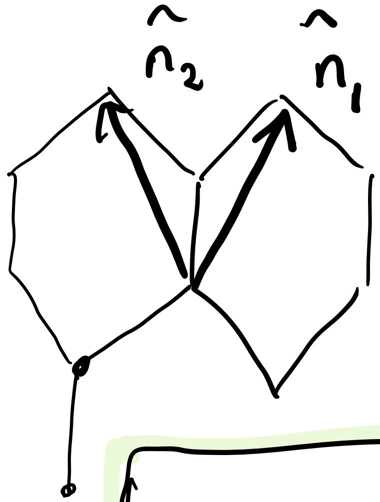
THE TORIC CODE

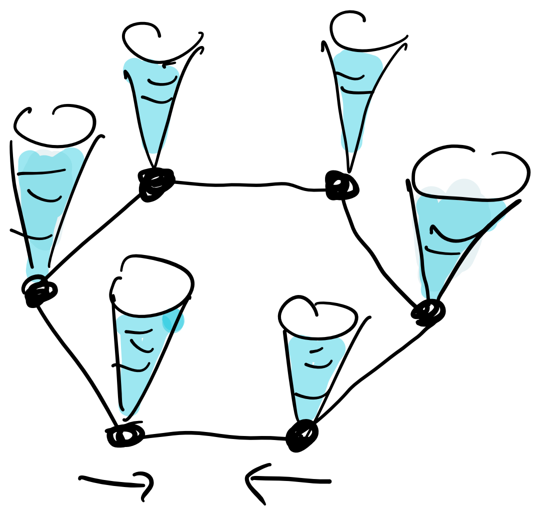
Last time we saw how the Kitaev model can be exactly diagonalized, leading to an exotic ground state with two kinds types of bulk excitation, and, if time reversal is broken, there are also topological edge states.



$$\hat{n}_1 = \left(\frac{\sqrt{3}}{2}a, \frac{3}{2}a\right)$$

$$\hat{n}_2 = \left(-\frac{\sqrt{3}}{2}a, \frac{3}{2}a\right)$$

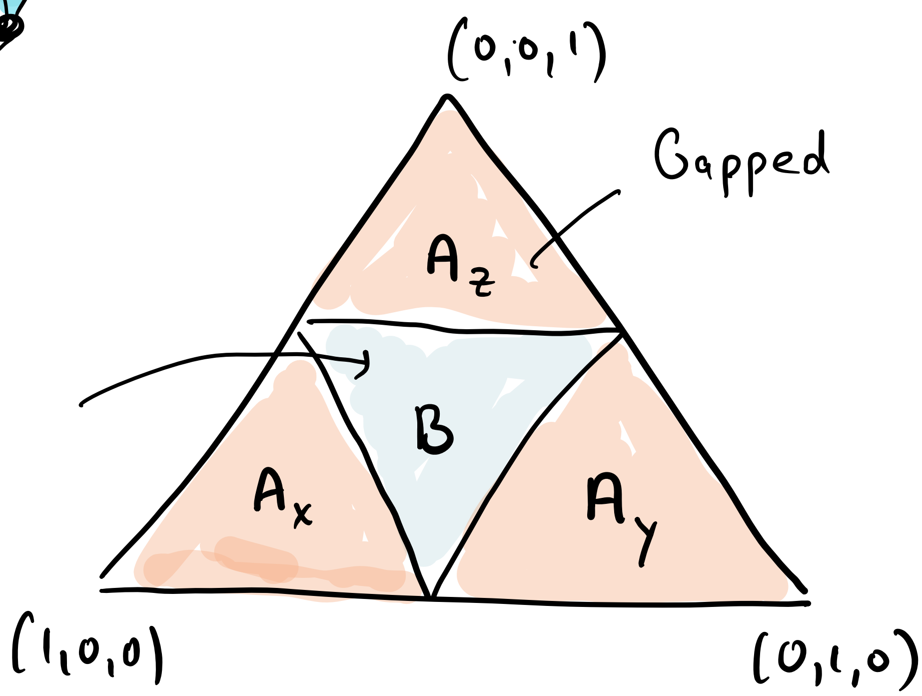
$$\boxed{\begin{aligned} \gamma(k) &= i\left(K_z + K_x e^{i\vec{k}\cdot\vec{n}_1} + K_y e^{i\vec{k}\cdot\vec{n}_2}\right) \\ &= i f_k \\ \epsilon(k) &= \pm|\gamma(k)| \end{aligned}}$$



$K_x = K_y = K_z$

$K_z > K_x = K_y$.

Gapless.



FERMION QPS

$$\begin{pmatrix} 0 & if_{\vec{k}} \\ -if^*_{\vec{k}} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \mp i e^{-i\theta_k} \end{pmatrix} = \begin{pmatrix} 0 & i|\gamma|e^{i\theta} \\ -i|\gamma|e^{-i\theta} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \mp i e^{-i\theta} \end{pmatrix}$$

$$f_k = |\gamma_k| e^{i\theta_{\vec{k}}} \qquad = \pm|\gamma_k|\begin{pmatrix} 1 \\ \mp i e^{-i\theta_k} \end{pmatrix}$$

$$|\psi_\pm\rangle = |i\rangle\langle i|\psi\rangle \Rightarrow a^\dagger_\pm = c^\dagger_i \langle i|\psi_\pm\rangle$$

$$a^\dagger_\pm(k) = \frac{1}{\sqrt{2}}\left(c^\dagger_{1k} \mp i e^{-i\theta_k} c^\dagger_{2k}\right)$$

$$\hat{\mathcal{H}} = \frac{K}{2}\sum_{k\in\bigcirc}\left[a^\dagger_+(\vec{k})a_+(\vec{k}) - a^\dagger_-(\vec{k})\, a_-(\vec{k})\right]$$

$$f_k = f^*_{-k} \qquad a^\dagger_-(-k) = \frac{1}{\sqrt{2}}\left(c^\dagger_{1-k} + i e^{+i\theta_k} c^\dagger_{2-k}\right)$$

$$\theta_k = -\theta_{-k}$$

$$= a_+(k)$$

$$\frac{1}{2}\left[a^\dagger_+(k)a_+(k) - a_+(k)\, a^\dagger_+(k)\right] = \left(a^\dagger_+(k)a_+(k) - \frac{1}{2}\right)$$

$$\mathcal{H} = \sum_{k \in BZ} |\gamma_k| \left[a_+^\dagger(k) a_+(k) - \frac{1}{2} \right]$$

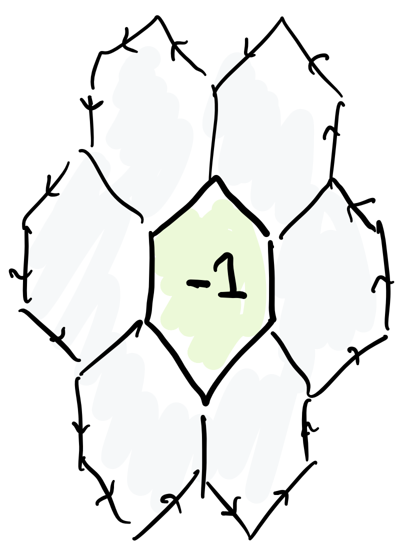
- Only positive energy excitations : MAJORANA's

