14 Superconductivity and BCS theory

14.1 Introduction: early history

Superconductivity, the phenomenon whereby the resistance of a metal spontaneously drops to zero upon cooling below its critical temperature, was discovered over a hundred years ago by Heike Kamerlingh Onnes in 1911. However, it took another 46 years for the development of the conceptual framework required to understand this collective phenomenon as a condensation of electron pairs. During this time, many great physicists, including Bohr, Einstein, Heisenberg, Bardeen, and Feynman, had tried to develop a microscopic theory of the phenomenon. Today, superconductivity has been observed in a wide variety of materials (see Table 14.1), with transition temperatures reaching up as high as high as 134 K.

The development of the theory of superconductivity leading to BCS theory really had two parts – one phenomenological, the second microscopic. Let me mention some highlights:

- The discovery of the Meissner effect in 1933 by Walther Meissner and Robert Ochsenfeld [1]. When a metal is cooled in a small magnetic field, the flux is spontaneously excluded as the metal becomes superconducting (see Figure 14.1). The Meissner effect demonstrates that a superconductor is, in essence, a perfect diamagnet.
- Rigidity of the wavefunction. In 1937 Fritz London, working at Oxford [2, 3], proposed that a persistent supercurrent is a property of the *ground state* associated with its *rigidity* against the application of a field. London's idea applies to the full many-body wavefunction, but he initially developed it using a phenomenological one-particle wavefunction $\psi(x)$ that today we call the *superconducting order parameter*. He noted that the quantum mechanical current contains a *paramagnetic* and a *diamagnetic* component, writing

$$j = \frac{\hbar e}{2im}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) - \left(\frac{e^2}{m}\right)\psi^*\psi\vec{A}. \tag{14.1}$$

  In the ground state in the absence of a field ($\vec{A} = 0$), the current vanishes, so the ground-state wavefunction ψ_0 must be uniform. Normally, the wavefunction is highly sensitive to an external magnetic field, but London reasoned that, if the wavefunction is somehow *rigid* and hence unchanged to linear order in the magnetic field, $\psi(x) = \psi_0(x) + O(B^2)$, where ψ_0 is the ground-state wavefunction, then, to leading order in a field, the current carried by the uniform quantum state is

Table 14.1 Selected superfluids/superconductors.

Symmetry	Superfluid/superconductor	T_c	Mechanism
s	Hg	4.2 K	Phonon-mediated
	Pb	7.2 K	
	$NbGe_3$	23 K	
	MgB_2	39 K	
p	^3He	2.5 mK	Magnetic interactions
	UPt_3	0.51 K	
	Sr_2RuO_4	0.93 K	
d	$CeCu_2Si_2$	0.65 K	
	$PuCoGa_7$	18.5 K	
	$HgBa_2Ca_2Cu_3O_8$	134 K	
$s^{\pm}$	$Sr_{0.5}Sm_{0.5}FeAsF$	56 K	

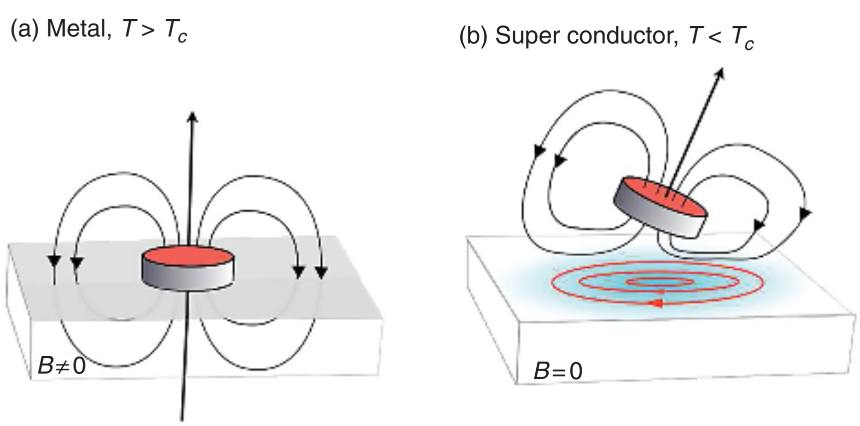


Fig. 14.1 (a) A magnet rests on top of a normal metal, with its field lines penetrating the metal. (b) Once cooled below T_c, the superconductor spontaneously excludes magnetic fields, generating persistent supercurrents at its surface, causing the magnet to levitate.

$$\vec{j} = -\frac{e^2}{m}|\psi_0|^2\vec{A} + \cdots . \tag{14.2}$$

In London's equation we see a remarkable convergence of the classical and the quantum: it is certainly a classical equation of motion in that it involves purely macroscopic variables, yet on the other hand it contains a naked vector potential $\vec{A}$ rather than the magnetic field $\vec{B} = \nabla \times \vec{A}$, a feature which reflects the broken gauge symmetry of the quantum ground state.

London's equation provides a natural explanation of the Meissner effect. To see this, we use Ampère's relation $\vec{j} = \mu_0^{-1}\nabla \times \vec{B}$ to rewrite the current in London's terms of the magnetic field:

$$\nabla \times \vec{B} = -\frac{1}{\lambda_L^2}\vec{A} \qquad \left(\frac{1}{\lambda_L^2} = \mu_0\frac{e^2}{m}|\psi_0|^2\right), \tag{14.3}$$

where the quantity λ_L defined above is the *London penetration depth*. Taking the curl of (14.3), we eliminate the vector potential to obtain

$$\overbrace{\nabla \times \nabla \times \vec{B}}^{-\nabla^2 \vec{B}} = -\frac{1}{\lambda_L^2} \overbrace{\nabla \times \vec{A}}^{\vec{B}} \tag{14.4}$$

or

$$\nabla^2 B = \frac{1}{\lambda_L^2} \vec{B}, \tag{14.5}$$

where we have substituted $\nabla \times (\nabla \times \vec{B}) = \vec{\nabla}(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = -\nabla^2 \vec{B}$, using the divergence-free nature of the magnetic field. The solutions of this equation describe magnetic fields $B(x) \sim B_0 e^{\pm x/\lambda_L}$ which decay inside the superconductor over a London penetration depth. This exclusion of magnetic fields inside superconductors is precisely the Meissner effect.

- Ginzburg–Landau theory [4]. In 1950, Lev Landau and Vitaly Ginzburg in Moscow reinterpreted London's phenomenological wavefunction $\psi(x)$ as a *complex-order parameter*. Using arguments of gauge invariance, they reasoned that the free energy must contain a gradient term that instills the rigidity of the order parameter:

$$f = \int d^3x \frac{1}{2m^*} |(-i\hbar\vec{\nabla} - e^*\vec{A})\psi|^2. \tag{14.6}$$

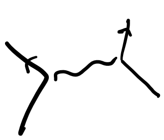
(At this stage, the identification of $e^* = 2e$ as the Cooper pair charge had not been made.) The vitally important aspect of this gauge-invariant functional (see Section 11.5) is that, once $\psi \neq 0$, the electromagnetic field develops a mass, giving rise to a super-current

$$\vec{j}(x) = -\delta f/\delta\vec{A}(x) = -\frac{(e^*)^2}{m^*} |\psi_0|^2 \vec{A}(x) \tag{14.7}$$

for a uniform $\psi = \psi_0$.

Following the Second World War, physicists set to work to try to develop a microscopic theory of superconductivity. The development of quantum field theory and new experimental techniques, such as microwaves – a biproduct of radar – and the availability of isotopes after the Manhattan Project, meant that a new intellectual offensive could begin. The landmark events included:

- Theory of the electron–phonon interaction. In 1949–1950, Herbert Fröhlich at Purdue and Liverpool universities [5] formulated the electron–phonon interaction as a direct analogue of photon exchange in electromagnetism. He showed that it gives rise to a low-energy attractive interaction,



$$V_{eff}(\mathbf{k}, \mathbf{k}') = -g^2_{\mathbf{k}-\mathbf{k}'} \frac{2\omega_{\mathbf{k}-\mathbf{k}'}}{\omega^2_{\mathbf{k}-\mathbf{k}'} - (\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2}, \tag{14.8}$$

where $\epsilon_{\mathbf{k}}$ and $\epsilon_{\mathbf{k}'}$ are the energies of incoming and outgoing electrons, while $\omega_{\mathbf{q}}$ is the phonon frequency. $V_{eff}(\mathbf{k}, \mathbf{k}')$ is attractive for low-energy transfer, $|\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}| << \omega_{\mathbf{k}-\mathbf{k}'}$.
- Discovery of the isotope effect. In 1950, Emanuel Maxwell at the National Bureau of Standards [7] and the group of Bernard Serin at Rutgers University [8] observed a reduction in the superconducting transition temperature with the isotopic mass in mercury.

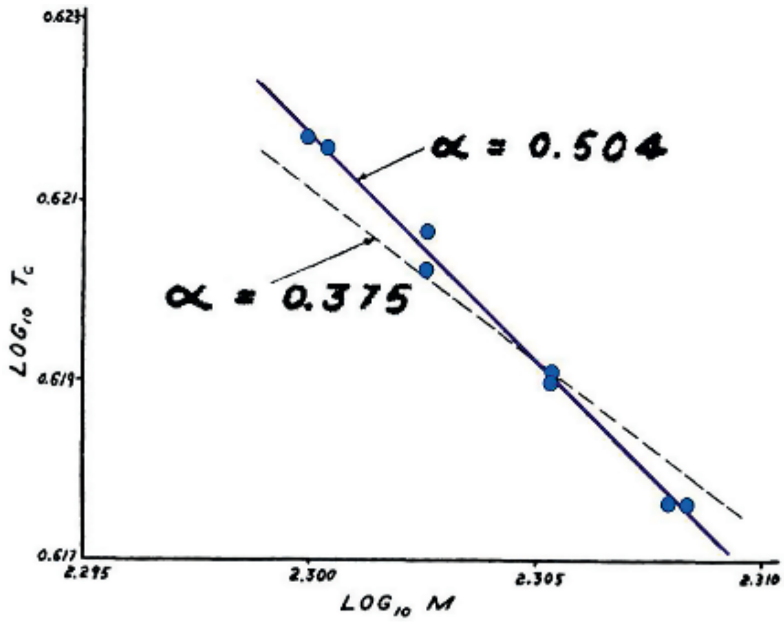
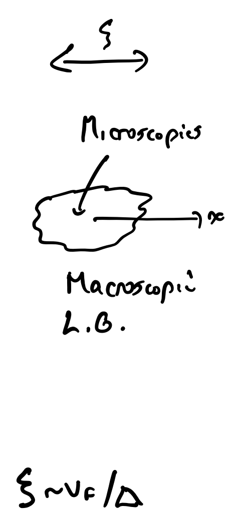


Fig. 14.2 Superconducting transition temperature as a function of isotopic mass for mercury, showing the $-\frac{1}{2}$ exponent, implying phonon-driven superconductivity. Reprinted with permission from B. Serin, *et al.*, *Phys. Rev.*, vol. 80, p. 761, 1950.

