_{Local}(i\omega_n)$

But $\quad i g_{k-k'} \longrightarrow (i g_{k-k'} \tau_3)$

$$H_{EP} = \sum \alpha_q \, c^{+}_{k-q\sigma} \, c_{k\sigma} \left(a^{+}_{q} + a_{-q} \right)$$

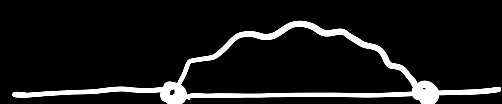
$$= \sum_{q,k} \alpha_q \left(c^{+}_{k-q\uparrow} c_{k\uparrow} - c_{k\downarrow} c^{+}_{k-q\downarrow} \right)\left(a^{+}_{q} + a_{-q} \right)$$

$$= \sum_{q,k} \alpha_q \left(c^{+}_{k-q\uparrow} c_{k\uparrow} - c_{-k\downarrow} c^{+}_{-k-q\downarrow} \right)\left(a_{-q} + a^{+}_{q} \right)$$

$$\left.\begin{array}{l} -k \to -k+q \\ -k-q \to -k \end{array}\right\}$$

$$= \sum \alpha_q \left(c^{+}_{k-q\uparrow} c_{k\uparrow} - c_{-k+q\downarrow} c^{+}_{-k\downarrow} \right)\left(a_{-q} + a^{+}_{q} \right)$$

$$= \sum \alpha_q \left(c^{+}_{k-q} \tau_3 c_k \right) \left(a^{+}_{q} + a_{-q} \right)$$



$$\Sigma(k, i\omega_n) = -T \sum_{q} D(q) \, \tau_3 \, G(k-q) \, \tau_3 \, \alpha_q^2$$

$$\to \quad -T \sum_{i\nu_n} \left[D_{Local}(i\nu_n) \, \tau_3 \, G_{Local}(i\omega_n - i\nu_n) \, \tau_3 \right]$$

Local P.P

$$D_{Local}(i\nu_n) = \int d\nu \, \frac{\alpha^2 F(\nu) \, \mathrm{sgn}\, \nu}{(i\nu_n - \nu)}.$$

$$G_{local}(i\omega_n) = \frac{-\pi}{\sqrt{\Delta^2+\omega_n^2}}\left(i\omega_n + \Delta\tau_1\right)$$

$$\boxed{\Sigma(i\omega_n) = -T\sum_{i\nu_n}\int d\nu \,\mathrm{sgn}\,\nu\; \alpha^2(\nu)F(\nu)\,\frac{1}{(i\nu_n-\nu)}\,\tau_3\, G^L(i\omega_n - i\nu_n)\tau_3}$$

$$\boxed{(1-Z)\,i\omega_n + Z(i\omega_n)\Delta(i\omega_n)\tau_1}$$

(e-p spectrum)

$$G_{local}(i\omega_n - i\nu_n) = \int\frac{d\omega'}{\pi}\,\frac{1}{i\omega_n - i\nu_n - \omega'}\,\mathrm{Im}\,G_L(\omega' - i\delta)$$

$$\boxed{\begin{aligned}&(1-Z(\omega))\omega + Z(\omega)\Delta(\omega)\tau_1\\ &= \int\frac{d\omega'}{\pi}\int d\nu\;\alpha^2(\nu)F(\nu)\,\mathrm{sgn}\,\nu\;\frac{1+n(\nu)-f(\omega')}{(z-\nu-\omega')}\left[\mathrm{Im}\,G_L(\omega')\right]\end{aligned}}$$

Local electron propagator

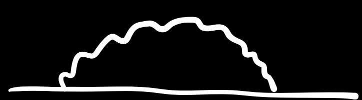
$$\xi(\omega) = (1-Z(\omega))\omega = -\int_{-\infty}^{\infty} d\omega'\;\mathrm{Im}\left[\frac{\omega'}{\sqrt{\Delta^2(\omega')-\omega'^2}}\right]\times\int_{-\infty}^{\infty} d\nu\;\alpha^2(\nu)F(\nu)\,\mathrm{sgn}\,\nu$$

$$\mathrm{Re}\left[\frac{|\omega'|}{\sqrt{\omega'^2-\Delta^2(\omega')}}\right] \times \frac{1}{\omega-(\omega'+\nu)}$$

$$Z(\omega) = 1 + \frac{1}{\omega}\int_{-\omega_D}^{\omega_D} d\omega' \, \mathrm{Re}\left[\frac{|\omega'|}{\sqrt{\omega'^2-\Delta^2(\omega')}}\right] \int_{-\omega_D}^{\omega_D} d\nu \, \frac{\alpha^2(\nu) F(\nu)\, \mathrm{sgn}\,\nu}{[\omega-(\omega'+\nu)]}$$

$$\mathrm{Re}\left[\frac{\Delta \, \mathrm{sgn}\,\omega}{\sqrt{\omega^2-\Delta^2(\omega)}}\right] \qquad \times \left[1 + \underbrace{n(\nu)}_{\text{vanishes at } T=0} - f(\omega')\right]$$

$$Z(\omega)\Delta(\omega) = \int_{-\omega_D}^{\omega_D} d\omega' \, \mathrm{Im}\left[\frac{\Delta(\omega')}{\sqrt{\Delta(\omega')^2-\omega'^2}}\right] \times \int d\nu \, \alpha^2 F \, \mathrm{sgn}\,\nu \times \frac{[1+n(\nu)-f(\omega')]}{\omega-(\omega'+\nu)}$$