- G.S.E $\quad E_g = -\frac{1}{2} \sum_k |\gamma_k|.$

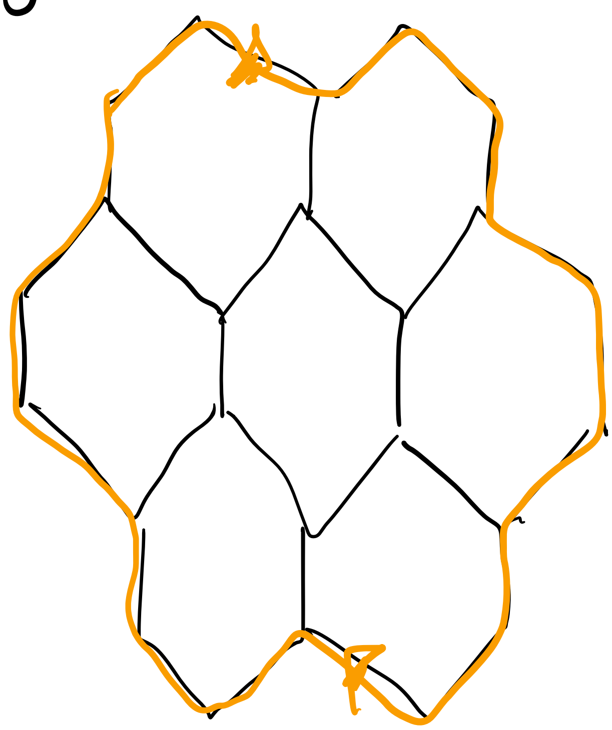
Types of excitation

- Fermions

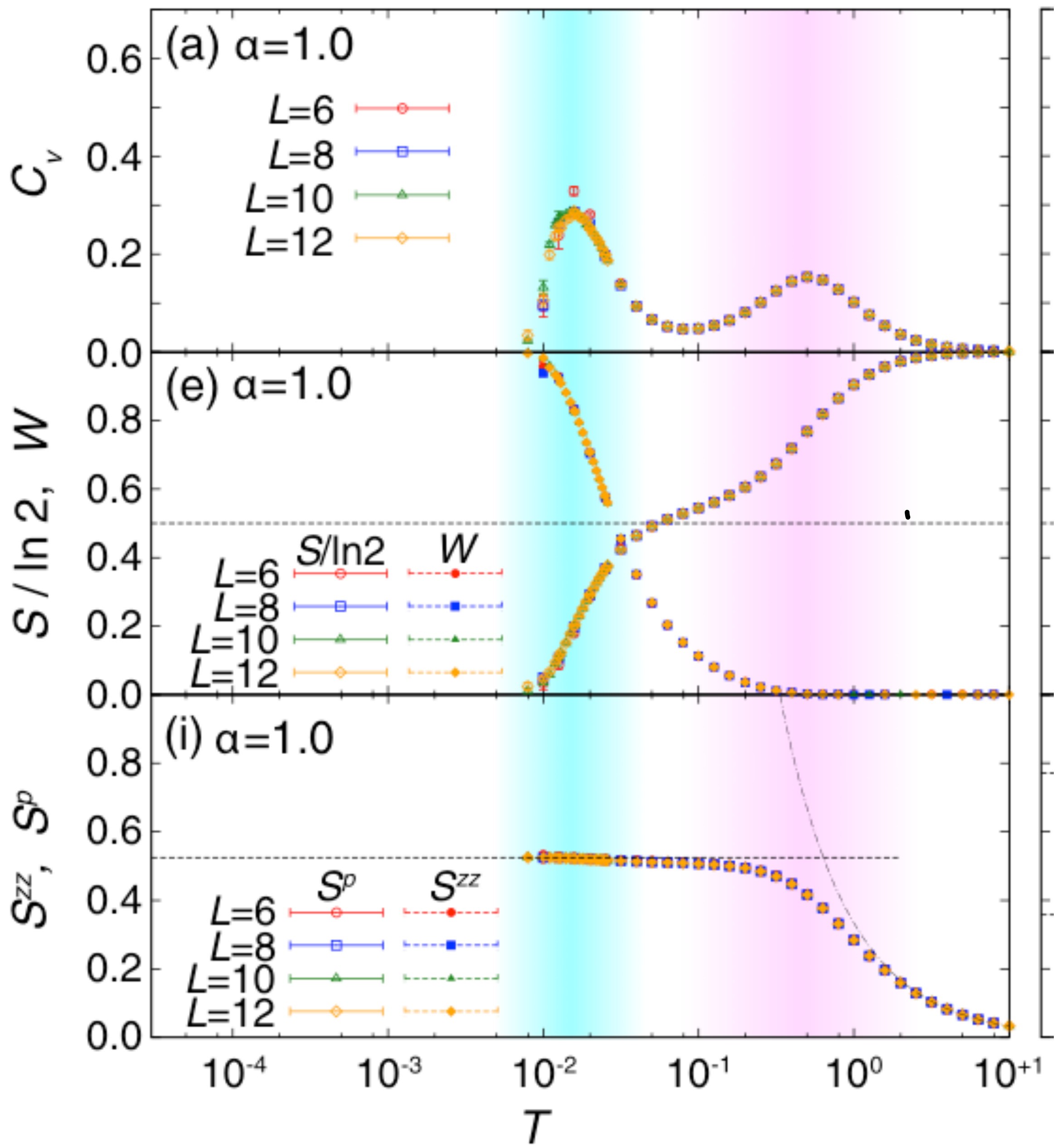
- Vortices.

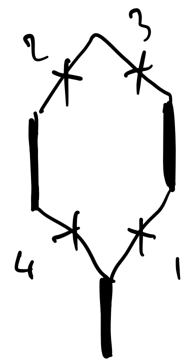


- Edge excitations (broken T.R.)



SPECIFIC HEAT





$$E_V \sim K_{eff} \sim \frac{K_x^2 K_y^2}{16 K_z^3}$$

Factor of $\frac{1}{16} = \frac{8}{4.8.4}$

$$W = \langle W_p \rangle,$$

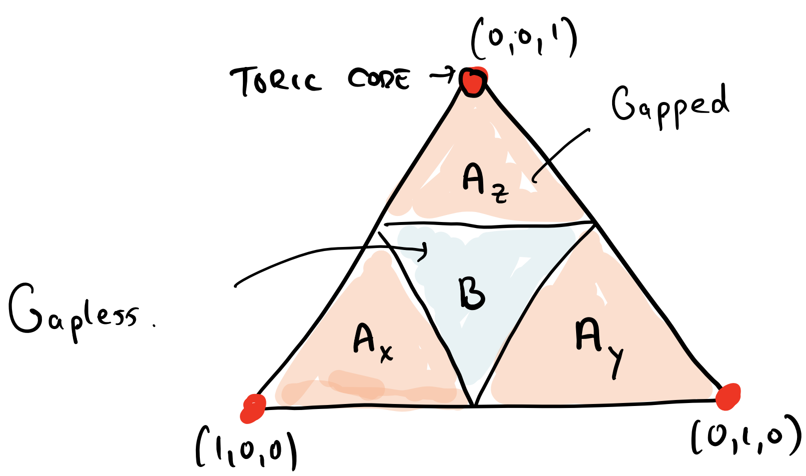
Vortex order

$$S^{\ell\ell} = \frac{2}{N} \sum \langle \sigma_j^\ell \sigma_k^\ell \rangle$$

ISING ORDER

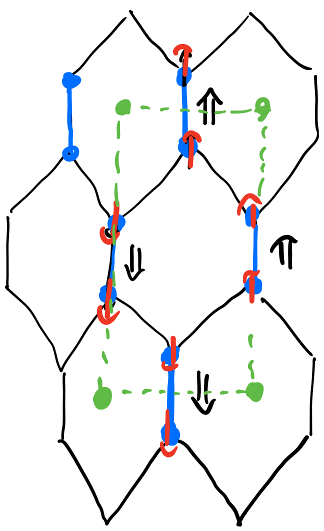
Nasu, Udayana & Motome, PRB 91, 115122 (2015)

Properties of gapped phases



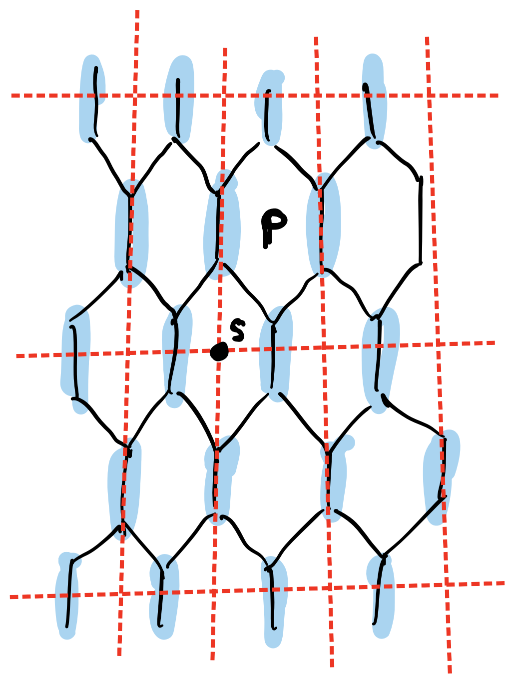
- Around a vortex $\prod u_{ij} = -1$

- Additional insight is gained from the extreme limit $J_x = J_y = 0$, which maps onto the "TORIC CODE"

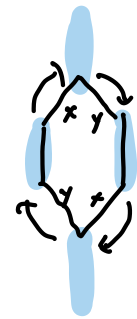


- $$\left.\begin{aligned} |\Uparrow\rangle &= |\uparrow\uparrow\rangle \\ |\Downarrow\rangle &= |\downarrow\downarrow\rangle \end{aligned}\right\} \text{effective spin.}$$

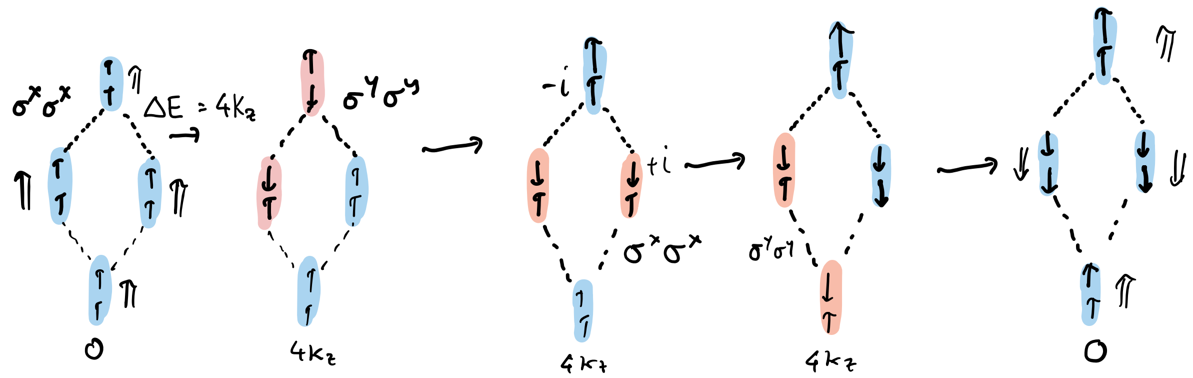
- Lie on bonds of a new lattice.