It now became clear that the electron–phonon interaction provided the key to superconductivity. Indeed, in *any* theory in which the transition temperature is proportional to the Debye temperature, the expected dependence on isotopic mass M is given by [9]

$$T_c \propto \omega_D \sim \frac{1}{\sqrt{M}} \Rightarrow \frac{d \ln T_c}{d \ln M} = -\frac{1}{2}. \tag{14.9}$$

Careful analysis showed agreement with the $-\frac{1}{2}$ exponent [6] (see Figure 14.2), but what was the mechanism?



- Discovery of the coherence length. In 1953 Brian Pippard at the Cavendish Laboratory in Cambridge [10, 11] proposed, based on his thesis work on the anomalous skin depth in dirty superconductors, that the character of superconductivity changes at short distances, below a scale he named the *coherence length* ξ. Pippard showed that, at these short distances, the local London relation between current and vector potential is replaced by a non-local relationship. Pippard's result means that Ginzburg–Landau theory is inadequate at distances shorter than the coherence length ξ, demanding a microscopic theory.
- Gap hypothesis. In 1955 John Bardeen, who had recently resigned from Bell Laboratories to pursue his research into the theory of superconductivity at the University of Illinois Urbana-Champaign, proposed that if a gap Δ developed in the electron spectrum this would account for the wavefunction rigidity proposed by London and would also give rise to Pippard's coherence length $\xi \sim v_F/\Delta$, where v_F is the Fermi velocity [12]. What was now needed was a model and mechanism to create the gap.
- Bardeen–Pines Hamiltonian. In 1955 John Bardeen and David Pines at the University of Illinois Urbana-Champaign [13] rederived the Fröhlich interaction as a second-quantized

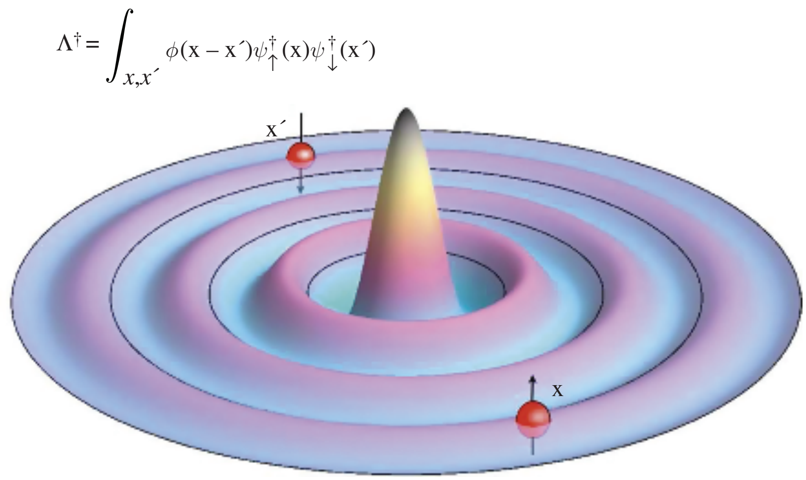


Fig. 14.3 Illustration of a Cooper pair. (Note: the location of the electrons relative to the pair wavefunction involves artistic license since the wavefunction describes the *relative* position of the two electrons.)

model, incorporating the effects of the Coulomb interaction in a "Jellium model" in which the ions form a smeared positive background (see Section 7.7.3). The Bardeen–Pines effective interaction takes the form

$$V_{BP}(\mathbf{q}, \nu) = \frac{e^2}{\epsilon_0(q^2 + \kappa^2)}\left[1 + \frac{\omega_{\mathbf{q}}^2}{\nu^2 - \omega_{\mathbf{q}}^2}\right], \tag{14.10}$$

where κ^{-1} is the Thomas–Fermi screening length and the phonon frequency $\omega_{\mathbf{q}}$ is related to the plasma frequency of the ions $\Omega_p^2 = (Ze)^2 n_{ion}/(\epsilon_0 M)$ via the relation $\omega_{\mathbf{q}} = (q/[q^2 + \kappa^2]^{1/2})\Omega_p$. The Bardeen–Pines interaction is seen to contain two terms: a frequency-independent Coulomb interaction, and a strongly frequency-dependent electron–phonon interaction. In the time domain, the former corresponds to an instantaneous Coulomb repulsion, while the latter is a highly retarded attractive interaction. This interaction became the basis for BCS theory.

The stage was set for Bardeen–Cooper–Schrieffer (BCS) theory.

14.2 The Cooper instability

In the fall of 1956, Bardeen's postdoc Leon Cooper, at the University of Illinois Urbana-Champaign, solved one of the most famous "warm-up" problems of all time. Considering two electrons moving above the Fermi surface of a metal, Cooper found that an arbitrarily weak electron–electron attraction induces a two-particle bound state that will destabilize the Fermi surface [14].

Cooper imagined adding a pair of electrons above the Fermi surface in a state with no net momentum, described by the wavefunction

$$|\Psi\rangle = \Lambda^\dagger|FS\rangle, \tag{14.11}$$

where

$$\Lambda^\dagger = \int d^3x d^3x' \phi(\mathbf{x}-\mathbf{x}')\psi^\dagger_\downarrow(\mathbf{x})\psi^\dagger_\uparrow(\mathbf{x}') \tag{14.12}$$

creates a pair of electrons, while $|FS\rangle = \prod_{k<k_F} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}|0\rangle$ defines the filled sea. If we Fourier transform the fields, writing $\psi^\dagger_\sigma(\mathbf{x}) = \frac{1}{\sqrt{V}}\sum_\mathbf{k} c^\dagger_{\mathbf{k}\sigma} e^{-i\mathbf{k}\cdot\mathbf{x}}$, then the pair creation operator can be recast as a sum over pairs in momentum space:

$$\Lambda^\dagger = \sum_\mathbf{k} \phi_\mathbf{k} c^\dagger_{\mathbf{k}\downarrow} c^\dagger_{-\mathbf{k}\uparrow}, \tag{14.13}$$

Cooper pair creation operator

where

$$\phi_\mathbf{k} = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{x}) \tag{14.14}$$

is the Fourier transform of the spatial pair wavefunction. This result tells us that a real-space pair of fermions can be decomposed into a sum of momentum-space pairs, weighted by the amplitude $\phi_\mathbf{k}$. The properties of the pair (and the superconductor it will give rise to) are encoded in the pair wavefunction $\phi_\mathbf{k}$. In the phonon-mediated superconductors considered by BCS, $\phi_\mathbf{k} \sim f(k)$ is an isotropic s-wave function, but in a rapidly growing class of *anisotropically paired superfluids* of great current interest, including superfluid ^{3}He, heavy-fermion, and iron- and copper-based high-temperature superconductors $\phi_\mathbf{k}$ is anisotropic changing sign *somewhere* in momentum space to lower the repulsive interaction energy, giving rise to a *nodal pair wavefunction*.

When an electron pair is created, electrons can only be added above the Fermi surface, so that

$$|\Psi\rangle = \Lambda^\dagger|FS\rangle = \sum_{|\mathbf{k}|>k_F} \phi_\mathbf{k}|\mathbf{k}_P\rangle, \tag{14.15}$$

where $|\mathbf{k}_P\rangle \equiv |\mathbf{k}\uparrow, -\mathbf{k}\downarrow\rangle = c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}|FS\rangle$. Now suppose that the Hamiltonian has the form

$$H = \sum_\mathbf{k} \epsilon_\mathbf{k} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \hat{V}, \tag{14.16}$$

where $\hat{V}$ contains the details of the electron–electron interaction; if $|\Psi\rangle$ is an eigenstate with energy E, then

$$H|\Psi\rangle = \sum_{|\mathbf{k}|>k_F} 2\epsilon_\mathbf{k}\phi_\mathbf{k}|\mathbf{k}_P\rangle + \sum_{|\mathbf{k}|,\ |\mathbf{k}'|>k_F} |\mathbf{k}_P\rangle\langle\mathbf{k}_P|\hat{V}|\mathbf{k}'_P\rangle\phi_{\mathbf{k}'}. \tag{14.17}$$

Identifying this with $E|\Psi\rangle = E\sum_\mathbf{k}\phi_\mathbf{k}|\mathbf{k}_P\rangle$, so comparing the amplitudes to be in the state $|\mathbf{k}_P\rangle$,

$$E\phi_\mathbf{k} = 2\epsilon_\mathbf{k}\phi_\mathbf{k} + \sum_{|\mathbf{k}'|>k_F} \langle\mathbf{k}_P|\hat{V}|\mathbf{k}'_P\rangle\phi_{\mathbf{k}'}. \tag{14.18}$$

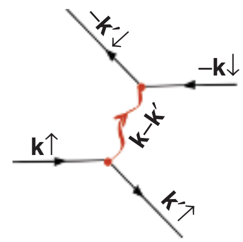


Fig. 14.4 Virtual phonon exchange process responsible for the BCS interaction. The process $|\mathbf{k}\uparrow, -\mathbf{k}\downarrow\rangle \rightarrow |\mathbf{k}'\uparrow, -\mathbf{k}'\downarrow\rangle$ can be thought of as the consequence of Bragg diffraction of a virtual standing wave: one electron in the pair $|\mathbf{k}\uparrow, -\mathbf{k}\downarrow\rangle$ diffracts from $\mathbf{k} \rightarrow \mathbf{k}'$, creating a virtual standing wave (phonon) of momentum $\mathbf{k} - \mathbf{k}'$. Later, the second diffracts from $-\mathbf{k} \rightarrow -\mathbf{k}'$, reabsorbing the virtual phonon.

