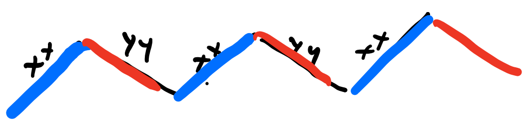
Kitaev Models in 1, 2 & 3 Dimensions

The Kitaev model is a quantum Ising model, in which the presence of competing anisotropic Ising interactions give rise to strong quantum fluctuations.

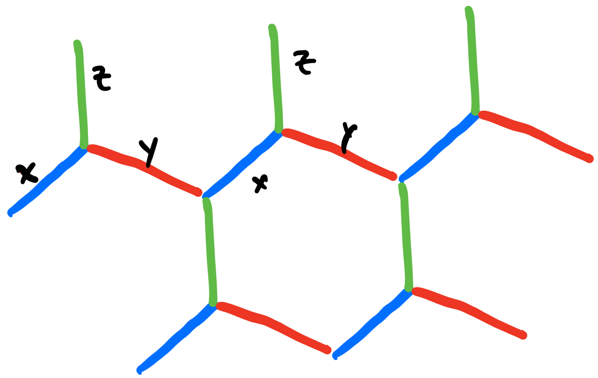
1D



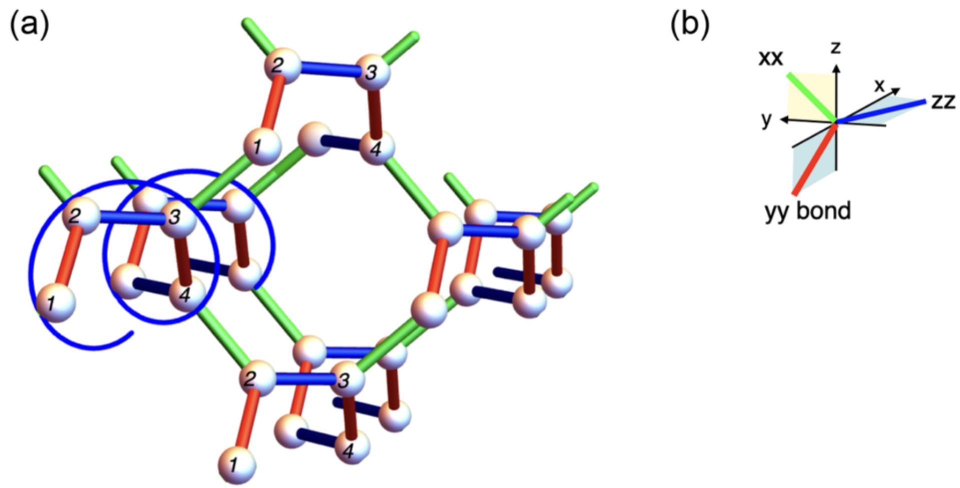
$$\mathcal{H} = J \sum_{\langle i,j\rangle} \sigma_i^{\alpha_{ij}} \sigma_j^{\alpha_{ij}}$$

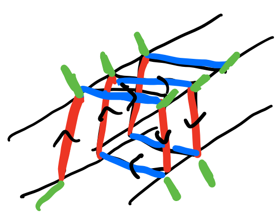
$\alpha_{ij} = xx\,,\ yy.$

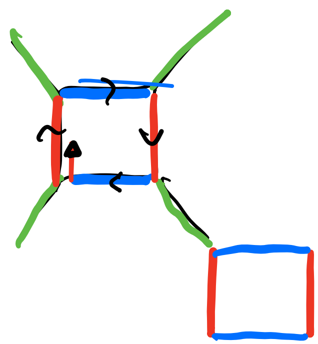
2D



3D







Majorana Primer

Just as bosons can be decomposed into their real components, q & p

$$b = \frac{q + ip}{\sqrt{2}} \qquad [q,p] = i \iff [b, b^\dagger] = 1$$

fermions can be decomposed into their real components, called Majorana fermions

$$c = \frac{a + ib}{\sqrt{2}} \qquad a = a^\dagger, \; b = b^\dagger$$

From this we see that $a = \frac{c + c^\dagger}{\sqrt{2}}$ $\quad b = \frac{c - c^\dagger}{\sqrt{2}i}$ and therefore

$$\{a, a\} = 1, \quad \{a, b\} = 0, \quad \{b, b\} = 1$$

so that the Majorana components of a "Dirac" fermion also satisfy canonical anticommutation rules.