- Doing perturbation theory on the weak links, acting sequentially around a hexagon $\quad H_{eff}^{(4)} = V G_0 V G_0 V G_0 V$



$$V = \sigma^x_j \sigma^x_k \quad \text{or} \quad \sigma^y_j \sigma^y_k$$

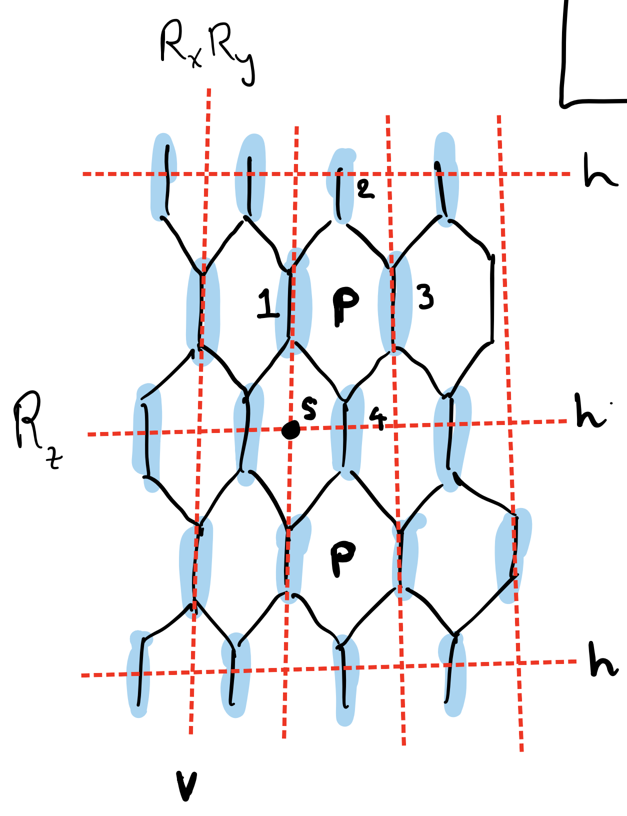


$$-\frac{8}{64}$$

Gives

$$\frac{8}{64} - \frac{8}{64} + \frac{8}{128}$$

$$\boxed{\begin{gathered} H_{eff}^{(4)} = \text{const} - \frac{K_x^2 K_y^2}{16 K_z^3} \sum_p Q_p \\ Q_p = \sigma^y_{p_1} \sigma^z_{p_2} \sigma^y_{p_3} \sigma^z_{p_4} \end{gathered}}$$



"Plaquet terms" $\quad B_p$

"Star terms" $\quad A_s$

Rotate about z	$\sigma_y \to \sigma_x$, $\sigma_z \to \sigma_z$	$R_z = X$ on horizontal links
Rotate about y then x	$\sigma_y \to \sigma_z$, $\sigma_z \to \sigma_x$	$R_x R_y = Y$ on vertical links

$$\mathcal{H} \longrightarrow U^{\dagger} \mathcal{H} U \qquad U = \underbrace{\sum X_j}_{\text{horizontal}} \ \underbrace{\sum Y_j}_{\text{vertical}}$$

$$X_j = R_z(j) = e^{i\pi/4\, \sigma_z(j)}$$

$$Y_j = e^{i\pi/4\, \sigma_y(j)} e^{i\pi/4\, \sigma_x(j)}$$

$$H_T = -J_e \sum_s A_s \qquad - J_m \sum_p B_p$$

ELECTRIC MAGNETIC

$A_s = \prod \sigma_j^x$

star

(FLIP SPINS)

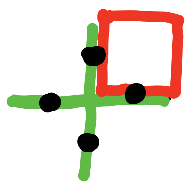
$B_p = \prod \sigma_j^z$

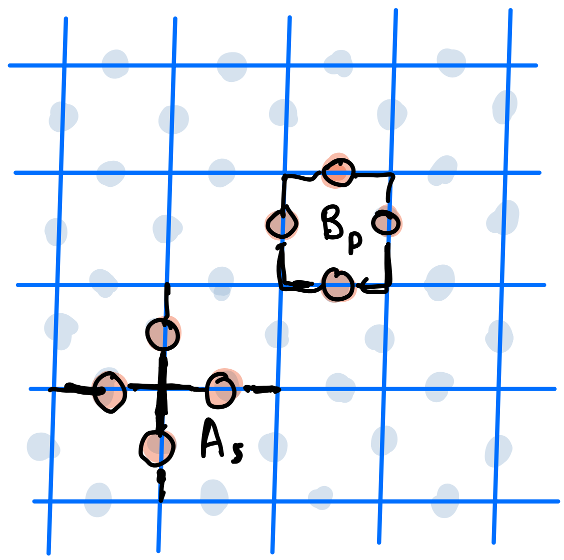
plaquet

TORIC CODE

- A's & B's all commute + each other

(SHARE ZERO OR TWO EDGES

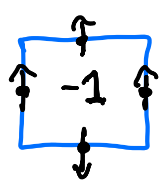




- Vortices correspond to loops with an odd # of spin flips

$\hat{\sigma}_j^z |s\rangle = s_j |s\rangle$

$s_j = \pm 1$



$$\boxed{W_p = \prod_{j \in \partial p} s_j}$$

$W_p = -1$ VORTEX.

Minimize energy $\Rightarrow$ $B_p |\psi\rangle = |\psi\rangle$

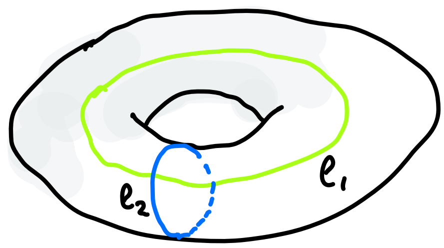
Requires states with <u>NO VORTICES</u>

$$\Rightarrow \quad |\psi\rangle = \sum_{\{s:\, W_p(s)=1\ \forall p\}} c_s |s\rangle$$

The condition $A_s|\psi\rangle = |\psi\rangle$ implies the state is unaffected by flipping spins on one star.

$\Rightarrow$ c_s equal for all configurations with no vortices.

TOPOLOGICAL G.S



LARGE CYCLES.

$$W_\ell(s) = \prod_{j\in \ell} s_j \,, \quad \ell = \ell_1, \ell_2$$

Absence of vortices means the $W_\ell(s)$ are the same for all paths ℓ_1 & all paths ℓ_2.

$W_\ell(s) = \pm 1$

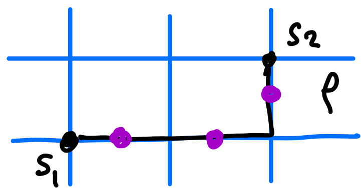
"Cohomology class of vortex free states"

$$|\psi\rangle = \sum c_{W_{\ell_1} W_{\ell_2}} |s\rangle$$

A_s star operator does not affect $W_\ell(s)$.

$$\boxed{D = (4)^g}$$

EXCITATIONS



"Electric Charges"

$$\hat{W}_\ell^{(E)} = \prod_{j \in \ell} \sigma_j^z$$ ← creates an electric vortex at s_1 & s_2

$$[W_\ell^{(E)}, B_p] = 0$$

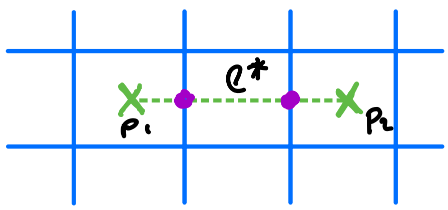
$$W_\ell^E A_{s_1} = -A_{s_1} W_\ell^E \qquad W_\ell^E A_{s_2} = -A_{s_2} W_\ell^E.$$

$$|\psi^E_{s_1, s_2}\rangle = \hat{W}_\ell^E |\psi_0\rangle$$

costs energy $\Delta E = 2J_e$. Independent of separation

CHARGES ARE "DECONFINED"

"Magnetic Charges"

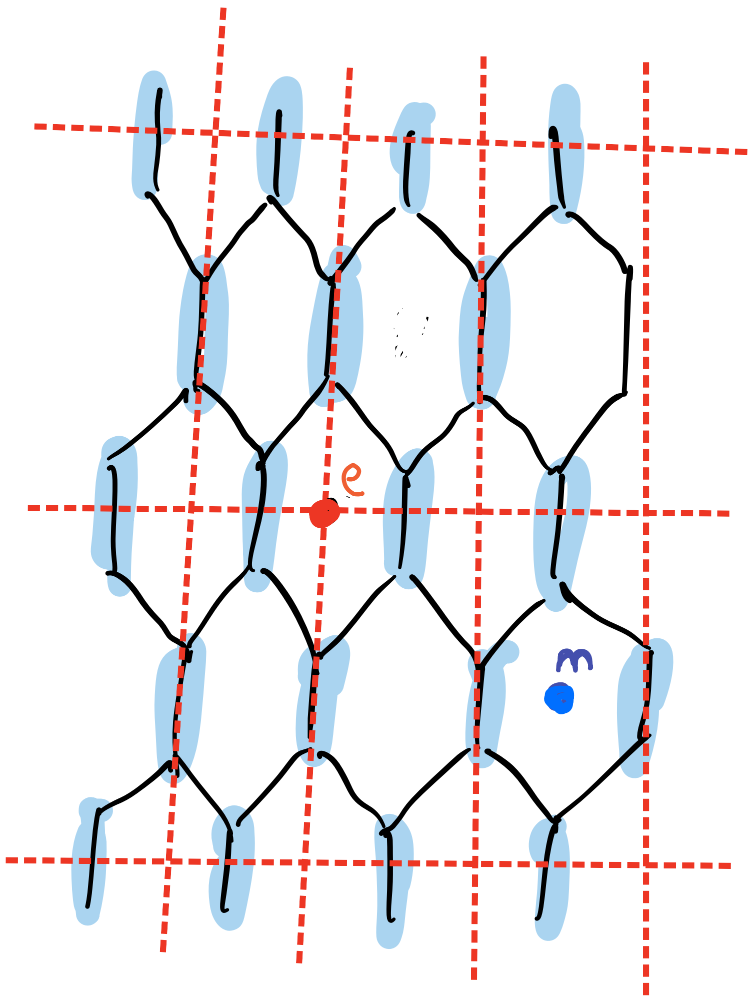


$$W^M_{e^*} = \prod_{j \in e^*} \sigma^z_e$$

$$\{B_{p_1}, W_{e^*}\} = 0$$

$$|\Psi^M_{p_1 p_2}\rangle = W_{e^*} |\Psi_0\rangle \qquad \Delta E = 2J_m$$

In the Honeycomb model, these excitations are degenerate because $J_m = J_e = \dfrac{K_x^2 K_y^2}{16 K_z^3}$.