The beauty of this equation is that the details of the electron interactions are entirely contained in the pair scattering matrix element $V_{\mathbf{k},\mathbf{k}'} = \langle \mathbf{k}_P|\hat{V}|\mathbf{k}'_P\rangle$. Microscopically, this scattering is produced by the exchange of virtual phonons (in conventional superconductors), and the scattering matrix element is determined by the electron–phonon propagator

$$V_{\mathbf{k},\mathbf{k}'} = g^2_{\mathbf{k}-\mathbf{k}'} D(\mathbf{k}' - \mathbf{k}, \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}), \tag{14.19}$$

as illustrated in Figure 14.4. Cooper noted that this matrix element is not strongly momentum-dependent, only becoming attractive within an energy ω_D of the Fermi surface, and this motivated a simplified model interaction in which

$$V_{\mathbf{k},\mathbf{k}'} = \begin{cases} -g_0/V & (|\epsilon_{\mathbf{k}}|,\ |\epsilon_{\mathbf{k}'}| < \omega_D) \\ 0 & (\text{otherwise}). \end{cases} \tag{14.20}$$

This is a piece of pure physics *haiku*, a brilliant simplification that makes BCS theory analytically tractable. Much more is to come, but for the moment it enables us to simplify (14.18):

$$(E - 2\epsilon_{\mathbf{k}})\phi_{\mathbf{k}} = -\frac{g_0}{V} \sum_{0<\epsilon_{\mathbf{k}'}<\omega_D} \phi_{\mathbf{k}'}, \tag{14.21}$$

so that by solving for $\phi_{\mathbf{k}}$,

$$\phi_{\mathbf{k}} = -\frac{g_0/V}{E - 2\epsilon_{\mathbf{k}}} \sum_{0<\epsilon_{\mathbf{k}'}<\omega_D} \phi_{\mathbf{k}'}, \tag{14.22}$$

then summing both sides over $\mathbf{k}$ and factoring out $\sum_{\mathbf{k}} \phi_{\mathbf{k}}$, we obtain the self-consistent equation

$$1 = -\frac{1}{V} \sum_{0<\epsilon_{\mathbf{k}}<\omega_D} \frac{g_0}{E - 2\epsilon_{\mathbf{k}}}. \tag{14.23}$$

Replacing the summation by an integral over energy, $\frac{1}{V}\sum_{0<\epsilon_{\mathbf{k}}<\omega_D} \to N(0)\int_0^{\omega_D}$, where $N(0)$ is the density of states per spin per unit volume at the Fermi energy, the resulting equation gives

$$1 = g_0 N(0)\int_0^{\omega_D}\frac{d\epsilon}{2\epsilon - E} = -\frac{1}{2}g_0 N(0)\ln\left[\frac{2\omega_D - E}{-E}\right] \approx -\frac{1}{2}g_0 N(0)\ln\left[\frac{2\omega_D}{-E}\right], \tag{14.24}$$

where, anticipating the smallness of $|E| << \omega_D$, we have approximated $2\omega_D - E \approx 2\omega_D$. In other words, the energy of the Cooper pair is given by

$$E = -2\omega_D e^{-\frac{2}{g_0 N(0)}}. \tag{14.25}$$

Remarks

- The Cooper pair is a bound state beneath the particle–hole continuum (see Figure 14.5).
- In his seminal paper, Cooper notes that the Cooper pair is a boson, an operator governed by a bosonic (commutator) algebra. (We will see shortly that it can be regarded as the transverse component of a very large isospin.) This changes everything, for, as pairs, electrons can *condense macroscopically*.
- A generalization of the above calculation to finite momentum (see Example 14.1) shows that the Cooper pair has a *linear* dispersion $E_{\mathbf{p}} - E = \mathrm{v}_F p$ (see Figure 14.5), reminiscent of a collective mode.

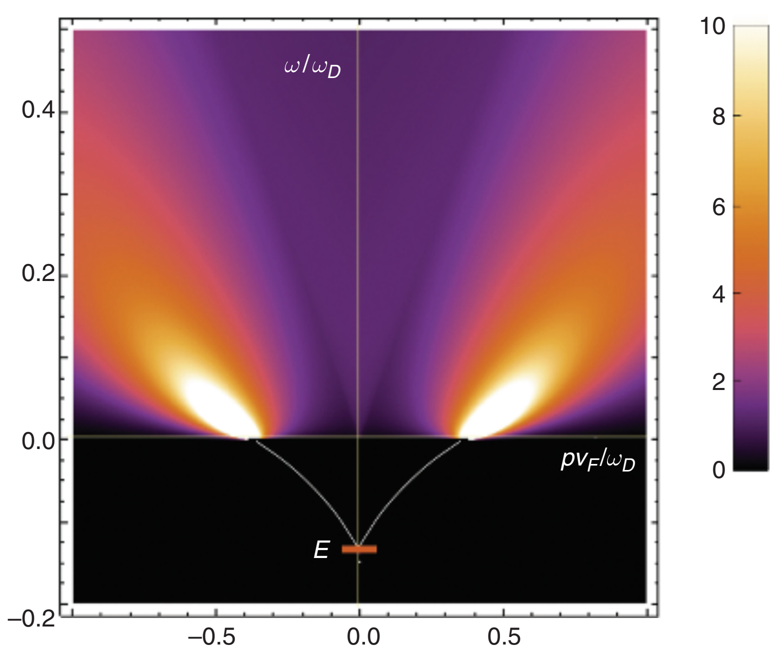


Fig. 14.5 Formation of a Cooper pair beneath the two-particle continuum. This density plot shows the density of states of pair excitations obtained from the imaginary part of the pair susceptibility $\chi''(E, \mathbf{p})$ (see Example 14.1). At a finite momentum, the Cooper pair energy defines a collective bosonic mode beneath the quasiparticle continuum with dispersion $E_{\mathbf{p}} \approx E(0) + \mathrm{v}_F|p|$.

Example 14.1 Generalize Cooper's calculation to a pair with finite momentum. In particular:

(a) Show that the operator that creates a Cooper pair at a finite momentum $\mathbf{p}$,

$$\Lambda^\dagger(\mathbf{p}) = \int d^3x d^3x' \phi(\mathbf{x}-\mathbf{x}')\psi^\dagger_\uparrow(\mathbf{x})\psi^\dagger_\downarrow(\mathbf{x}')e^{i\mathbf{p}\cdot(\mathbf{x}+\mathbf{x}')/2}, \tag{14.26}$$

can be rewritten in the form

$$\Lambda^\dagger(\mathbf{p}) = \sum_{\mathbf{k}} \phi(\mathbf{k}) c^\dagger_{\mathbf{k}+\mathbf{p}/2\uparrow} c^\dagger_{-\mathbf{k}+\mathbf{p}/2\downarrow}. \tag{14.27}$$

(b) Show that the energy $E_{\mathbf{p}}$ of the pair state $\Lambda^\dagger(\mathbf{p})|FS\rangle$ is given by the roots $z = E_{\mathbf{p}}$ of the equation

$$1 + \frac{g_0}{V} \sum_{0<\epsilon_{\mathbf{k}\pm\mathbf{p}/2}<\omega_D} \frac{1}{z - (\epsilon_{\mathbf{k}+\mathbf{p}/2} + \epsilon_{\mathbf{k}-\mathbf{p}/2})} = 0. \tag{14.28}$$

Demonstrate that this equation predicts a linear dispersion given by

$$E_{\mathbf{p}} = -2\omega_D e^{-\frac{2}{g_0 N(0)}} + \mathrm{v}_F|p|. \tag{14.29}$$

Solution

(a) Introducing center-of-mass variables $\mathbf{X} = (\mathbf{x}+\mathbf{x}')/\mathbf{2}$ and $\mathbf{r} = \mathbf{x}-\mathbf{x}'$, using $d^3x d^3x' = d^3X d^3r$, we rewrite the Cooper pair creation operator in the form

$$\Lambda^\dagger(\mathbf{p}) = \int d^3r d^3X \, e^{i\mathbf{p}\cdot\mathbf{X}}\phi(\mathbf{r}) \, \psi^\dagger_\uparrow(\mathbf{X}+\mathbf{r}/2)\psi^\dagger_\downarrow(\mathbf{X}-\mathbf{r}/\mathbf{2}). \tag{14.30}$$

If we substitute $\psi^\dagger_\sigma(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} e^{-i\mathbf{k}\cdot\mathbf{x}}$, we then obtain

$$\begin{aligned}
\Lambda^\dagger(p) &= \frac{1}{V}\int d^3r d^3X e^{i\mathbf{p}\cdot\mathbf{X}}\phi(\mathbf{r}) \sum_{\mathbf{k}_1,\mathbf{k}_2} c^\dagger_{\mathbf{k}_1\uparrow} c^\dagger_{\mathbf{k}_2\downarrow} e^{-i\mathbf{k}_1\cdot(\mathbf{X}+\mathbf{r}/2)} e^{i\mathbf{k}_2\cdot(\mathbf{X}-\mathbf{r}/2)} \\
&= \sum_{\mathbf{k}_1,\mathbf{k}_2} c^\dagger_{\mathbf{k}_1\uparrow} c^\dagger_{-\mathbf{k}_2\downarrow} \overbrace{\int d^3r \, \phi(\mathbf{r}) e^{i\mathbf{r}\cdot(\mathbf{k}_1+\mathbf{k}_2)/2}}^{\phi((\mathbf{k}_1+\mathbf{k}_2)/2)} \overbrace{\frac{1}{V}\int d^3R \, e^{i[\mathbf{p}-(\mathbf{k}_1-\mathbf{k}_2)]\cdot\mathbf{X}}}^{\delta_{\mathbf{p}-(\mathbf{k}_1-\mathbf{k}_2)}} \\
&= \sum_{\mathbf{k}} \phi(\mathbf{k}) \, c^\dagger_{\mathbf{k}+\mathbf{p}/2\uparrow} c^\dagger_{-\mathbf{k}+\mathbf{p}/2\downarrow},
\end{aligned} \tag{14.31}$$

where we have replaced $(\mathbf{k}_1+\mathbf{k}_2)/2 \to \mathbf{k}$ in the last step.

(b) Denote a Cooper pair with momentum $\mathbf{p}$ by

$$\Lambda^\dagger(\mathbf{p})|FS\rangle \equiv |\psi(\mathbf{p})\rangle = \sum_{\mathbf{k}} \phi_{\mathbf{k}}|\mathbf{k},\mathbf{p}\rangle, \tag{14.32}$$

where $|\mathbf{k},\mathbf{p}\rangle = c^\dagger_{\mathbf{k}+\mathbf{p}/2\uparrow} c^\dagger_{-\mathbf{k}+\mathbf{p}/2\downarrow}|FS\rangle$. Applying $H|\Psi(\mathbf{p})\rangle = E_{\mathbf{p}}|\Psi(\mathbf{p})\rangle$, using (14.16),

$$E_{\mathbf{p}} \sum_{\mathbf{k}} \phi_{\mathbf{k}}|\mathbf{k},\mathbf{p}\rangle = \sum_{|\mathbf{k}\pm\frac{\mathbf{p}}{2}|>k_F} (\epsilon_{\mathbf{k}+\mathbf{p}/2}+\epsilon_{\mathbf{k}-\mathbf{p}/2})\,\phi_{\mathbf{k}}|\mathbf{k},\mathbf{p}\rangle + \sum_{|\mathbf{k}|,\,|\mathbf{k}'|>k_F} |\mathbf{k},\mathbf{p}\rangle\langle\mathbf{k},\mathbf{p}|\hat{V}|\mathbf{k}',\mathbf{p}\rangle\phi_{\mathbf{k}'}.$$

Assume that $\langle \mathbf{k}, \mathbf{p}|\hat{V}|\mathbf{k}', \mathbf{p}\rangle \phi_{\mathbf{k}'} = -g_0/V$ is independent of $\mathbf{p}$. Comparing coefficients of $|\mathbf{k}, \mathbf{p}\rangle$,

$$E_{\mathbf{p}}\phi_{\mathbf{k}} = (\epsilon_{\mathbf{k}+\mathbf{p}/2} - \epsilon_{\mathbf{k}-\mathbf{p}/2})\,\phi_{\mathbf{k}} - \frac{g_0}{V} \sum_{0<\epsilon_{\mathbf{k}'\pm\mathbf{p}/2}<\omega_D} \phi_{\mathbf{k}'}. \tag{14.33}$$