It takes a while to get used to Majorana fermions.

Here are a few useful properties.

① $a^2 = b^2 = 1/2$ (some use a unit normalization)

② Consider $\mathcal{H} = \epsilon \left(f^\dagger f - \frac{1}{2} \right)$ $\updownarrow \epsilon$

$\epsilon/2 \quad Z = 1$

$-\epsilon/2 \quad Z = -1$

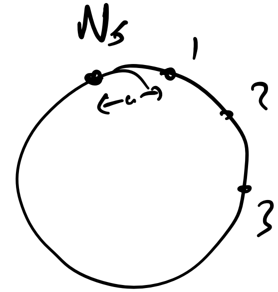
This becomes

$$\mathcal{H} = \epsilon\left(\frac{(a-ib)(a+ib)}{2} - \frac{1}{2}\right) = \frac{\epsilon}{2}\big(\overbrace{2i\hat{a}\hat{b}}^{Z=\pm 1}\big)$$

Z_2 variable

(3) $$\mathcal{D}_f = 2 = \mathcal{D}_a \mathcal{D}_b \Rightarrow \mathcal{D}_a = \sqrt{2} \;!$$

Entropy @ high T $\quad \ln W = \frac{1}{2}\ln 2.$



(4) Momentum space $\quad \mathcal{H} = t\sum_j i\,\hat{a}_{j+1}\hat{a}_j$

$$a_k = \frac{1}{\sqrt{N_s}}\sum a_j e^{-ikR_j}$$

$$a_k^{+} = \frac{1}{\sqrt{N_s}}\sum \underset{a_j = a_j^{+}}{a_j} e^{ikR_j} = a_{-k} \;!$$

$R_j = ja.$

$k = j\left(\frac{2\pi}{N_s a}\right)$

$$a_j = \frac{1}{\sqrt{N_s}} \sum_k a_k e^{ikR_j} \qquad a_{j+1} = \frac{1}{\sqrt{N}} \sum \underset{\underset{a_k^+}{\shortparallel}}{a_{k'}} \, e^{-ik(R_j+a)}$$

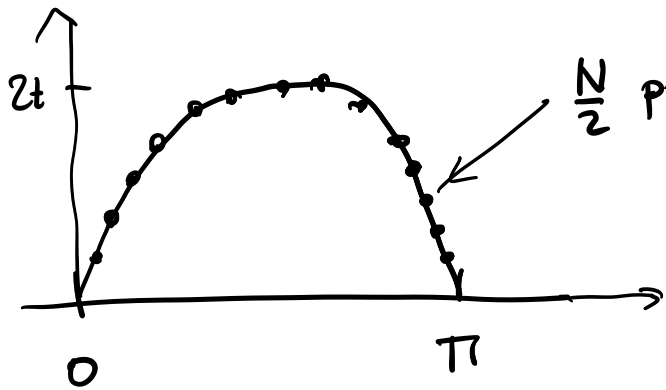
$$\mathcal{H} = it \sum_k a_k^+ a_k e^{-ika}$$

$$a_{-k}^+ a_{-k} e^{ika}$$
$$= a_k a_k^+ e^{ika}$$
$$= (-a_k^+ a_k + 1) e^{ika}$$

$$= \sum_k + t \left(\frac{e^{+ika} - e^{-ika}}{2i} \right) a_k^+ a_k$$

$$\sum e^{ika} = 0$$

$$= \sum_{k \in \frac{1}{2} BZ} 2t \sin ka \left(a_k^+ a_k - \frac{1}{2} \right)$$



$$E_{gs} = -L \int_0^{\pi/a} 2t \sin ka \frac{dk}{2\pi}$$

$$= -\frac{2}{\pi} t N_s$$

5) Hamiltonian + Path integral

$$H = \frac{1}{2} \sum_{\alpha \neq \beta} c_\alpha \mathcal{H}_{\alpha\beta} c_\beta \qquad \mathcal{H}_{\alpha\beta} = -\mathcal{H}_{\beta\alpha}$$