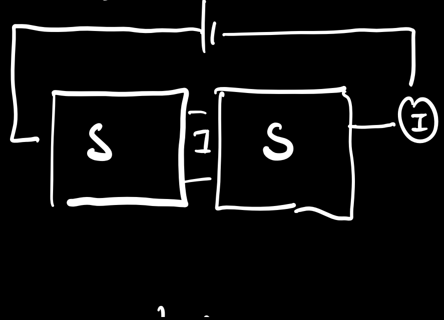


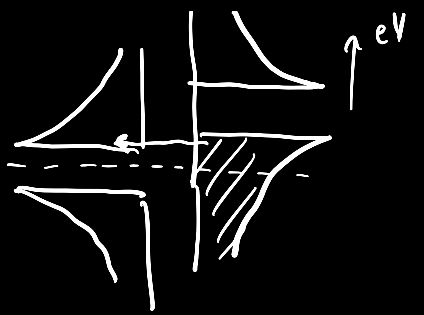
$$H_I = H_{EP} + \sum \overset{[\mu/N(0)]}{V_{k,k'}} (c^+_k c^+_{-k})(c_{-k'} c_{k'})$$

ϵ_k — Landau Fermi liquid dispersion. $m \to m^*$

$$\boxed{H_{\mathrm{Coulomb}} = \frac{\mu^*}{N(0)} \sum_{k,k',\ |\epsilon_k| < \omega_D} (c^+_k c^+_{-k})(c_{-k'} c_{-k'})}$$

$$Z(\omega)\Delta(\omega) = \int_{-\omega_D}^{\omega_D} d\omega' \, \mathrm{Re}\, \frac{\Delta(\omega')\,\mathrm{sgn}(\omega')}{\sqrt{\omega'^2-\Delta^2(\omega')}} \left(\overbrace{\int d\nu \, \alpha^2(\nu) F(\nu)\, \mathrm{sgn}\,\nu \left[\frac{1+n(\nu)-f(\omega')}{\omega-(\omega'+\nu)}\right]}^{\lambda(\omega';\nu)} - \mu^* \right)$$





$$\boxed{I(V)} \propto \int d\omega \, N_s(\omega) N_s(\omega - V)$$

$$= \int d\omega \, N_s(\omega) N_s(V-\omega)$$

$$= \int d\omega \, N_s(V-\omega) N_s(\omega)$$

Convolution

Guess $\alpha^2 F \rightarrow$ adjust by taking Functional derivative!

MacMillan + Dynes
Bell Labs 1966

$$\delta \alpha^2 F \longrightarrow \delta N_s(\omega)$$

$$N_s^{calc}\Big[\overset{\substack{guess\\ \downarrow}}{\alpha^2 F} + \overset{\downarrow}{\delta}[\alpha^2 F] \Big] = N_s^{exp}[\omega].$$

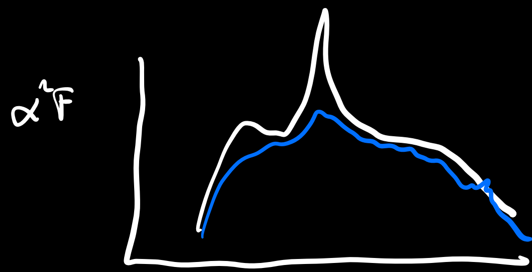
$$N_s^{calc} + \frac{\delta N_s^{calc}[\alpha^2 F(\omega)]}{\delta[\alpha^2(\omega') F(\omega')]} = N_s^{exp}[\omega]$$

cal

$$\frac{\delta N_s^{cal}(\omega)}{\delta \alpha^2 F(\omega')} = N_s^{exp}[\omega'] - N_s^{calc}[\omega']$$

$$\delta(\alpha^2 F)(\omega) = \int d\omega' \underbrace{\frac{\delta \alpha^2 F(\omega)}{\delta N(\omega')}} \; N_s^{exp}[\omega'] - N_s^{calc}[\omega']$$

$$\delta[\alpha^2 F(\omega)] \quad \int d\omega' \left(\frac{\delta N(\omega')}{\delta \alpha^2 F(\omega)} \right)^{-1} \delta N_s(\omega')$$



Neutrons $\leftrightarrow$ Tunneling Match!

Pb, Hg.