Solving for $\phi_{\mathbf{k}}$,

$$\phi_{\mathbf{k}} = \frac{g_0/V}{\epsilon_{\mathbf{k}+\mathbf{p}/2} + \epsilon_{\mathbf{k}-\mathbf{p}/2} - E_{\mathbf{p}}} \sum_{0<\epsilon_{\mathbf{k}'\pm\mathbf{p}/2}<\omega_D} \phi_{\mathbf{k}'}. \tag{14.34}$$

Summing both sides over momentum $\mathbf{k}$ and removing the common factor $\sum_{\mathbf{k}} \phi_{\mathbf{k}}$, we then obtain

$$1 - \frac{g_0}{V} \sum_{0<\epsilon_{\mathbf{k}\pm\mathbf{p}/2}<\omega_D} \frac{1}{\epsilon_{\mathbf{k}+\mathbf{p}/2} + \epsilon_{\mathbf{k}-\mathbf{p}/2} - E_{\mathbf{p}}} = 0. \tag{14.35}$$

It is convenient to cast this as the zero of the function $\mathcal{G}^{-1}[E_{\mathbf{p}}, \mathbf{p}] = 0$, where

$$\mathcal{G}^{-1}[z, \mathbf{p}] = 1 - g_0\chi_0(z, \mathbf{p}), \tag{14.36}$$

and

$$\chi_0(z, \mathbf{p}) = \frac{1}{V} \sum_{0<\epsilon_{\mathbf{k}\pm\mathbf{p}/2}<\omega_D} \frac{1}{\epsilon_{\mathbf{k}+\mathbf{p}/2} + \epsilon_{\mathbf{k}-\mathbf{p}/2} - z} \tag{14.37}$$

can be interpreted as the bare pair susceptibility of the conduction sea. Now, taking $\epsilon_{\mathbf{k}} = k^2/2m - \mu$ in the momentum summation, we must impose the condition

$$\epsilon_{\mathbf{k}\pm\mathbf{p}/2} = \epsilon_{\mathbf{k}} \pm \frac{\mathbf{p}\cdot\mathbf{v_F}}{2} + \frac{p^2}{8m} > 0, \tag{14.38}$$

or $\epsilon_k > \frac{p v_F}{2}|\cos\theta| - \frac{p^2}{8m}$. Replacing the momentum summation by an integral over energy and angles,

$$\begin{aligned}
\chi_0[z, p] &= \frac{N(0)}{2} \int_{-1}^{1} \frac{d\cos\theta}{2} \int_{\frac{p v_F}{2}|\cos\theta| - p^2/8m}^{\omega_D} \frac{d\epsilon}{2\epsilon + p^2/4m - z} \\
&= \frac{N(0)}{2} \int_0^1 d\cos\theta \ln\left[\frac{2\omega_D}{p v_F\cos\theta - z}\right].
\end{aligned} \tag{14.39}$$

Finally, carrying out the integral over θ, one obtains

$$\chi_0(z, p) = \frac{N(0)}{2} \tilde{\chi}_0\left[\frac{z}{2\omega_D}, \frac{p v_F}{2\omega_D}\right], \tag{14.40}$$

where

$$\tilde{\chi}_0[\tilde{z}, \tilde{p}] = \ln\left(\frac{1}{\tilde{p} - \tilde{z}}\right) + \left[1 + \frac{\tilde{z}}{\tilde{p}} \ln\left(1 - \frac{\tilde{p}}{\tilde{z}}\right)\right]. \tag{14.41}$$

Thus for small $v_F p << |E|$, using (14.36),

$$\mathcal{G}^{-1}[E, p] = 1 - \frac{g_0 N(0)}{2} \ln\left[\frac{2\omega_D}{v_F p - E}\right], \tag{14.42}$$

so the bound-state pole occurs at $\mathcal{G}^{-1}(E_{\mathbf{p}}, \mathbf{p}) = 0$ or

$$E_{\mathbf{p}} = -2\omega_D \exp\left[-\frac{2}{g_0 N(0)}\right] + \mathrm{v}_F p. \tag{14.43}$$

The linear spectrum is a signature of a collective bosonic mode. Incidentally, the quantity

$$\chi''(E, \mathbf{p}) = \mathrm{Im}[\chi_0(z, \mathbf{p})/(1 - g_0\chi_0(z, \mathbf{p}))]|_{z=E-i\delta} \tag{14.44}$$

can be interpreted as a spectral function giving the density of Cooper pairs above and below the particle–particle continuum. It is this quantity that is plotted in Figure 14.5.

14.3 The BCS Hamiltonian

After Cooper's discovery, it took a further six months of intense exploration of candidate wavefunctions before Bardeen, Cooper and Schrieffer succeeded in formulating the theory of superconductivity in terms of a *pair condensate* [15]. It was the graduate student in the team, J. Robert Schrieffer, who took the next leap.[1] Schrieffer's insight was to identify the superconducting ground state as a coherent state of the Cooper pair operator:

$$|\psi_{BCS}\rangle = \exp[\Lambda^\dagger]|0\rangle, \tag{14.45}$$

where $|0\rangle$ is the electron vacuum and $\Lambda^\dagger = \sum_{\mathbf{k}} \phi_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}$ is the Cooper pair creation operator (14.13). If we expand the exponential as a product in momentum space,

$$|\psi_{BCS}\rangle = \prod_{\mathbf{k}} \exp[\phi_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}]|0\rangle = \prod_{\mathbf{k}} (1 + \phi_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle. \tag{14.46}$$

BCS wavefunction

In the second step, we have truncated the exponential to linear order because all higher powers of the pair operator vanish: $(c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})^n = 0\ (n > 1)$. This remarkable coherent state mixes states of different particle number, giving rise to a state of off-diagonal long-range order in which

$$\langle\psi_{BCS}|c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}|\psi_{BCS}\rangle \propto \phi_{\mathbf{k}}. \tag{14.47}$$

[1] Following a conference at the Stevens Institute of Technology on the many–body problem, inspired by a wavefunction that Tomonaga had derived, Schrieffer wrote down a candidate wavefunction for the ground-state superconductivity. He recalls the event in his own words [16]:

> So I guess it was on the subway, I scribbled down the wave function and I calculated the beginning of that expectation value and I realized that the algebra was very simple. I think it was somehow in the afternoon and that night at this friend's house I worked on it. And the next morning, as I recall, I did the variational calculation to get the gap equation and I solved the gap equation for the cutoff potential.

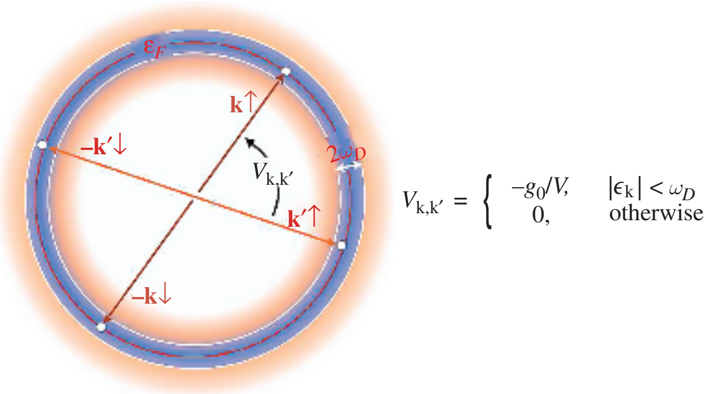


Fig. 14.6 In the BCS Hamiltonian, the matrix $V_{\mathbf{k},\mathbf{k}'}$ acts attractively on pairs of electrons within ω_D of the Fermi surface. Provided the repulsive interaction at higher energies is not too large, a superconducting instability results.

But what Hamiltonian explicitly gives rise to pairing? A clue came from the Cooper instability, which depends on the scattering amplitude $V_{\mathbf{k},\mathbf{k}'} = \langle \mathbf{k}_P | \hat{V} | \mathbf{k}'_P \rangle$ between *zero-momentum pairs*. BCS [15] incorporated this feature into a model Hamiltonian:

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}. \quad (14.48)$$

BCS Hamiltonian

In the universe of possible superconductors and superfluids, the interaction $V_{\mathbf{k},\mathbf{k}'}$ can take a wide variety of symmetries, but in its s-wave manifestation it is simply an isotropic attraction that develops within a narrow energy shell of electrons within a Debye energy of the Fermi surface, ω_D (Figure 14.6):

$$V_{\mathbf{k},\mathbf{k}'} = \begin{cases} -g_0/V & (|\epsilon_{\mathbf{k}}| < \omega_D) \\ 0 & \text{(otherwise)}. \end{cases} \quad (14.49)$$

The s-wave BCS Hamiltonian then takes the form

$$H = \sum_{|\epsilon_{\mathbf{k}}|<\omega_D,\, \sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \frac{g_0}{V} A^\dagger A.$$
$$A^\dagger = \sum_{|\epsilon_{\mathbf{k}}|<\omega_D} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}, \qquad A = \sum_{|\epsilon_{\mathbf{k}'}|<\omega_D} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}. \quad (14.50)$$

s-wave BCS Hamiltonian

Remarks

- The BCS Hamiltonian is a *model* Hamiltonian capturing the low-energy pairing physics.

- The normalizing factor $1/V$ is required in the interaction so that the interaction energy is extensive, growing linearly rather than quadratically with volume V.
- The BCS interaction takes place exclusively at zero momentum, and as such involves an infinite-range interaction between pairs. This long-range aspect of the model permits the exact solution of the BCS Hamiltonian using mean-field theory. In the more microscopic Fröhlich model the effective interaction (Figure (14.6)) is attractive within a narrow momentum shell $|\Delta \mathbf{p}| \sim \omega_D/\mathrm{v}_F$, corresponding to a spatial interaction range of order $1/|\Delta \mathbf{p}| \sim \mathrm{v}_F/\omega_D \sim O(\epsilon_F/\omega_D) \times a$, where a is the lattice spacing. This length scale is typically hundreds of lattice spacings, so the infinite-range mean-field theory is a reasonable rendition of the underlying physics.

14.3.1 Mean-field description of the condensate

The key consequence of the BCS model is the development of a state with off-diagonal long-range order (see Section 11.4.2). The pair operator $\hat{A}$ is extensive, and in a superconducting state its expectation value is proportional to the volume of the system $\langle \hat{A} \rangle \propto V$. The pair density

$$\Delta = |\Delta| e^{i\phi} = -\frac{g_0}{V}\langle \hat{A} \rangle = -g_0 \int_{|\epsilon_{\mathbf{k}}|<\omega_D} \frac{d^3k}{(2\pi)^3} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle \tag{14.51}$$

is an intensive, macroscopic property of superconductors that has both an amplitude $|\Delta|$ *and* a phase ϕ. This is the order parameter. It sets the size of the gap in the excitation spectrum and gives rise to the emergent phase variable whose rigidity supports superconductivity.