$e: \prod_{+} \sigma_j^x = -1$

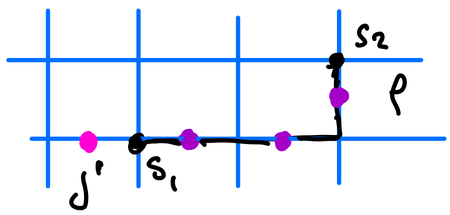
$m: \prod_{\square} \sigma_j^z = -1$

"Abelian Anyons"

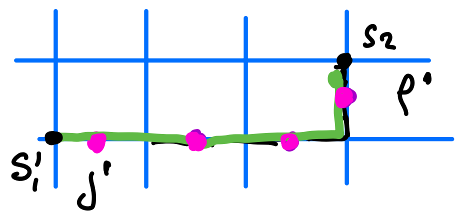
ABELIAN ANYONS : BRAIDING, SUPERSELECTION & FUSION.

The anyons of the Toric code are examples of "Abelian" anyons. When we braid them, the braiding processes commute. We can "move" the e & m anyons by simply extending the string creation operators, e.g

$$W^E_{e'} = \sigma^z_{j'} \prod_{j \in e} \sigma^z = \prod_{j \in e'} \sigma^z_j$$

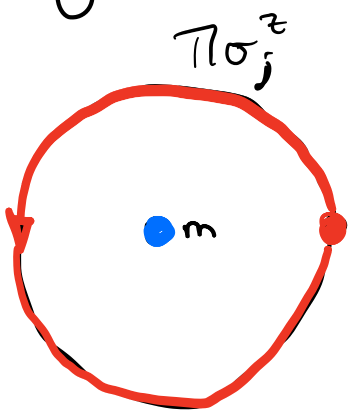


W^E_e

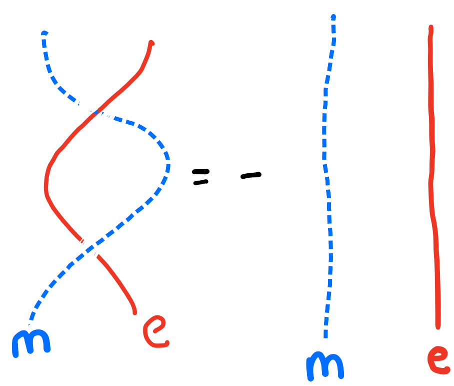


$W^E_{e'}$

If we take an e around an m, we pick up a string of $\prod_{\square\ \text{loop}} \sigma^z = -1$, so that moving an e around an m yields -1.



$= -1$,

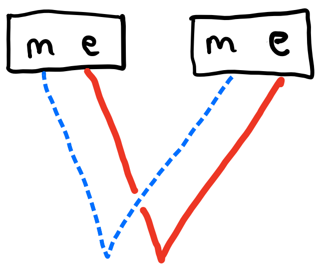


$$\prod_{\square} \sigma^z = +1$$

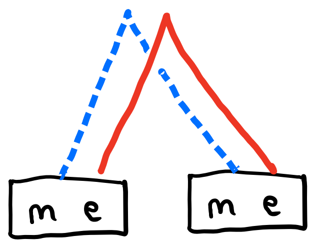
loop

Similarly taking e's around e's produce +1, likewise with m's, so they are bosons to themselves. However the combination of an m & an e, which is called an ε behaves as a fermion!

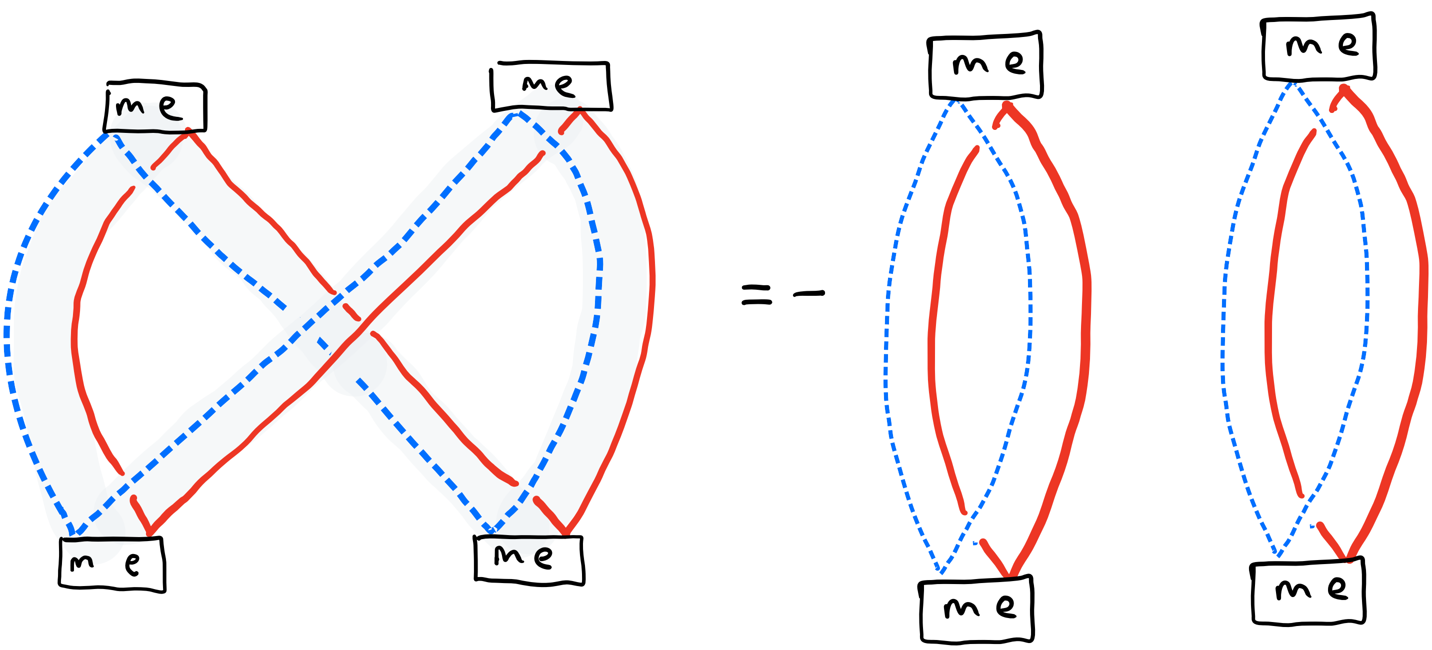
creation =

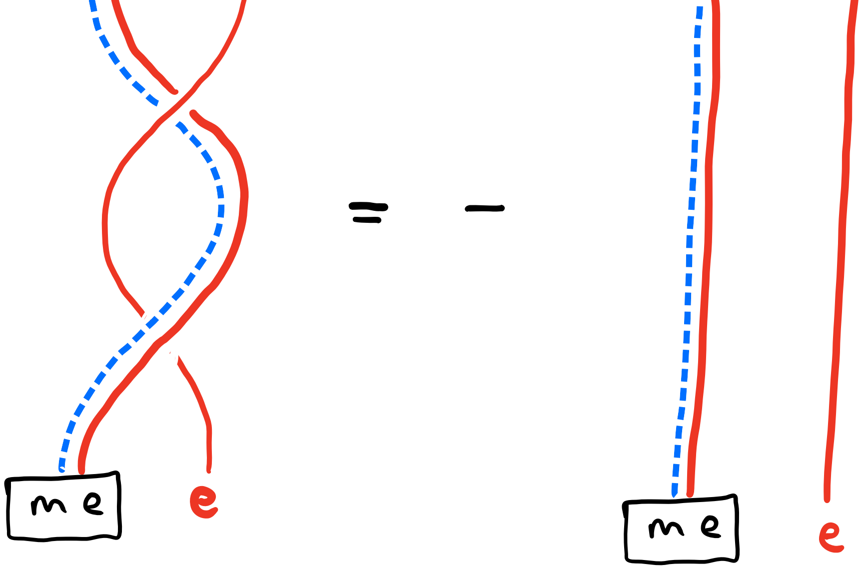


annihilation =



$\varepsilon \equiv$ Fermion





SUPER-SELECTION

Class of states that can be transformed into one-another by local operators. e.g the current operator can fuse a positron + electron into a photon.

$$e^+ \times e^- = \gamma$$

Toric code: 1 (vacuum), e, m, ε.

TORIC CODE : 1

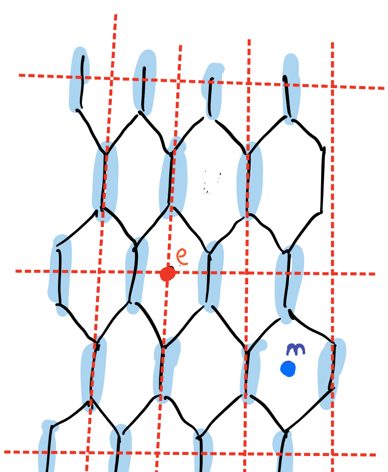
Fusion Rules : How particles (or "superselection sectors") fuse together.

For the toric code

$$e \times e = m \times m = \varepsilon \times \varepsilon = 1 .$$

$$e \times m = \varepsilon, \qquad e \times \varepsilon = m, \qquad m \times \varepsilon = e .$$

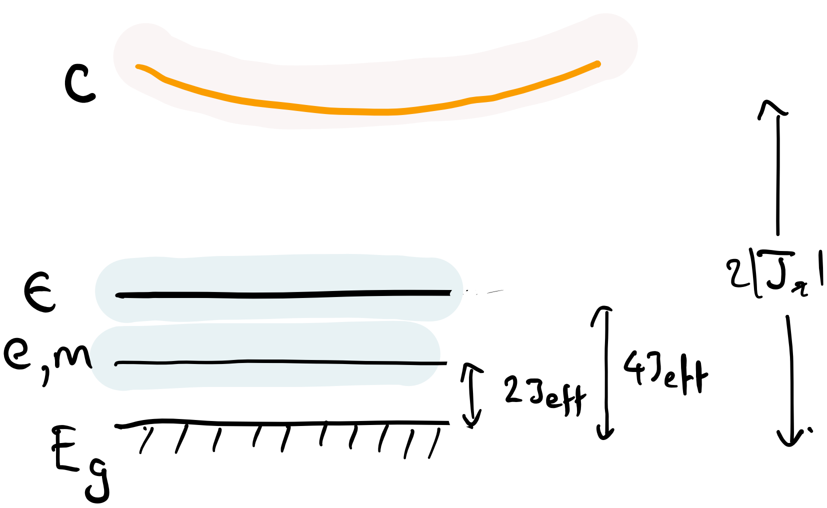
IDENTIFICATION OF PARTICLES IN THE HONEYCOMB LATTICE.