Consider

$$H_f = \sum f_\alpha^\dagger \mathcal{H}_{\alpha\beta} f_\beta = \frac{1}{2} \sum (a_\alpha - i b_\alpha) \mathcal{H}_{\alpha\beta} (a_\beta + i b_\beta)$$

$$f_\beta = \frac{a_\beta + i b_\beta}{\sqrt{2}} \qquad = \frac{1}{2} \sum \left(a_\alpha \mathcal{H}_{\alpha\beta} a_\beta + b_\alpha \mathcal{H}_{\alpha\beta} b_\beta \right)$$

Seperable Hilbert Spaces.

$$\sum (b_\alpha \mathcal{H}_{\alpha\beta} a_\beta - a_\alpha \mathcal{H}_{\alpha\beta} b_\beta) = \sum \Big(-\overbrace{\mathcal{H}_{\alpha\beta}}^{\mathcal{H}_{\beta\alpha}} a_\beta b_\alpha - a_\alpha b_\beta \mathcal{H}_{\alpha\beta}$$

$$= \sum \mathcal{H}_{\beta\alpha} a_\beta b_\alpha - \sum \mathcal{H}_{\alpha\beta} a_\alpha b_\beta = 0$$

$$S = \sum f^+_\alpha (\partial_\tau + \mathcal{H}_{\alpha\beta}) f_\beta$$

$$= \frac{1}{2} \sum \left(a_\alpha (\delta_{\alpha\beta} \partial_\tau + \mathcal{H}_{\alpha\beta}) a_\beta + b_\alpha (\delta_{\alpha\beta} \partial_\tau + \mathcal{H}_{\alpha\beta}) b_\beta \right)$$

$$\int \mathcal{D}[f^+ f] e^{-S} = \det[\partial_\tau + \mathcal{H}]$$

$$= \int \mathcal{D}[a] \, e^{-\frac{1}{2} a (\partial_\tau + \mathcal{H}) a} \int \mathcal{D}[b] \, e^{-\frac{1}{2} b (\partial_\tau + \mathcal{H}) b}$$

$$= D_a D_b$$

$$\Rightarrow \quad \int \mathcal{D}[a] e^{-\frac{1}{2} a_\alpha (\partial_\tau + \mathcal{H}) a_\beta} = \sqrt{\det(\partial_\tau + \mathcal{H})}$$

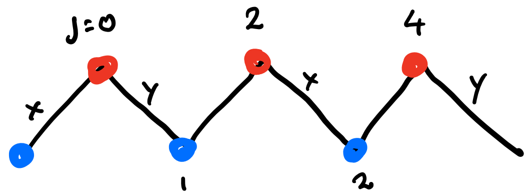
$$= \mathrm{Pf}[(\partial_\tau + \mathcal{H})]$$

Two methods of solution (1) Jordan-Wigner Transformation

(2) Ancilliary qubit approach.

Jordan Wigner Approach to 1D Kitaev Chain

1D



$$H = \frac{1}{2} \sum_{j\,\text{even}} \left[K^x \sigma_j^x \sigma_{j-1}^x + K^y \sigma_{j+1}^y \sigma_j^y \right]$$

$$\left.\begin{aligned} \sigma_j^+ &= \frac{\sigma_j^x + i\sigma_j^y}{2} = f_j^\dagger P_j \\ \sigma_j^- &= \frac{\sigma_j^x - i\sigma_j^y}{2} = f_j P_j \end{aligned}\right\}$$