Like the pressure in a gas, the order parameter Δ is an emergent many-body property. Just as fluctuations in pressure $\langle \delta P^2 \rangle \sim O(1/V)$ become negligible in the thermodynamic limit, fluctuations in Δ can be similarly ignored. Of course, the reasoning needs to be refined to encompass a quantum variable, formally requiring a path-integral approach. The important point is that the change in action $\delta S[\delta\Delta] = S[\Delta + \delta\Delta_0] - S[\Delta]$ associated with a small variation in Δ about a stationary point scales extensively in volume: $\delta S[\delta\Delta] \sim V \times \delta\Delta^2$, so that the corresponding distribution function can be expanded as a Gaussian,

$$\mathcal{P}[\Delta] \propto e^{-S[\delta\Delta]} \sim \exp\left[-\frac{\delta\Delta^2}{O(1/V)}\right], \tag{14.52}$$

which is exquisitely peaked about $\Delta = \Delta_0$, with variance $\langle \delta\Delta^2 \rangle \propto 1/V$, justifying a mean-field treatment.

Let us now expand the BCS interaction in powers of the fluctuation operator $\delta\hat{A} = \hat{A} - \langle \hat{A} \rangle$:

$$-\frac{g_0}{V} A^\dagger A = \overbrace{\bar{\Delta} A + A^\dagger \Delta + V\frac{\bar{\Delta}\Delta}{g_0}}^{O(V)} - \overbrace{\frac{g_0}{V}\delta A^\dagger \delta A}^{O(1)}. \tag{14.53}$$

Now the first three terms are extensive in volume, but since $\langle \delta A^\dagger \delta A \rangle \sim O(V)$ the last term is intensive $O(1)$, and can be neglected in the thermodynamic limit. We shall shortly see how this same decoupling is accomplished in a path integral using a Hubbard–Stratonovich transformation. The resulting mean-field Hamiltonian for BCS theory is then

$$H_{MFT} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left[\bar{\Delta} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \Delta \right] + \frac{V}{g_0} \bar{\Delta}\Delta, \quad (14.54)$$

BCS theory: mean-field Hamiltonian

in which Δ needs to be determined self-consistently by minimizing the free energy.

14.4 Physical picture of BCS theory: pairs as spins

Let us discuss the physical meaning of the pairing terms in the BCS mean-field Hamiltonian (14.54)

$$H_P(\mathbf{k}) = \left(\bar{\Delta} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \Delta \right). \quad (14.55)$$

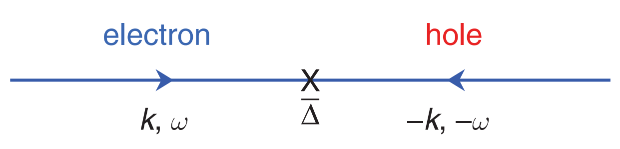
On the one hand, the term $\bar{\Delta} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}$ converts two particles into the condensate:

$$\text{Pair creation :} \qquad e^- + e^- \rightleftharpoons \text{pair}^{2-}. \quad (14.56)$$

Alternatively, by writing $c_{-\mathbf{k}\downarrow} = h^\dagger_{\mathbf{k}\downarrow}$ as a hole creation operator, we see that $H_P(\mathbf{k}) \equiv (h^\dagger_{\mathbf{k}\uparrow} \bar{\Delta}) c_{\mathbf{k}\uparrow} + \text{H.c.}$ describes the scattering of a single electron into a condensed pair (represented by $\bar{\Delta}$) and a hole, a process called *Andreev reflection*, named after its discoverer, Alexander Andreev:

$$\text{Andreev reflection :} \qquad e^- \rightleftharpoons \text{pair}^{2-} + h^+. \quad (14.57)$$

While the first process builds the condensate, the second coherently mixes particle and holes. We will denote the Andreev scattering process by a Feynman diagram:



Andreev reflection differs from conventional reflection in that

- it elastically scatters electrons into holes, reversing *all* components of the velocity[2]

[2] Andreev noticed that, although the momentum of the hole is the same as the incoming electron, its group velocity $\nabla_{\mathbf{k}}(-\epsilon_{-\mathbf{k}}) = \nabla_{\mathbf{k}}(-\epsilon_{\mathbf{k}}) = -\nabla_{\mathbf{k}}\epsilon_{\mathbf{k}}$ is reversed. This led him to predict that such scattering at the interface of a superconductor leads to non-specular reflection of electrons, which scatter back as holes moving in the opposite direction to the incoming electrons.

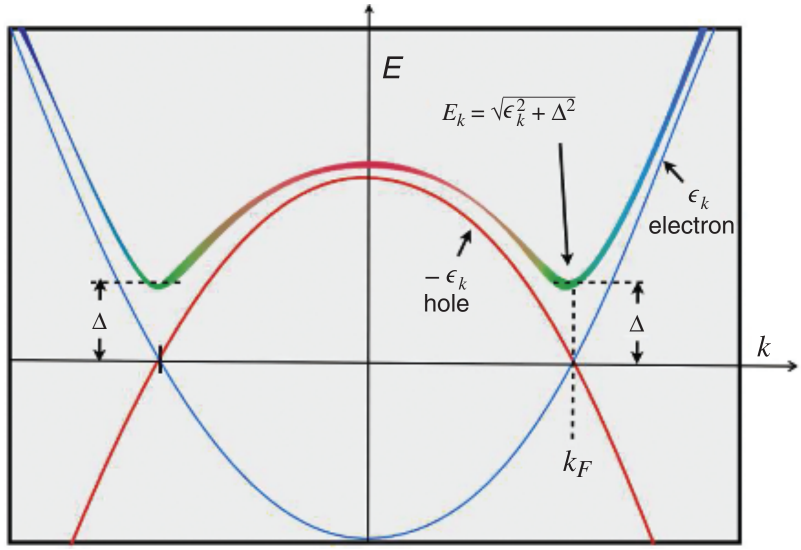


Fig. 14.7 Illustrating the excitation spectrum of a superconductor. Andreev scattering mixes the electron excitation spectrum (blue) with the hole excitation spectrum (red), producing the gap Δ in the quasiparticle excitation spectrum. The quasiparticles at the Fermi momentum are linear combinations of electrons and holes, with an indefinite charge.

- it *conserves* spin, momentum, *and* current, for a hole in the state $(-\mathbf{k}, \downarrow)$ has spin up, momentum $+\mathbf{k}$, and carries a current $I = (-e) \times (-\nabla\epsilon_{\mathbf{k}}) = e\nabla_{\mathbf{k}}\epsilon_{\mathbf{k}}$.

Now the particle and hole dispersions are given by

$$\begin{array}{rr} \text{particle:} & \epsilon_{\mathbf{k}} \\ \text{hole:} & -\epsilon_{-\mathbf{k}}, \end{array} \tag{14.58}$$

as denoted by the blue and red lines, respectively, in Figure 14.7. These lines intersect at the Fermi surface, so that the Andreev mixing between electrons and holes in a superconductor opens up a gap that eliminates the Fermi surface, giving rise to a dispersion which, we will shortly show, takes the form

$$E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta|^2}, \tag{14.59}$$

as illustrated in Figure 14.7. The quasiparticle operators now become linear combinations of electron and hole states with corresponding quasiparticle operators

$$a^\dagger_{\mathbf{k}\sigma} = u_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} + \text{sgn}(\sigma) v_{\mathbf{k}} c_{-\mathbf{k}-\sigma}. \tag{14.60}$$

14.4.1 Nambu spinors

We now introduce Nambu's spinor formulation of BCS theory, which we'll employ to expose the beautiful magnetic analogy between pairs and spins, discovered by Yoichiro Nambu [17] working at the University of Chicago and Philip W. Anderson [18] at AT&T Bell Laboratories. The analogue of a superconductor is an antiferromagnet, for both superconductivity and antiferromagnetism involve an order parameter which (unlike ferromagnetism), does *not* commute with the Hamiltonian. Superconductivity involves an analogous quantity to spin, which we will call *isospin*, which describes orientations in charge space. The pairing field Δ can be regarded as a transverse field in isospin space.

To bring out this physics, it is convenient to introduce the charge analogue of the electron spinor, the *Nambu spinor*, defined as

$$\psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}_{-\mathbf{k}.\downarrow} \end{pmatrix} \qquad \begin{matrix} \text{electron} \\ \text{hole} \end{matrix} \tag{14.61}$$

with the corresponding Hermitian conjugate

$$\psi^{\dagger}_{\mathbf{k}} = \left(c^{\dagger}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow} \right). \tag{14.62}$$

Nambu spinors behave like conventional electron fields, with an algebra

$$\{\psi_{\mathbf{k}\alpha},\ \psi^{\dagger}_{\mathbf{k}'\beta}\} = \delta_{\alpha\beta}\delta_{\mathbf{k},\mathbf{k}'}, \tag{14.63}$$

but instead of up and down electrons, they describe electrons and holes. These spinors enable us to unify the kinetic and pairing energy terms into a single *vector* field, analogous to a magnetic field, that acts in isospin space.

The kinetic energy can be written as

$$\sum_{\mathbf{k}} \epsilon_{\mathbf{k}}(c^{\dagger}_{\mathbf{k}\uparrow}c_{\mathbf{k}\uparrow} - c_{-\mathbf{k}\downarrow}c^{\dagger}_{-\mathbf{k}\downarrow} + 1) = \left(c^{\dagger}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow} \right) \begin{bmatrix} \epsilon_{\mathbf{k}} & 0 \\ 0 & -\epsilon_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ \bar{c}^{\dagger}_{-\mathbf{k}\downarrow} \end{pmatrix} + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}, \tag{14.64}$$

where the sign reversal in the lower component derives from anticommuting the down-spin electron operators. The energy $-\epsilon_{\mathbf{k}}$ is the energy to create a hole. We will drop the constant remainder term $\sum_{\mathbf{k}} \epsilon_{\mathbf{k}}$. We can now combine the kinetic and pairing terms into a single matrix:

$$\begin{aligned} \epsilon_{\mathbf{k}} \sum_{\sigma} c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + \left[\bar{\Delta} c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow} + c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}\Delta \right] &= \left(c^{\dagger}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow} \right) \begin{bmatrix} \epsilon_{\mathbf{k}} & \Delta \\ \bar{\Delta} & -\epsilon_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}_{-\mathbf{k}\downarrow} \end{pmatrix} \\ &= \psi^{\dagger}_{\mathbf{k}} \begin{bmatrix} \epsilon_{\mathbf{k}} & \Delta_1 - i\Delta_2 \\ \Delta_1 + i\Delta_2 & -\epsilon_{\mathbf{k}} \end{bmatrix} \psi_{\mathbf{k}} \\ &= \psi^{\dagger}_{\mathbf{k}}[\epsilon_{\mathbf{k}}\tau_3 + \Delta_1\tau_1 + \Delta_2\tau_2]\psi_{\mathbf{k}}, \end{aligned} \tag{14.65}$$

where we denote $\Delta = \Delta_1 - i\Delta_2$, $\bar{\Delta} = \Delta_1 + i\Delta_2$ and we have introduced the *isospin matrices*

$$\vec{\tau} = (\tau_1, \tau_2, \tau_3) = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right). \tag{14.66}$$

By convention the symbol $\vec{\tau}$ is used to distinguish a Pauli matrix in charge space from a spin $\boldsymbol{\sigma}$ acting in spin space. Putting this all together, the mean-field Hamiltonian can now be rewritten

$$H = \sum_{\mathbf{k}} \psi^{\dagger}_{\mathbf{k}} (\vec{h}_{\mathbf{k}} \cdot \vec{\tau})\psi_{\mathbf{k}} + V\frac{\bar{\Delta}\Delta}{g_0}, \tag{14.67}$$

where

$$\vec{h}_{\mathbf{k}} = (\Delta_1, \Delta_2, \epsilon_{\mathbf{k}}) \tag{14.68}$$

plays the role of a Zeeman field acting in isospin space.