Vortices $\equiv$ e & m

Majorana fermions c $\equiv$ ε superselection sector.

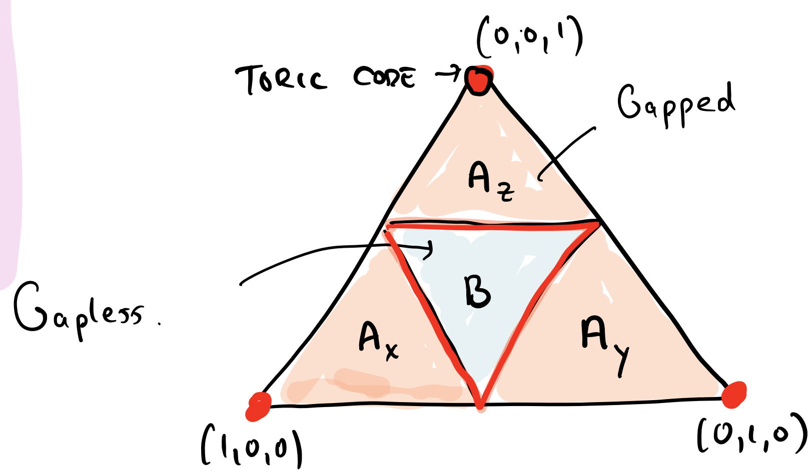
In limit $J_z \gg J_x, J_y$,



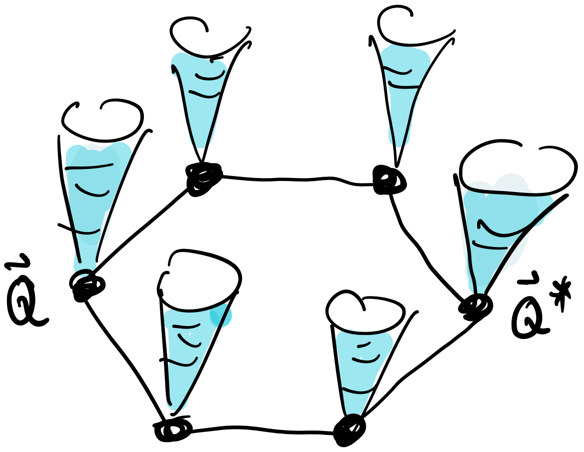
$$E(c) \sim 2|J_z|$$

$$E(e\text{-}m) = 4J_{eff} \ll |J_z|$$

PHASE B IN A MAGNETIC FIELD.



In phase B, the gapless fermions prevent the vortices from being moved adiabatically, so they no longer have well defined statistics.



$|Q - Q^*| \sim L^{-1}$

Leads to a kind of RKKY interaction between vortices that is of order $\epsilon(q \sim L^{-1}) \sim L^{-1}$ that oscillates at wavevector

$Q^* - Q \equiv 2Q^*$. This produces a non universal phase

$\Delta\phi \sim L^{-1} t \sim \frac{1}{\text{velocity}}$. As $v \to 0$, $\Delta\phi \to$ very large $\therefore$ no well defined statistics

But! Broken time reversal symmetry can induce a gap in the Fermion spectrum!

$$V = -\sum_j \vec{h} \cdot \vec{\sigma}_j \qquad \left(\frac{1}{E_0 - \mathcal{H}}\right)' = (1-P)\frac{1}{E_0 - \mathcal{H}}(1-P)$$

$$H_{eff} = P\left[V + V\left(\frac{1}{E_0 - \mathcal{H}}\right)'V + V\left(\frac{1}{E_0 - \mathcal{H}}\right)'V\left(\frac{1}{E_0 - \mathcal{H}}\right)'V + \ldots\right]P$$

$\overset{\shortparallel}{0}$

Preserves T-rev

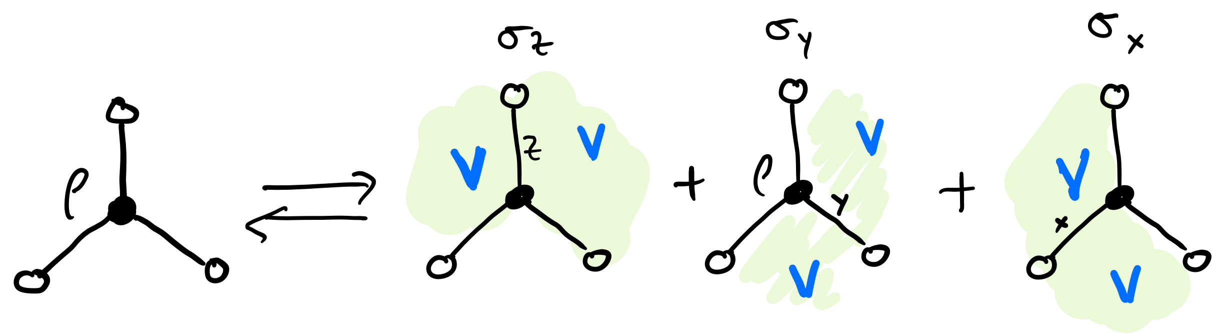
(Poor Man's R.G)

V creates two adjacent vortices.

$$\left(\frac{1}{E_0-\mathcal{H}}\right)' \sim -\frac{1}{J}$$

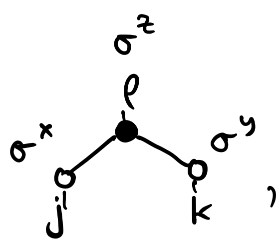
$$V = \sum -i\left(h^x b_j^x + h^y b_j^y + h^z b_j^z\right) c_j$$

FLIPS

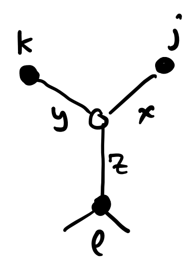


$$\boxed{H_{eff} \sim -\frac{h_x h_y h_z}{K^2} \sum_{j,k,l} \sigma_j^x \sigma_k^y \sigma_l^z}$$

& symmetry equivalents.



$\sigma_j^x \sigma_l^z \sigma_k^y$



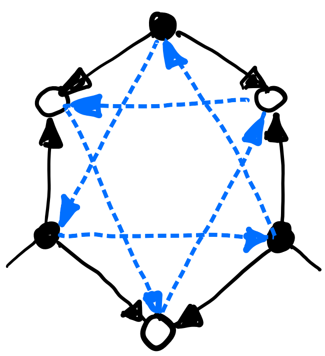
six fermion term

$$\sim -i\delta\, b_j^x c_j\, b_\ell^z c_\ell\, b_k^y c_k$$

$$\sim +i\delta\, c_j \underbrace{b_j^x b_\ell^x}_{u_{j\ell}} \underbrace{(b_\ell^x b_\ell^z b_\ell^y c_\ell)}_{D_\ell} \underbrace{b_\ell^y b_k^y}_{u_{\ell k}} c_k$$

$$= i(D_\ell\, \hat{u}_{j\ell}\, \hat{u}_{\ell k})\, c_j c_k \sim -i\, c_j c_k\,.$$

NEXT NEAREST NEIGHBOR HOPPING!



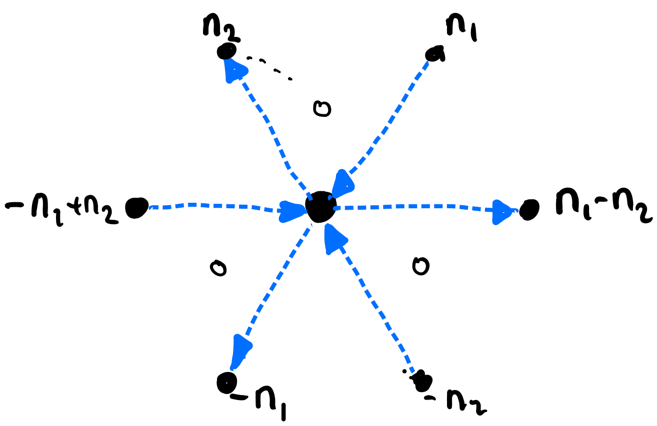
$$H_{eff} = i \sum A_{jk} c_j c_k$$

$$A = 2K(\leftarrow) + 2K'(\leftarrow)$$

$$\tilde{K} \sim \frac{h_x h_y h_z}{J^2}$$

$$H_{eff} = \sum_{k \in \frac{1}{2}BZ} \psi_k^\dagger\, iA(\vec{q})\, \psi_k$$

$$iA(\vec{q}) = \begin{pmatrix} \Delta(q) & if(q) \\ -if^*(q) & -\Delta(q) \end{pmatrix}$$

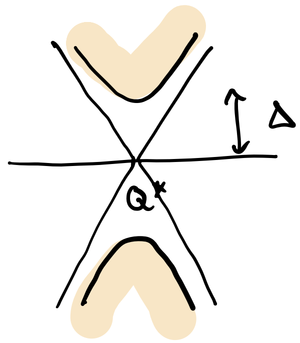


$$\Delta(q) = i2\tilde{K}\left(e^{i\vec{q}\cdot\vec{n}_2} + e^{-i\vec{q}\cdot\vec{n}_1} + e^{iq\cdot(\vec{n}_1-\vec{n}_2)} - h.c.\right)$$

$$= 4\tilde{K}\left(\sin(\vec{q}\cdot\hat{n}_1) - \sin(\vec{q}\cdot n_2) + \sin(\vec{q}\cdot(\vec{n}_1-\vec{n}_2))\right)$$

$$\Delta = \Delta(Q^*) \sim K' \sim h_x h_y h_z$$ GAP

$\Delta = \Delta(Q^*) \;\; \ldots \;\; \frac{\ldots}{K^2}$



$\epsilon(\vec{q}) \sim \pm \sqrt{3K^2 \delta q^2 + \Delta^2}.$

The gapped spectrum has a non zero Chern number.