$$P_j = e^{i\pi \Phi_j}$$ Jordan Wigner

$$\Phi_j = \pi \sum_{j'<j} n_{j'}$$

$$\sigma_j^x \sigma_{j-1}^x = (f_j + f_j^\dagger)(f_{j-1} - f_{j-1}^\dagger)$$

$$\sigma_{j+1}^y \sigma_j^y = (f_{j+1}^\dagger - f_{j+1})(f_j^\dagger + f_j)$$

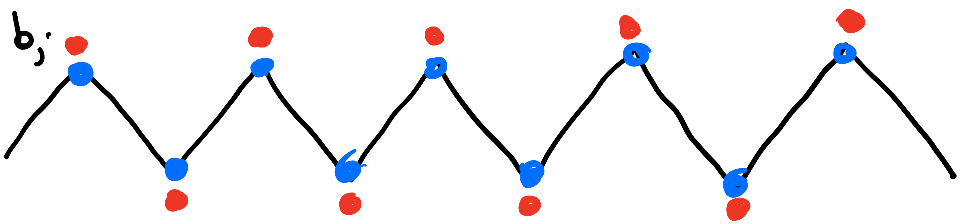
$$H = \frac{1}{2} \sum_{j \in \text{even}} \left[k^x (f_j + f_j^\dagger)(f_{j-1} - f_{j-1}^\dagger) + k^y (f_{j+1}^\dagger - f_{j+1})(f_j + f_j^\dagger) \right]$$

$f_j = (c_j - i b_j)/\sqrt{2}$ even $\qquad f_j + f_j^\dagger = \sqrt{2}\, c_j$ even j

$f_j = (b_j + i c_j)/\sqrt{2}$ odd $\qquad f_{j\pm1} - f_{j\pm1}^\dagger = \sqrt{2}\, i c_{j\pm1}$ odd j.

$$\boxed{H = \sum_{j \in \text{even}} \left(i k^x c_j \underset{\uparrow\,\text{odd}}{c_{j-1}} + i k^y \underline{c_{j+1} c_j} \right)}$$

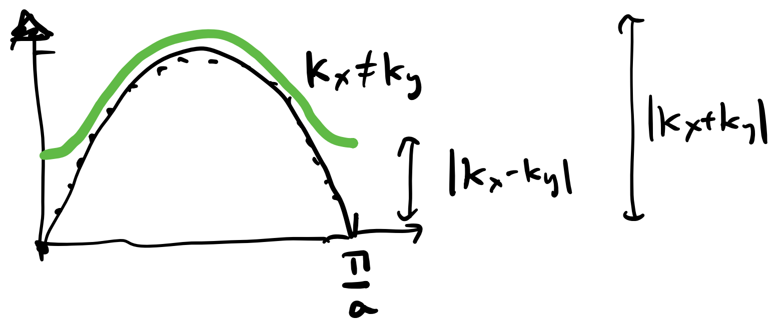
$\boxed{b \text{ is decoupled}}$



Take $k^x = k^y = k$

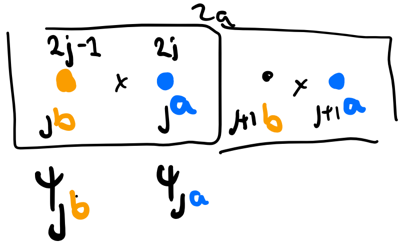
$$H = +ik \sum c_{j+1} c_j \equiv \sum_{k \in \frac{1}{2}BZ} 2k \sin ka \left(c_k^\dagger c_k - \tfrac{1}{2}\right)$$