14.4.2 Anderson's domain-wall interpretation of BCS theory

Anderson noted that the isospin operators $\psi^\dagger_{\mathbf{k}}\vec{\tau}\psi_{\mathbf{k}}$ have the properties of spin-$\frac{1}{2}$ operators acting in charge space. The z component of the isospin is

$$\tau_{3\mathbf{k}} = \psi^\dagger_{\mathbf{k}}\tau_3\psi_{\mathbf{k}} = (c^\dagger_{\mathbf{k}\uparrow}c_{\mathbf{k}\uparrow} - c_{-\mathbf{k}\downarrow}c^\dagger_{-\mathbf{k}\downarrow}) = (n_{\mathbf{k}\uparrow} + n_{-\mathbf{k}\downarrow} - 1), \tag{14.69}$$

so the up and down states correspond to the doubly occupied and empty pair state, respectively:

$$\begin{array}{lll} \tau_{3\mathbf{k}} = +1: & |\Uparrow_{\mathbf{k}}\rangle & \equiv & |2\rangle = c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}|0\rangle \\ \tau_{3\mathbf{k}} = -1: & |\Downarrow_{\mathbf{k}}\rangle & \equiv & |0\rangle. \end{array} \tag{14.70}$$

By contrast, the transverse components of the isospin describe pair creation and annihilation:

$$\begin{aligned} \hat{\tau}_{1\mathbf{k}} &= \psi^\dagger_{\mathbf{k}}\tau_1\psi_{\mathbf{k}} = \quad c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow} + c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow} \\ \hat{\tau}_{2\mathbf{k}} &= \psi^\dagger_{\mathbf{k}}\tau_2\psi_{\mathbf{k}} = -i(c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow} - c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}). \end{aligned} \tag{14.71}$$

In a normal metal, the isospin points "up" in the occupied states below the Fermi surface, and "down" in the empty states above the Fermi surface (Figure 14.8(a)). Now since the Hamiltonian is $H = \sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}}(\vec{h}_{\mathbf{k}} \cdot \vec{\tau})\psi_{\mathbf{k}}$, the quantity

$$\vec{B}_{\mathbf{k}} = -\vec{h}_{\mathbf{k}} = -(\Delta_1, \Delta_2, \epsilon_{\mathbf{k}}) \tag{14.72}$$

is thus a momentum-dependent Weiss field, setting a natural quantization axis for the electrons at momentum **k**: in the ground state, the fermion isospins line up with this field. In the normal state, the natural isospin quantization axis is the charge or "z-axis," but in the

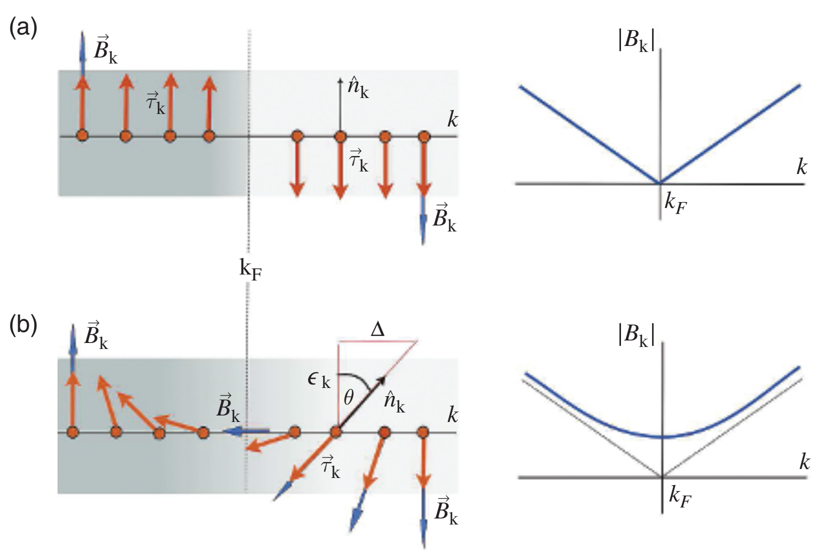


Fig. 14.8 Showing the domain-wall configuration of the isospin $\vec{\tau}_{\mathbf{k}}$ and direction of pairing field $\hat{n}_{\mathbf{k}}$ near the Fermi momentum: (a) a normal metal, in which the Weiss field $B_{\mathbf{k}}$ vanishes linearly at the Fermi energy, and (b) a superconductor in which the Weiss field remains finite at the Fermi energy, giving rise to a gap in the excitation spectrum.

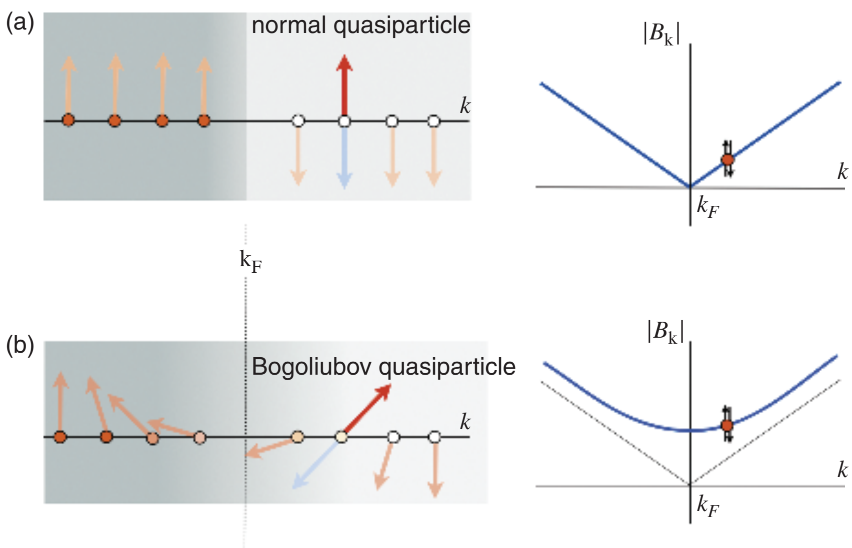


Fig. 14.9

Illustrating how the excitation of quasiparticle pairs corresponds to an “isospin flip,” which forms a pair of up and down quasiparticles with energy $2|B_{\mathbf{k}}|$: (a) quasiparticle pair formation in the normal state where the quasiparticle spectrum is gapless; (b) formation of a Bogoliubov quasiparticle pair in the superconducting state where the excitation spectrum is gapped.

superconductor, the presence of a pairing condensate tips the quantization axis, mixing particle and hole states (Figure 14.8(b)).

With this analogy one can identify the reversal of an isospin out of its ground-state configuration as the creation of a pair of quasiparticles “above” the condensate. Since this costs an energy $2|\vec{B}_{\mathbf{k}}|$, the magnitude of the Weiss field

$$E_{\mathbf{k}} \equiv |\vec{B}_{\mathbf{k}}| = \sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta|^2} = \text{quasiparticle energy} \tag{14.73}$$

must correspond to the energy of a single quasiparticle. In a metal ($\Delta = 0$), the Weiss field vanishes at the Fermi surface so it costs no energy to create a quasiparticle there (Figure 14.9(a)), but in a superconductor the Weiss field has magnitude $|\Delta|$ so the quasiparticle spectrum is now gapped (Figure 14.9(b)).

Let us write $\vec{B}_{\mathbf{k}} = -E_{\mathbf{k}}\hat{n}_{\mathbf{k}}$, where the unit vector

$$\hat{n}_{\mathbf{k}} = \left(\frac{\Delta_1}{E_{\mathbf{k}}}, \frac{\Delta_2}{E_{\mathbf{k}}}, \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \tag{14.74}$$

points upwards far above the Fermi surface, and downwards far beneath it. In a normal metal, $\hat{n}_{\mathbf{k}}$ (see Figure 14.8) reverses at the Fermi surface forming a sharp “Ising-like” domain wall, but in a superconductor the $\hat{n}$ vector is aligned at an angle θ to the $\hat{z}$ axis, where

$$\cos\theta_{\mathbf{k}} = \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}}. \tag{14.75}$$

This angle rotates continuously as one passes through the Fermi energy, so the domain wall is now spread out over an energy range of order Δ, forming a kind of Bloch domain wall in isospin space, as shown in Figure 14.8.

In the ground state each isospin will align parallel to the field $\vec{B}_{\mathbf{k}} = -E_{\mathbf{k}}\hat{n}_{\mathbf{k}}$, i.e.

$$\langle \psi^{\dagger}_{\mathbf{k}} \vec{\tau} \psi_{\mathbf{k}} \rangle = -\hat{n}_{\mathbf{k}} = -(\sin\theta_{\mathbf{k}}, 0, \cos\theta_{\mathbf{k}}), \tag{14.76}$$

where we have taken the liberty of choosing the phase of Δ so that $\Delta_2 = 0$. In a normal ground state ($\Delta = 0$) the isospin aligns along the z-axis, $\langle \tau_{3\mathbf{k}} \rangle = \langle n_{\mathbf{k}\uparrow} + n_{-\mathbf{k}\downarrow} - 1 \rangle = \mathrm{sgn}(k_F - k)$, but in a superconductor the isospin quantization axis is rotated through an angle $\theta_{\mathbf{k}}$ so that the z component of the isospin is

$$\langle \tau_{3\mathbf{k}} \rangle = \langle n_{\mathbf{k}\uparrow} + n_{-\mathbf{k}\downarrow} - 1 \rangle = -\cos\theta_{\mathbf{k}} = -\frac{\epsilon_{\mathbf{k}}}{\sqrt{\epsilon_{\mathbf{k}}^2 + \Delta^2}}, \tag{14.77}$$

which smears the occupancy around the Fermi surface, while the transverse isospin component, representing the pairing, is now finite:

$$\langle \tau_{1\mathbf{k}} \rangle = \langle (c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) \rangle = -\sin\theta_{\mathbf{k}} = -\frac{\Delta}{\sqrt{\epsilon_{\mathbf{k}}^2 + \Delta^2}}. \tag{14.78}$$