$A_{\vec{k}} = i \psi_k^* \nabla_k \psi_k$

$B_k = (\vec{\nabla}_k \times \vec{A}_k)_z$

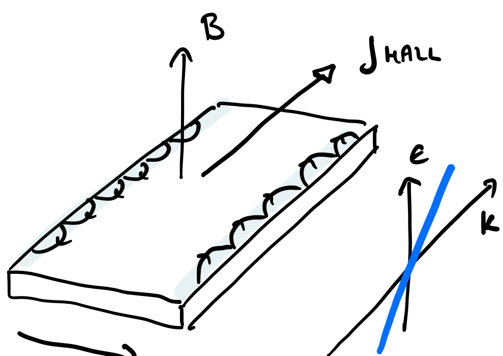
$\nu = \frac{1}{2\pi} \int d^2k \, B_k = \pm 1.$

$\Rightarrow$ Edge modes.

$\boxed{\nu_{edge} = (\#\text{ right movers} - \#\text{ left movers}) = \nu}$

Quantized Thermal Hall Effect

Recall that in the IQHE, the presence of edge states led to a quantized Hall conductance.



$\sigma_{xy} = \frac{e^2}{h} \nu \; ; \quad \sigma_{xx} = 0$

$\nu = \#$ of edge states.

$E, \nabla T$

Now under rather general conditions

$$\frac{1}{T} \kappa_{ab} = L \, \sigma_{ab} , \qquad L = \frac{\pi^2 k_B^2}{3 e^2} ,$$

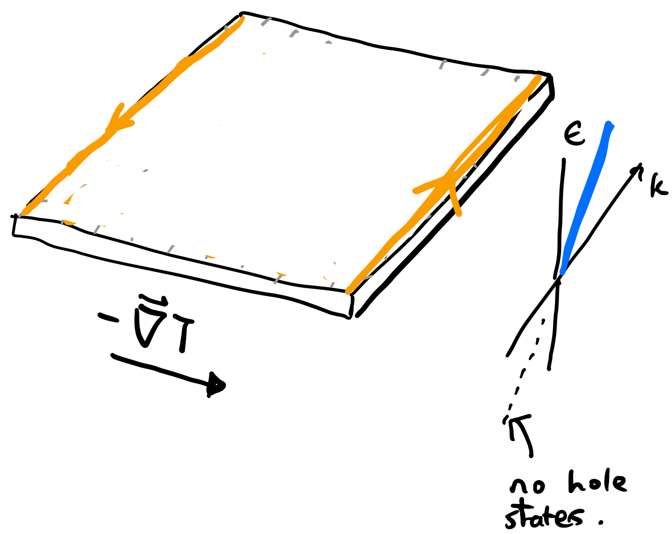
where κ is the thermal conductivity tensor. So for the IQHE, we expect

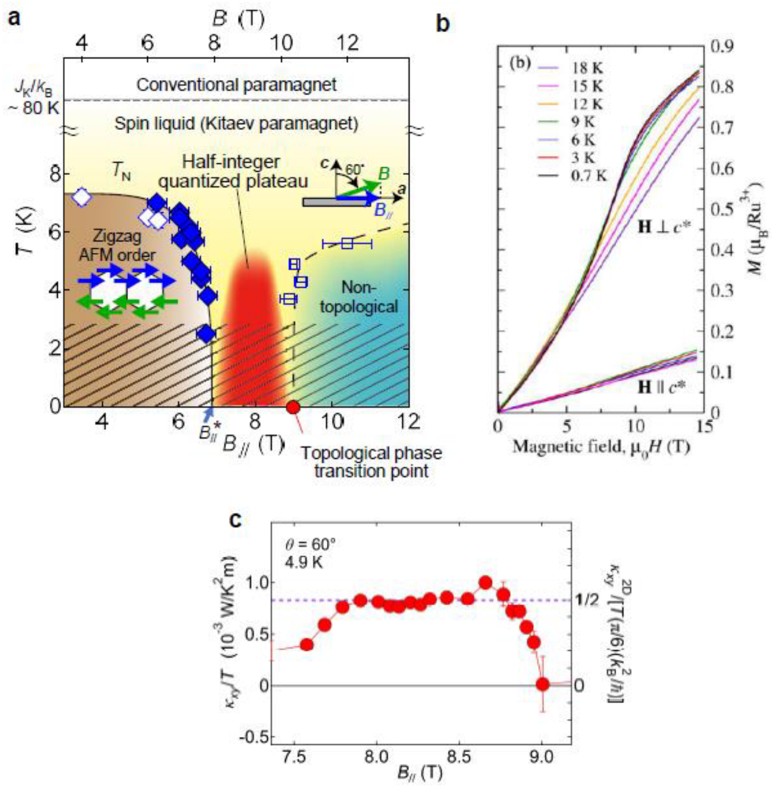
$$\frac{\kappa_{xy}}{T} = \frac{\pi^2 k_B^2}{3e^2} \times \frac{e^2}{h} \nu = \left(\frac{\pi k_B^2}{6 \hbar} \right) \nu$$

A Majorana edge state carries ½ the heat current of an electron edge state

$$\Rightarrow \boxed{ \frac{\kappa_{xy}}{T} = \frac{1}{2} \frac{\pi k_B^2}{6 \hbar} }$$

A PURE THERMAL HALL EFFECT.





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$$\varepsilon \times \varepsilon = 1 , \quad \varepsilon \times \sigma = \sigma , \quad \sigma \times \sigma = 1 + \varepsilon$$

Anyons now have NON-ABELIAN braiding rules.

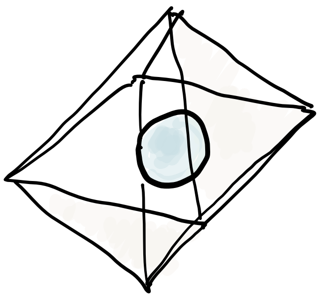
KITAEV SPIN LIQUIDS

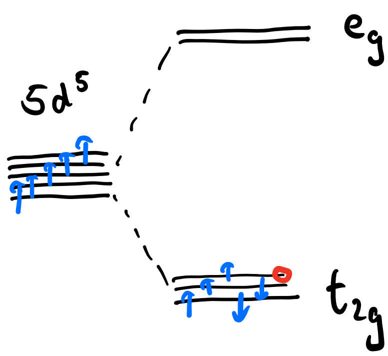
In 2009, George Jackeli & Giniyat Khaliulin proposed that Iridium atoms inside octahedra would develop Kitaev-like Ising interactions with their neighbors. Suddenly, the Kitaev honeycomb model was no longer a "toy": it might be realized in solid state quantum materials.

Proposed systems : spin orbit coupled transition metals (e.g Ru, Ir) in an Octahedral environment

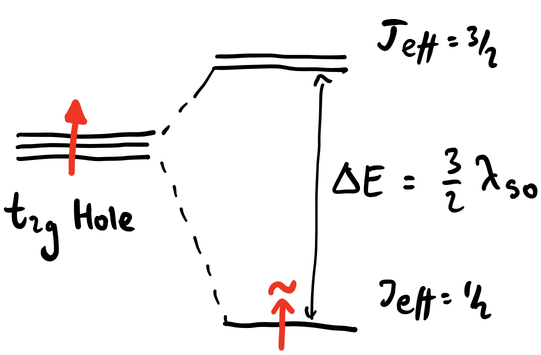
Ir^{4+} $5d^5$

Ru $4d^5$.





One hole.



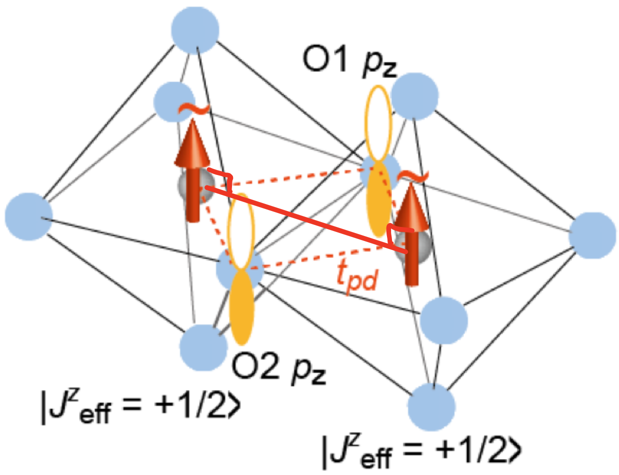
$$\sqrt{\tfrac{1}{3}}\,|\ell^z=0,\uparrow\rangle + \sqrt{\tfrac{2}{3}}\,|\ell^z=+1,\downarrow\rangle$$

Spin orbit splits the t_{2g} orbitals into a low + high spin J-state.

"Mottness"

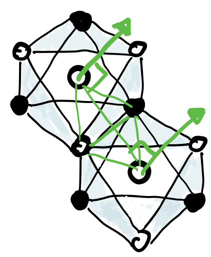
Edge sharing octahedra.

e.g IrO_6 or $RuCl_6$.