$k = j\,\frac{2\pi}{L}$



For $k_x \neq k_y$,

$$\epsilon_k = \sqrt{(k_x - k_y)^2 + 4 k_x k_y \sin ka}$$



$$\psi_j = \begin{pmatrix} c_{2j} \\ c_{2j-1} \end{pmatrix} = \begin{pmatrix} \psi_{ja} \\ \psi_{jb} \end{pmatrix}$$

$$H = \sum_j \left(i k^x \psi_{ja} \psi_{jb} + i k^y \psi_{j+1b} \psi_{ja} \right)$$

$$= \sum_{\vec{k}} \left[-i k^x \psi_b^\dagger(k) \psi_a(k) + i k^y \psi_{kb}^\dagger \psi_{ka} e^{-2ika} \right]$$

$$= \frac{1}{2} \sum \psi^\dagger(k) \begin{pmatrix} 0 & +ik_x - ik_y e^{2ika} \\ -ik_x + ik_y e^{-2ika} & 0 \end{pmatrix} \psi_k = \frac{1}{2} \sum \psi_k^\dagger \begin{pmatrix} 0 & \gamma^*(k) \\ \gamma(k) & 0 \end{pmatrix} \psi_k$$

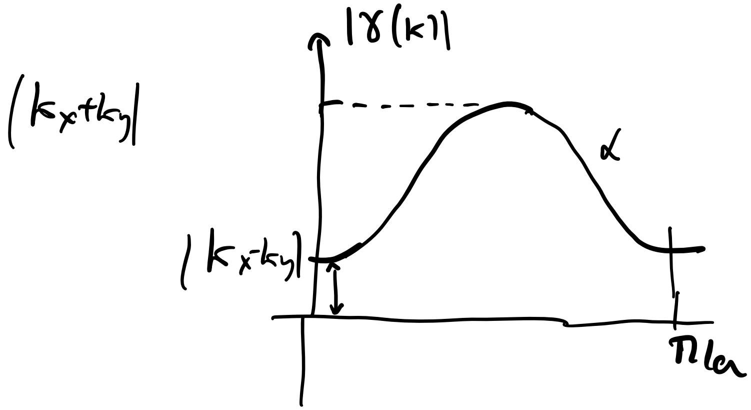
$$\gamma(k) = (-ik_x + ik_y \cos 2ka) + k_y \sin(2ka)$$

$$|\gamma(k)|^2 = (k_x - k_y c)^2 + (k_y s)^2$$

$$= k_x^2 + k_y^2 - 2 \cos 2ka \cos 2ka$$

$$|\gamma(k)| = \sqrt{k_x^2 + k_y^2 - 2(1 - 2s^2) k_x k_y}$$

$$= \sqrt{(k_x - k_y)^2 + 4 k_x k_y \sin^2 k_x}$$



$$\begin{pmatrix} 0 & \gamma^* \\ \gamma & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = |\gamma|\begin{pmatrix} u \\ v \end{pmatrix} \qquad \begin{pmatrix} 0 & \gamma^* \\ \gamma & 0 \end{pmatrix}\begin{pmatrix} -v^* \\ u^* \end{pmatrix} = -|\gamma|\begin{pmatrix} -v^* \\ u^* \end{pmatrix}$$

$$\psi_k = \overbrace{\begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix}}^{U_k}\begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} \qquad (\alpha_k^+ \;\; \beta_k^+) = \psi_k^+ \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix}$$

$$\frac{1}{2}\sum_k \psi_k^+ H_k \psi_k = \frac{1}{2}\sum_k (\alpha_k^+ \;\; \beta_k^+)\overbrace{(U_k^+ H_k U_k^+)}^{\begin{pmatrix} |\gamma| & \\ & -|\gamma| \end{pmatrix}}\begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix}$$

$$= \frac{1}{2}\sum_{k \in \frac{1}{2}BZ \sim BZ' = \frac{1}{2}BZ} |\gamma|(\alpha_k^+\alpha_k - \beta_k^+\beta_k)$$

$$= \frac{1}{2}\sum_{k \in \frac{1}{2}BZ} |\gamma_k|(\alpha_k^+\alpha_k - \alpha_{-k}\alpha_{-k}^+)$$

$$= \sum_{k \in \frac{1}{2}BZ} |\gamma_k|\left(\alpha_k^+\alpha_k - \frac{1}{2}\right)$$