Now since we have chosen $\Delta_2 = 0$, $\langle \tau_{2\mathbf{k}} \rangle = -i\langle (c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} - c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) \rangle = 0$, it follows that $\langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle = -\frac{1}{2}\sin\theta_{\mathbf{k}}$. Imposing the self-consistency condition $\Delta = -\frac{g_0}{V}\sum_{\mathbf{k}} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle$ (14.51), one then obtains the *BCS gap equation*:

$$\Delta = \frac{g_0}{V}\sum_{\mathbf{k}} \frac{1}{2}\sin\theta_{\mathbf{k}} = g_0 \int_{|\epsilon_{\mathbf{k}}|<\omega_D} \frac{d^3k}{(2\pi)^3} \frac{\Delta}{2\sqrt{\epsilon_{\mathbf{k}}^2 + \Delta^2}}. \tag{14.79}$$

BCS gap equation ($T = 0$)

Since the momentum sum is restricted to a narrow region of the Fermi surface, one can replace the momentum sum by an energy integral, to obtain

$$1 = g_0 N(0) \int_{-\omega_D}^{\omega_D} d\epsilon \frac{1}{2\sqrt{\epsilon^2 + \Delta^2}} = g_0 N(0) \sinh^{-1}\left(\frac{\omega_D}{\Delta}\right) \approx g_0 N(0) \ln\left[\frac{2\omega_D}{\Delta}\right], \tag{14.80}$$

so, in the superconducting ground state, the BCS gap is given by

$$\Delta = 2\omega_D e^{-\frac{1}{g_0 N(0)}}. \tag{14.81}$$

Remarks

- Note the disappearance of the factor of 2 in the exponent that appeared in Cooper's original calculation (14.25).
- The magnetic analogy has many intriguing consequences. One can immediately see that, like a magnet, there must be collective pair excitations, in which the isospins fluctuate about their ground-state orientations. Like magnons, these excitations form quantized collective modes. In a neutral superconductor, this leads to a gapless "sound" (Bogoliubov or Goldstone) mode, but in a charged superconductor the condensate phase mixes with the electromagnetic vector potential via the Anderson–Higgs mechanism (see Section 11.6) to produce the massive photon responsible for the Meissner effect.

14.4.3 The BCS ground state

In the vacuum $|0\rangle$, electron isospin operators all point "down," $\tau_{3\mathbf{k}} = -1$. To construct the ground state in which the isospins are aligned with the Weiss field, we need to construct a state in which each isospin is rotated relative to the vacuum. This is done by rotating the isospin at each momentum $\mathbf{k}$ through an angle $\theta_{\mathbf{k}}$ about the y-axis, as follows:

$$\begin{aligned}|\theta_{\mathbf{k}}\rangle = \exp\left[-i\frac{\theta_{\mathbf{k}}}{2}\psi^{\dagger}_{\mathbf{k}}\tau_y\psi_{\mathbf{k}}\right]|\Downarrow_{\mathbf{k}}\rangle &= \left(\cos\frac{\theta_{\mathbf{k}}}{2} - i\sin\frac{\theta_{\mathbf{k}}}{2}\psi^{\dagger}_{\mathbf{k}}\tau_y\psi_{\mathbf{k}}\right)|\Downarrow_{\mathbf{k}}\rangle \\ &= \cos\frac{\theta_{\mathbf{k}}}{2}|\Downarrow_{\mathbf{k}}\rangle - \sin\frac{\theta_{\mathbf{k}}}{2}|\Uparrow_{\mathbf{k}}\rangle.\end{aligned} \tag{14.82}$$

The ground state is a product of these isospin states:

$$|BCS\rangle = \prod_{\mathbf{k}}|\theta_{\mathbf{k}}\rangle = \prod_{\mathbf{k}}\left(\cos\frac{\theta_{\mathbf{k}}}{2} + \sin\frac{\theta_{\mathbf{k}}}{2}c^{\dagger}_{-\mathbf{k}\downarrow}c^{\dagger}_{\mathbf{k}\uparrow}\right)|0\rangle, \tag{14.83}$$

where we have absorbed the minus sign by anticommuting the two electron operators. Following BCS, the coefficients $\cos\left(\frac{\theta_{\mathbf{k}}}{2}\right)$ and $\sin\left(\frac{\theta_{\mathbf{k}}}{2}\right)$ are labeled $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$, respectively, writing

$$|BCS\rangle = \prod_{\mathbf{k}}|\theta_{\mathbf{k}}\rangle = \prod_{\mathbf{k}}\left(u_{\mathbf{k}} + v_{\mathbf{k}}c^{\dagger}_{-\mathbf{k}\downarrow}c^{\dagger}_{\mathbf{k}\uparrow}\right)|0\rangle, \tag{14.84}$$

where

$$\begin{aligned}u_{\mathbf{k}} &\equiv \cos\left(\frac{\theta_{\mathbf{k}}}{2}\right) = \sqrt{\frac{1}{2}\Big[1 + \underbrace{\cos\theta_{\mathbf{k}}}_{\epsilon_{\mathbf{k}}/E_{\mathbf{k}}}\Big]} = \sqrt{\frac{1}{2}\left[1 + \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right]} \\ v_{\mathbf{k}} &\equiv \sin\left(\frac{\theta_{\mathbf{k}}}{2}\right) = \sqrt{\frac{1}{2}\Big[1 - \cos\theta_{\mathbf{k}}\Big]} = \sqrt{\frac{1}{2}\left[1 - \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right]}.\end{aligned} \tag{14.85}$$

Remarks

- Dropping the normalization, the BCS wavefunction can be rewritten as a coherent state (14.45),

$$|BCS\rangle = \prod_{\mathbf{k}}\left(1 + \phi_{\mathbf{k}}c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}\right)|0\rangle) = \exp\left[\sum_{\mathbf{k}}\phi_{\mathbf{k}}c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}\right]|0\rangle = \exp\left[\Lambda^{\dagger}\right]|0\rangle, \tag{14.86}$$

  where $\phi_{\mathbf{k}} = -\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}}$ determines the Cooper pair wavefunction.
- We can thus expand the exponential in (14.86) as a coherent sum of pair-states:

$$|BCS\rangle = \sum_n \frac{1}{n!}(\Lambda^{\dagger})^n|0\rangle = \sum_n \frac{1}{\sqrt{n!}}|n\rangle, \tag{14.87}$$

  where $|n\rangle = \frac{1}{\sqrt{n!}}(\Lambda^{\dagger})^n|0\rangle$ is a state containing n pairs.

The BCS wavefunction breaks gauge invariance, because it is not invariant under gauge transformations $c^\dagger_{\mathbf{k}\sigma} \to e^{i\alpha} c^\dagger_{\mathbf{k}\sigma}$ of the electron operators:

$$|BCS\rangle \to |\alpha\rangle = \prod_{\mathbf{k}} (1 + e^{2i\alpha} \phi_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle = \sum \frac{e^{i2n\alpha}}{\sqrt{n!}} |n\rangle. \tag{14.88}$$

Under this transformation, the order parameter $\Delta = -g_0/V \sum_{\mathbf{k}} \langle\alpha|c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}|\alpha\rangle$ acquires a phase $\Delta \to e^{2i\alpha}|\Delta|$. On the other hand, the energy of the BCS state is unchanged by a gauge transformation, so the states $|\alpha\rangle$ must form a family of degenerate broken-symmetry states.

The action of the number operator $\hat{N}$ on this state may be represented as a differential with respect to phase:

$$\hat{N}|\alpha\rangle = \sum \frac{1}{\sqrt{n!}} 2n e^{i2n\alpha} |n\rangle = -i \frac{d}{d\alpha} |\alpha\rangle, \tag{14.89}$$

so that

$$\hat{N} \equiv -i \frac{d}{d\alpha}. \tag{14.90}$$

In this way, we see that the particle number is the generator of gauge transformations. Moreover, the phase of the order parameter is conjugate to the number operator, $[\alpha, N] = i$, and like position and momentum, or energy and time, the two variables therefore obey an uncertainty principle,

$$\Delta\alpha \Delta N \gtrsim 1. \tag{14.91}$$

Just as a macroscopic object with a precise position has an ill-defined momentum, a pair condensate with a sharply defined phase (relative to other condensates) is a physical state of matter – a macroscopic Schrödinger cat state – with an *ill-defined particle number*.

For the moment, we're ignoring the charge of the electron, but once we restore it, we will have to keep track of the vector potential, which also changes under gauge transformations.

14.5 Quasiparticle excitations in BCS theory

Let us now construct the quasiparticles of the BCS Hamiltonian. Recall that, for any one-particle Hamiltonian $H = \psi^\dagger_\alpha h_{\alpha\beta} \psi_\beta$, we can transform to an energy basis where the operators $a^\dagger_k = \psi^\dagger_\beta \langle\beta|k\rangle$ diagonalize $H = \sum_k E_k a^\dagger_k a_k$. Now the $\langle\beta|k\rangle$ are the eigenvectors of $h_{\alpha\beta}$, since $\langle\alpha|\hat{H}|k\rangle = E_k \langle\alpha|k\rangle = h_{\alpha\beta}\langle\beta|k\rangle$, so to construct quasiparticle operators we must project the particle operators onto the eigenvectors of $h_{\alpha\beta}$, $a^\dagger_k = \psi^\dagger_\beta \langle\beta|k\rangle$.

We now seek to diagonalize the BCS Hamiltonian, written in Nambu form:

$$H = \sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} (\vec{h}_{\mathbf{k}} \cdot \vec{\tau}) \psi_{\mathbf{k}} + \frac{V}{g_0} \bar{\Delta}\Delta.$$

The two-dimensional Nambu matrix

$$\underline{h}_{\mathbf{k}} = \epsilon_{\mathbf{k}} \tau_3 + \Delta_1 \tau_1 + \Delta_2 \tau_2 \equiv E_{\mathbf{k}} \hat{n}_{\mathbf{k}} \cdot \vec{\tau} \tag{14.92}$$

has two eigenvectors with isospin quantized parallel and antiparallel to $\hat{n}_{\mathbf{k}}$, [3]

$$\hat{n}_{\mathbf{k}} \cdot \vec{\tau} \begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \end{pmatrix} = + \begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \end{pmatrix}, \qquad \hat{n}_{\mathbf{k}} \cdot \vec{\tau} \begin{pmatrix} -v_{\mathbf{k}}^* \\ u_{\mathbf{k}}^* \end{pmatrix} = - \begin{pmatrix} -v_{\mathbf{k}}^* \\ u_{\mathbf{k}}^* \end{pmatrix}, \tag{14.93}$$

and corresponding energies $\pm E_{\mathbf{k}} = \pm\sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$. We can combine (14.93) into a single equation,

$$(\hat{n}_{\mathbf{k}} \cdot \vec{\tau}) U_{\mathbf{k}} = U_{\mathbf{k}} \tau_3, \tag{14.94}$$

where

$$U_{\mathbf{k}} = \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}}^* \\ v_{\mathbf{k}} & u_{\mathbf{k}}^* \end{pmatrix} \tag{14.95}$$

is the unitary matrix formed from the eigenvectors of $\underline{h}_{\mathbf{k}}$. If we now project $\psi_{\mathbf{k}}^\dagger$ onto the eigenvectors of $h_{\mathbf{k}}$, we obtain the quasiparticle operators for the BCS Hamiltonian:

$$a_{\mathbf{k}\uparrow}^\dagger = \psi_{\mathbf{k}}^\dagger \cdot \begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \end{pmatrix} = c_{\mathbf{k}\uparrow}^\dagger u_{\mathbf{k}} + c_{-\mathbf{k}\downarrow} v_{\mathbf{k}}$$

$$a_{-\mathbf{k}\downarrow} = \psi_{\mathbf{k}}^\dagger \cdot \begin{pmatrix} -v_{\mathbf{k}}^* \\ u_{\mathbf{k}}^* \end{pmatrix} = c_{-\mathbf{k}\downarrow} u_{\mathbf{k}}^* - c_{\mathbf{k}\uparrow}^\dagger v_{\mathbf{k}}^*. \tag{14.96}$$