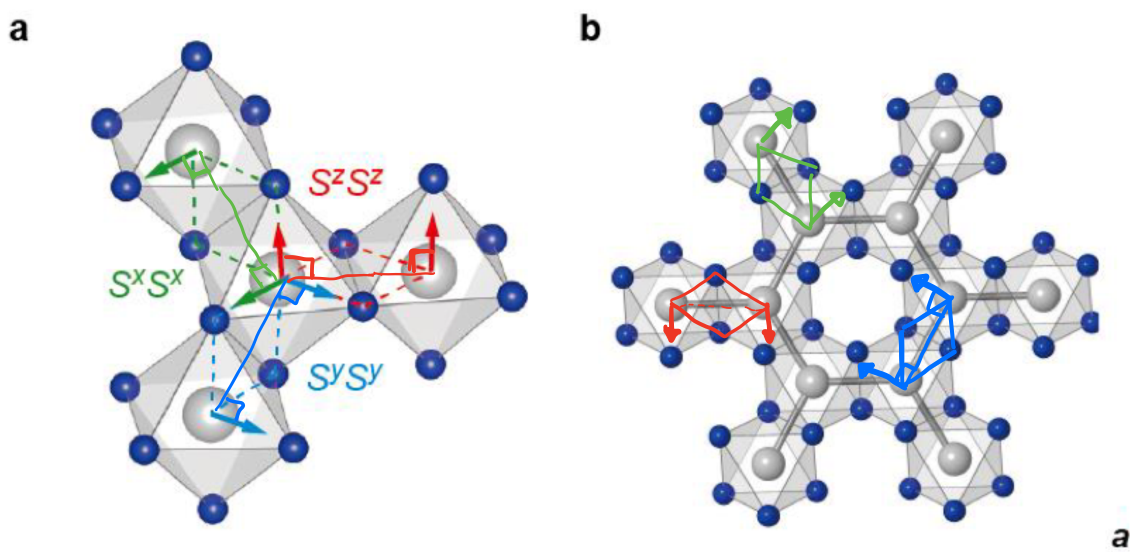


Hunds coupling lowers the energy of a virtual hop of two parallel spins, inducing an Ising coupling perpendicular to the plane shared by the edge and the Ir atoms

e



Octahedra can be laid down in a plane to form a honeycomb structure

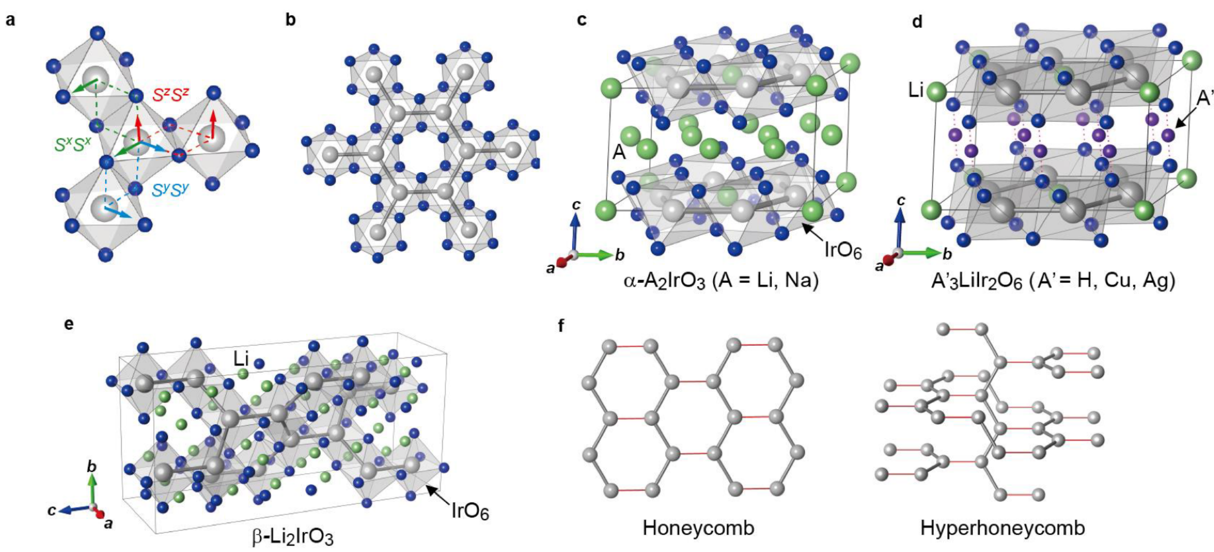


The three neighboring edge-sharing octahedra define three orthogonal planes, giving rise to the three directions for the Ising interactions. Interlayer couplings are weak, but there are additional Heisenberg & tensor interactions.

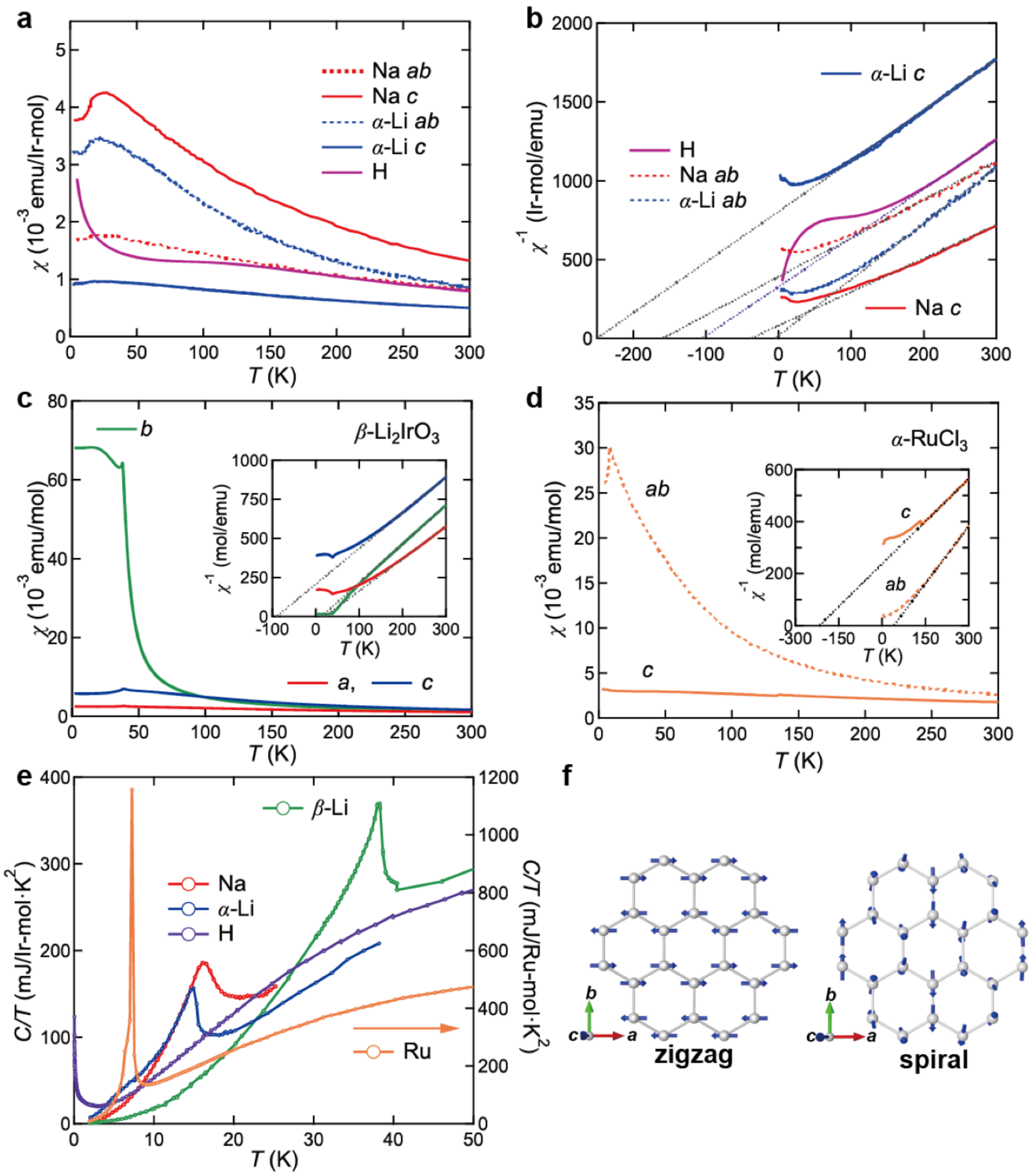
$$H = \sum_{\langle i,j \rangle_\gamma} \left\{ -K S_i^\gamma S_j^\gamma + \Gamma \left(S_i^\alpha S_j^\beta + S_i^\beta S_j^\alpha \right) + J \vec{S}_i \cdot \vec{S}_j \right\}$$