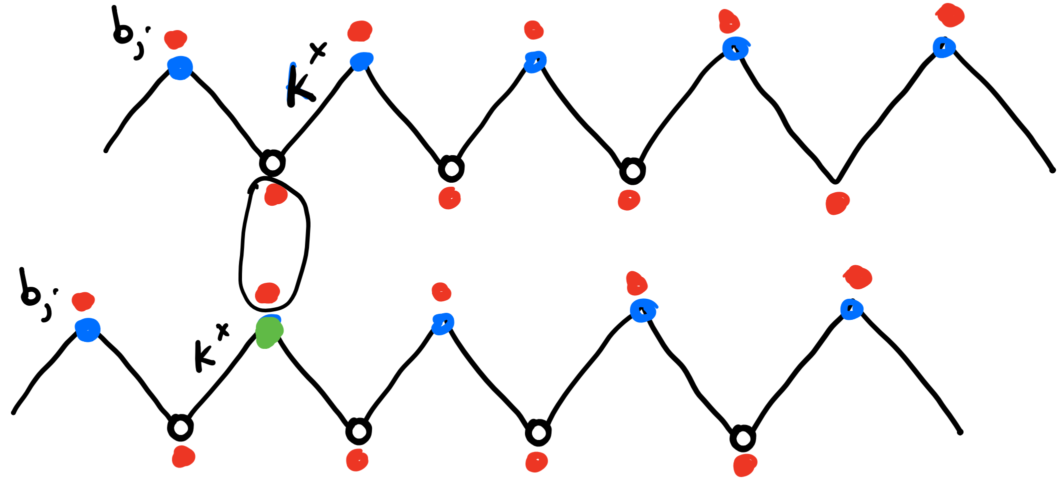
$$\gamma(-k) = \gamma^*(k)$$

$$\gamma(k) = |\gamma(k)|\, e^{-i\phi_k}$$

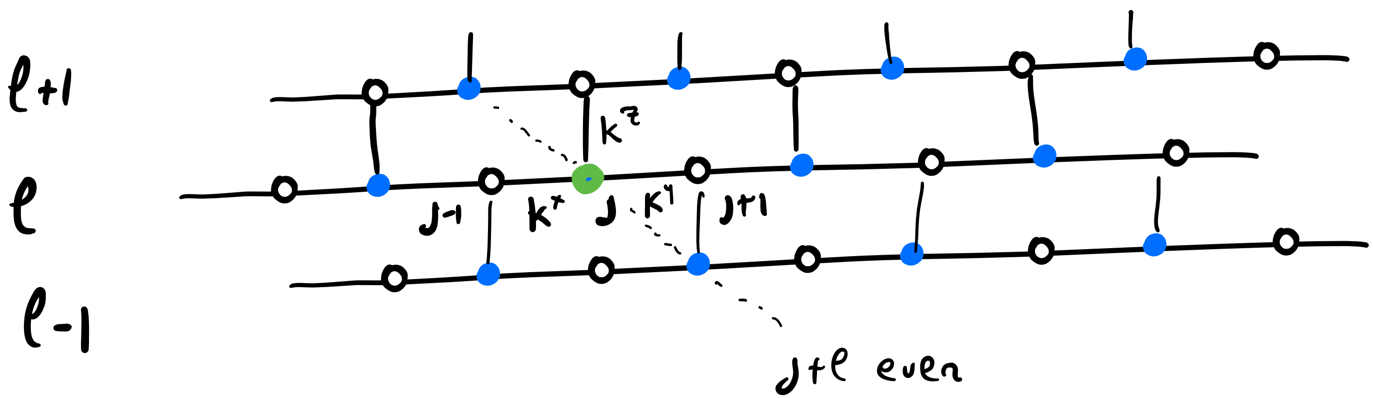
$$u(k) = e^{i\phi_k/2}/\sqrt{2}$$

$$v(k) = e^{-i\phi_k/2}/\sqrt{2}$$

2D Kitaev Model



$\sigma^z_{j\ell}\ \sigma^z_{j\ell+1}$



$$\mathcal{H} = \frac{1}{2} \sum_{j+\ell\ \text{even}} \left(K^x \sigma_{j-1\ell}\, \sigma^x_{j\ell} + K^y \sigma^y_{j\ell}\, \sigma^y_{j+1\ell} + K^z \sigma^z_{j\ell+1} \sigma^z_{j\ell} \right)$$