Bogoliubov transformation

This transformation, mixing particles and holes, is named after its inventor, Nikolai Bogoliubov. If one takes the complex conjugate of the quasihole operator and reverses the momentum, one obtains $a_{\mathbf{k}\downarrow}^\dagger = c_{\mathbf{k}\downarrow}^\dagger u_{\mathbf{k}} - c_{-\mathbf{k}\uparrow} v_{\mathbf{k}}$, which defines the spin-down quasiparticle. The general expression for the spin-up and spin-down quasiparticles can be written

$$a_{\mathbf{k}\sigma}^\dagger = c_{\mathbf{k}\sigma}^\dagger u_{\mathbf{k}} + \mathrm{sgn}(\sigma) c_{-\mathbf{k}-\sigma} v_{\mathbf{k}}. \tag{14.97}$$

Let us combine the two expressions (14.96) into a single Nambu spinor $a_{\mathbf{k}}^\dagger$:

$$a_{\mathbf{k}}^\dagger = (a_{\mathbf{k}\uparrow}^\dagger, a_{-\mathbf{k}\downarrow}) = \psi_{\mathbf{k}}^\dagger \overbrace{\begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}}^* \\ v_{\mathbf{k}} & u_{\mathbf{k}}^* \end{pmatrix}}^{U_{\mathbf{k}}} = \psi_{\mathbf{k}}^\dagger U_{\mathbf{k}}. \tag{14.98}$$

Taking the Hermitian conjugate $a_{\mathbf{k}} = U_{\mathbf{k}}^\dagger \psi_{\mathbf{k}}$, then $\psi_{\mathbf{k}} = U_{\mathbf{k}} a_{\mathbf{k}}$, since $UU^\dagger = 1$. Using (14.94),

$$\psi_{\mathbf{k}}^\dagger h_{\mathbf{k}} \psi_{\mathbf{k}} = a_{\mathbf{k}}^\dagger U_{\mathbf{k}}^\dagger \overbrace{h_{\mathbf{k}} U_{\mathbf{k}}}^{U_{\mathbf{k}} E_{\mathbf{k}} \tau_3} a_{\mathbf{k}} = a_{\mathbf{k}}^\dagger E_{\mathbf{k}} \tau_3 a_{\mathbf{k}}, \tag{14.99}$$

so that, as expected,

$$H = \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger E_{\mathbf{k}} \tau_3 a_{\mathbf{k}} + V \frac{\bar{\Delta}\Delta}{g_0} \tag{14.100}$$

[3] Here complex conjugation is required to ensure that the complex eigenvectors are orthogonal when the gap is complex.

is diagonal in the quasiparticle basis. Written out explicitly,

$$H = \sum_{\mathbf{k}} E_{\mathbf{k}} \left(a^{\dagger}_{\mathbf{k}\uparrow} a_{\mathbf{k}\uparrow} - a_{-\mathbf{k}\downarrow} a^{\dagger}_{-\mathbf{k}\downarrow} \right) + V \frac{\bar{\Delta}\Delta}{g_0}. \tag{14.101}$$

If we rewrite the Hamiltonian in the form

$$H = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \left(a^{\dagger}_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} - \frac{1}{2} \right) + V \frac{\bar{\Delta}\Delta}{g_0}, \tag{14.102}$$

we can interpret the excitation spectrum in terms of quasiparticles of energy $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta|^2}$ and a ground-state energy[4]

$$E_g = -\sum_{\mathbf{k}} E_{\mathbf{k}} + V \frac{\bar{\Delta}\Delta}{g_0}. \tag{14.104}$$

Now if the density of Bogoliubov quasiparticles per spin is $N_s(E)$, then, since the number of quasiparticle states is conserved, $N_s(E)dE = N_n(0)d|\epsilon|$ (where $N_n(0) = 2N(0)$ is the quasiparticle density of states in the normal state). It follows that

$$N_s^*(E) = N_n(0) \frac{d|\epsilon_{\mathbf{k}}|}{dE_{\mathbf{k}}} = N_n(0) \left(\frac{E}{\sqrt{E^2 - |\Delta|^2}} \right) \theta(E - |\Delta|), \tag{14.105}$$

where we have written $\epsilon_{\mathbf{k}} = \sqrt{E_{\mathbf{k}}^2 - |\Delta|^2}$ to obtain $d\epsilon_{\mathbf{k}}/dE_{\mathbf{k}} = E_{\mathbf{k}}/\sqrt{E_{\mathbf{k}}^2 - |\Delta|^2}$. The theta function describes the absence of states in the gap (see Figure 14.10(a)). Notice how the Andreev scattering causes states to pile up in a square-root singularity above the gap; this feature is called a *coherence peak*.

One of the most direct vindications of BCS theory derives from tunneling measurements of the excitation spectrum, in which the differential tunneling conductance is proportional to the quasiparticle density of states:

$$\frac{dI}{dV} \propto N_s(eV) = N_n(0) \frac{eV}{\sqrt{eV^2 - \Delta^2}} \theta(eV - |\Delta|). \tag{14.106}$$

The observation of such tunneling spectra in superconducting aluminum in 1960 by Ivar Giaever [19] provided the first direct confirmation of the energy gap predicted by BCS theory (see Figure 14.10(b)).

[4] Note that, if we were to restore the constant term $\sum_{\mathbf{k}} \epsilon_k$ dropped in (14.64), the ground-state energy becomes

$$E_g = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - E_{\mathbf{k}}) + V \frac{\bar{\Delta}\Delta}{g_0}. \tag{14.103}$$

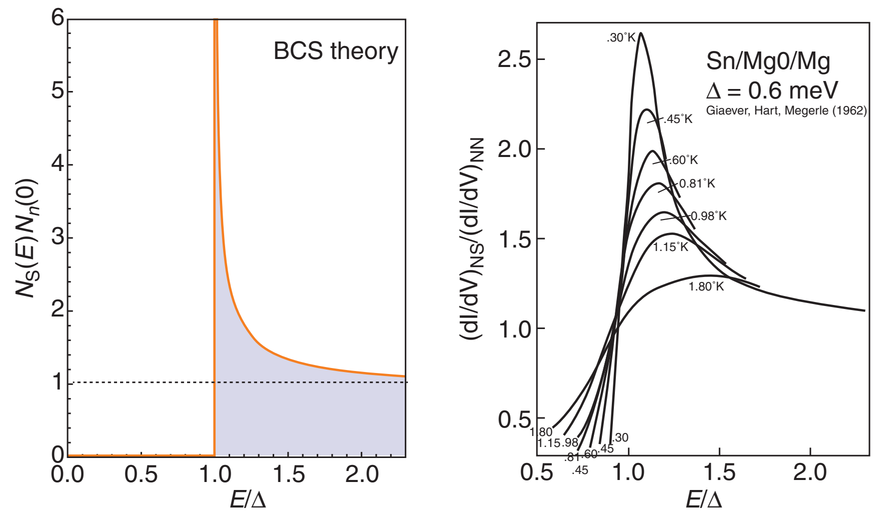


Fig. 14.10 Contrasting (a) quasiparticle density of states with (b) measured tunneling density of states in Sn-MgO-Mg superconducting-normal tunnel junctions. In practice, finite temperature, disorder, variations in gap size around the Fermi surface, and strong-coupling corrections to BCS theory lead to small deviations from the ideal ground-state BCS density of states. Reprinted with permission from I. Giaever, *et al.*, *Phys. Rev.*, vol. 126, p. 941, 1962. Copyright 1962 by the American Physical Society.

Example 14.2 Show that the BCS ground state is the vacuum for the Bogoliubov quasiparticles, i.e. that the destruction operators $a_{\mathbf{k}\sigma}$ annihilate the BCS ground state.

Solution

One way to confirm this is to directly construct the quasiparticle vacuum $|\psi\rangle$, by repeatedly applying the pair destruction operators to the electron vacuum, so that

$$\begin{aligned} |\psi\rangle &= \prod_{\mathbf{k}} a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} |0\rangle \\ \Rightarrow a_{\mathbf{k}\sigma} |\psi\rangle &= 0 \end{aligned} \tag{14.107}$$

for all $\mathbf{k}$, since the square of a destruction operator is zero, so $|\psi\rangle$ is the quasiparticle vacuum. Using the form (14.97),

$$\begin{aligned} a_{\mathbf{k}\uparrow} &= u_{\mathbf{k}} c_{\mathbf{k}\uparrow} + \mathrm{v}_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} \\ a_{-\mathbf{k}\downarrow} &= u_{\mathbf{k}} c_{-\mathbf{k}\downarrow} - \mathrm{v}_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow}, \end{aligned} \tag{14.108}$$

where for convenience we assume that $u_{\mathbf{k}}$ and $\mathrm{v}_{\mathbf{k}}$ are real, we find

$$\begin{aligned} \prod_{\mathbf{k}} a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} |0\rangle &= \prod_{\mathbf{k}} (u_{\mathbf{k}}\, c_{-\mathbf{k}\downarrow} - \mathrm{v}_{\mathbf{k}}\, c^{\dagger}_{\mathbf{k}\uparrow})(\cancel{u_{\mathbf{k}} c_{\mathbf{k}\uparrow}} + \mathrm{v}_{\mathbf{k}}\, c^{\dagger}_{-\mathbf{k}\downarrow})|0\rangle \\ &= \prod_{\mathbf{k}} (u_{\mathbf{k}} \mathrm{v}_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c^{\dagger}_{-\mathbf{k}\downarrow} - (\mathrm{v}_{\mathbf{k}})^2 c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow})|0\rangle \\ &= \prod_{\mathbf{k}} \mathrm{v}_{\mathbf{k}} \times \prod_{\mathbf{k}} (u_{\mathbf{k}} + \mathrm{v}_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} c^{\dagger}_{\mathbf{k}\uparrow})|0\rangle \propto |BCS\rangle, \end{aligned} \tag{14.109}$$