Hunds   d-d / d-p electron transfer   d-d hybridization

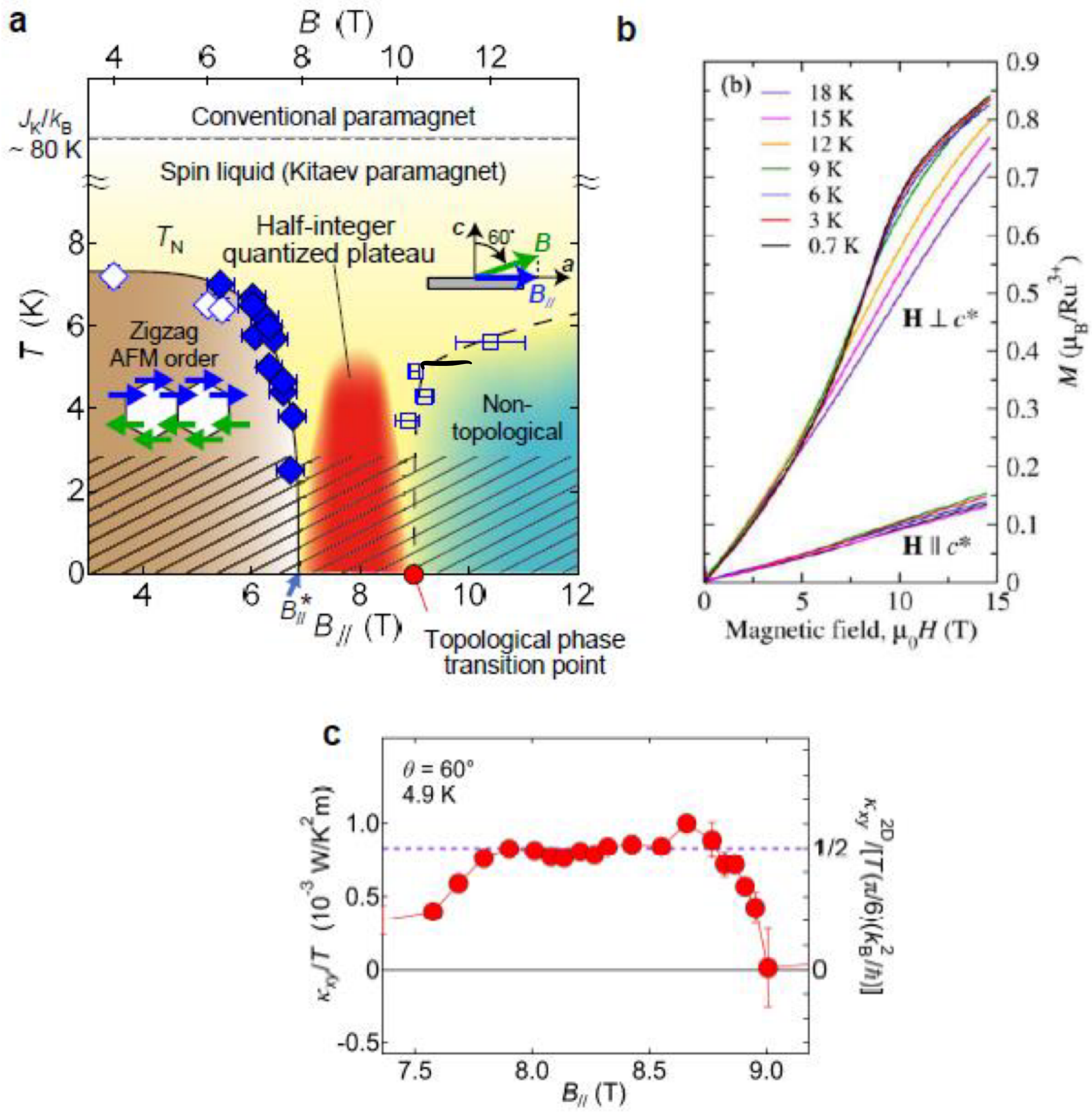
$(K > \Gamma \gg J)$.



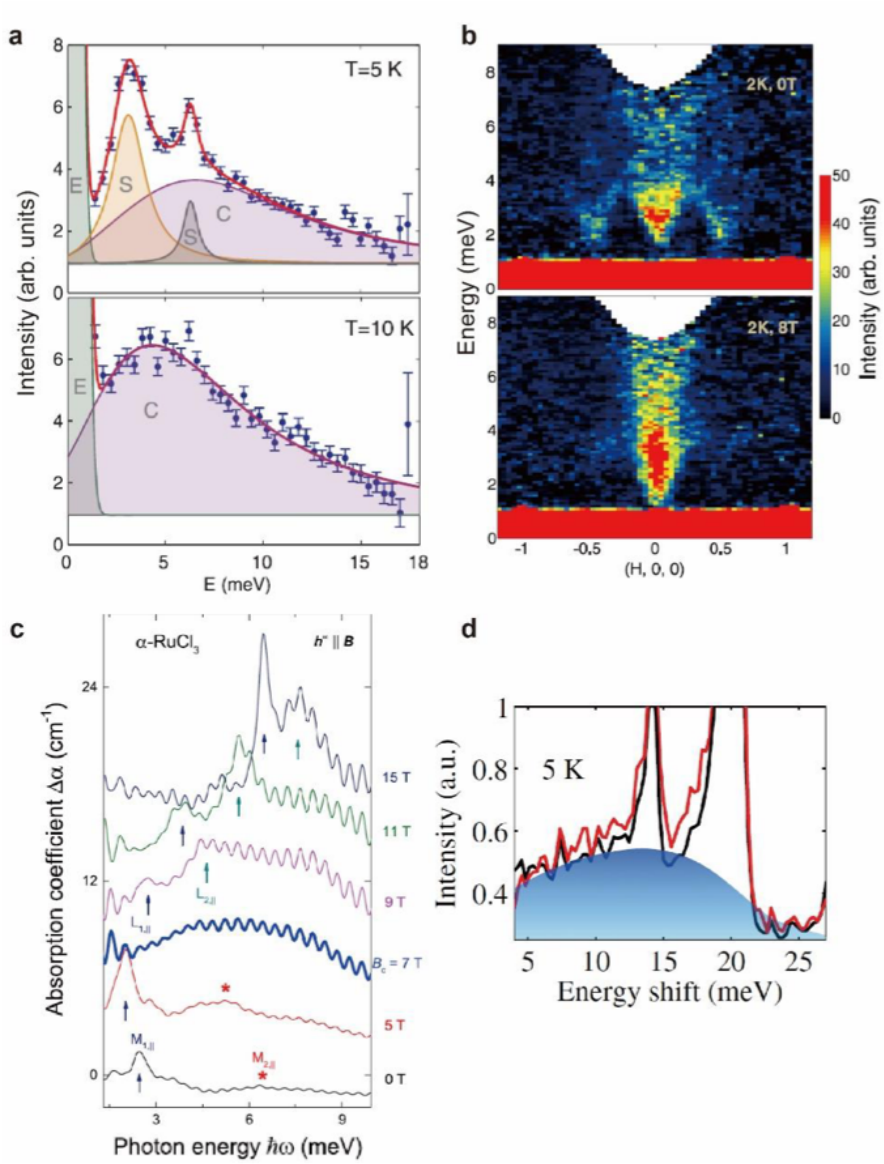
Materials	Crystal structure (Space group)	T_{mag}	anisotropy	p_{eff} (μ_B)	θ_{CW} (K)	Magnetic ground state	Ref.
Na_2IrO_3	2D ($C2/m$)	15 K	$\chi_c > \chi_{ab}$	1.81 (*ab*) 1.94 (*c*)	-176 (θ_{ab}) -40 (θ_c)	zigzag	40,57,66, 67
α-Li_2IrO_3	2D ($C2/m$)	15 K	$\chi_{ab} > \chi_c$	1.50 (*ab*) 1.58 (*c*)	+5 (θ_{ab}), -250 (θ_c)	Spiral	44,65,70
$H_3LiIr_2O_6$	2D ($C2/m$)	-	$\chi_{ab} > \chi_c$	1.60	-105	Spin-liquid	46
Cu_2IrO_3	2D ($C2/c$)	2.7 K	Not known	1.93(1)	-110	AF order or Spin-glass	42
$Cu_3LiIr_2O_6$	2D ($C2/c$)	15 K	Not known	2.1(1)	-145	AF order	49
$Ag_3LiIr_2O_6$	2D (R-3m*)	~12 K	Not known	1.77		AF order	48
α-$RuCl_3$	2D ($C2/m$ or $P3_112$,or R-3) ; T and sample dependent	7 K and/or, 14 K See text	$\chi_{ab} > \chi_c$	2.33 (*ab*), 2.71 (*c*)	+39.6(θ_{ab}), -216.4 (θ_c)	zigzag	51,64,68, 69, 131
β-Li_2IrO_3	3D ($Fddd$)	38 K	$\chi_b > \chi_c > \chi_a$	1.87 (*a*) 1.80 (*b*) 1.97 (*c*)	-90.2 (θ_a) +12.9 (θ_b) +21.6 (θ_c)	Spiral	52,71,92
γ-Li_2IrO_3	3D ($Cccm$)	39.5 K	$\chi_b > \chi_c > \chi_a$	~1.6	+40	Spiral	53,72


a
Na ab
Na c
α-Li ab
α-Li c
H
χ (10^-3 emu/Ir-mol)
T (K)
b
α-Li c
H
Na ab
α-Li ab
Na c
χ^-1 (Ir-mol/emu)
T (K)
c
b
β-Li2IrO3
χ^-1 (mol/emu)
T (K)
a, c
χ (10^-3 emu/mol)
T (K)
d
α-RuCl3
ab
c
χ^-1 (mol/emu)
T (K)
χ (10^-3 emu/mol)
T (K)
e
β-Li
Na
α-Li
H
Ru
C/T (mJ/Ir-mol·K^2)
C/T (mJ/Ru-mol·K^2)
T (K)
f
zigzag
spiral

HALF QUANTIZED THERMAL HALL IN $RuCl_3$



Signatures of a fractional continuum.



Neutron

$S^+ \to$ Majoranas
+ Vortex Pair

Raman

$S.S \to$ Majoranas

Figure 7. **Signature of fractional excitations in α-RuCl$_3$. a,** Inelastic neutron scattering in single-crystal α-RuCl$_3$ measured at temperatures of T = 5 K (top) and 10 K (bottom)[75]. The data is integrated over a small reciprocal space volume centered at the Γ point of the two-dimensional lattice. The letters designate the contributions from the elastic line "E", spin-waves "S", and continuum scattering "C". **b,** Inelastic neutron scattering is measured at T = 2 K (top) in zero external magnetic field and (bottom) in a field of 8 T in the honeycomb plane, large enough to suppress the magnetic order[90]. The color bar denotes the relative intensity. **c,** THz spectroscopy measurements in α-RuCl$_3$[113] in the presence of a magnetic field applied in the honeycomb plane, with the THz field parallel to the applied field direction. All measurements were carried out at T = 2.4 K. The arrows indicate locations of excitations inferred from the data. **d,** Detail of Raman measurements in α-RuCl$_3$ at T = 5 K[107]. The blue shaded area represents the magnetic continuum scattering. (Panel **a** reproduced with permission from Ref. 75, panel **b** reproduced with permission from Ref. 90, panel **c** reproduced with mission from Ref. 113, and panel **d** reproduced with permission from Ref. 107).