$$\sigma^+_{jl} = \frac{\sigma^x_{jl} + i\sigma^y_{jl}}{2} = f^+_{jl} P_{jl}$$

$$\sigma^-_{jl} = \frac{\sigma^x_{jl} - i\sigma^y_{jl}}{2} = f_{jl} P_{jl}$$

$$P_j = e^{i\pi \Phi_j}$$ Jordan Wigner

$$\Phi_j = \pi \sum_{j'<j} n_{j'l} + \pi \sum_{j'\, l'<l} n_{j'l'}$$

$$\{P_{jl}, f_{j'l'} P_{j'l'}\} = 0$$ if $j'<j$ $l'=l$, or if $l'<l$

$$\mathcal{H} = \sum_{l+j\in \text{even}} \left(i k^x c_{jl}\, \underset{\uparrow \text{ave}}{c_{j-1l}} + i k^y \underline{c_{j+1l}\, c_{jl}} \right) + \frac{k^z}{2} \sum (2n_{jl}-1)(2n^+_{jl+1}-1)$$

$f_j = (c_j - i b_j)/\sqrt{2}$ $j+l$ even

$f_j = (b_j + i c_j)/\sqrt{2}$ $j+l$ odd.

$$2n_{jl} - 1 = \left[2\left(\frac{c_{jl} + i b_{jl}}{\sqrt{2}}\right)\left(\frac{c_{jl} - i b_{jl}}{\sqrt{2}}\right)\right] - 1 = 2 i b_{jl} c_{jl}$$

$$2n_{jl+1} - 1 = 2\left[\left(\frac{b_{jl+1} - i c_{jl+1}}{\sqrt{2}}\right)\left(\frac{b_{jl+1} + i c_{jl+1}}{\sqrt{2}}\right)\right] - 1 = 2i\, b_{jl+1} c_{jl+1}$$

$$\frac{1}{2}(2n_{jl+1} - 1)(2n_{jl} - 1) = -2\, b_{jl+1} c_{jl+1} b_{jl} c_{jl}$$
$$= 2\, c_{jl+1} b_{jl+1} b_{jl} c_{jl} = i\, c_{jl+1} \overbrace{(-2i\, b_{jl+1} b_{jl})}^{u_{jl+1}} c_{jl}$$

$$H = \sum i\left(k^x c_{j-1l} + k^y c_{j+1l} + k^z c_{jl+1} u_{jl+1}\right) c_{jl}.$$

$$u_{jl+1} = -2i\, b_{jl+1} b_{jl}.$$ commutes with $\mathcal{H}$

$[\mathcal{H}, \hat{u}_{jl+1}] = 0 \Rightarrow$ static gauge field

We can see that this has the form

$$h = -i \sum K^{\alpha_{ij}} (c_i \, u_{ij} \, c_j)$$

where the $u_{ij} = 1$ for even $\to$ odd. in a horizontal rung

$$Ga\left(\frac{3}{4}+\frac{3}{4}\right)=2\pi$$

$$\vec{G}=\left(\frac{4\pi}{3a}\right)\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$$

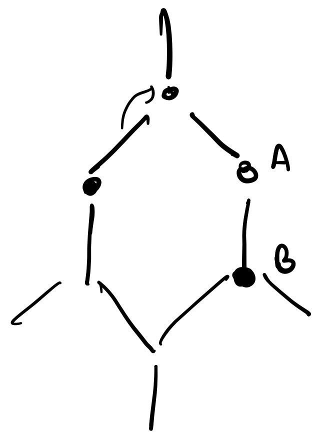
$$G=\left(\frac{4\pi}{3a}\right)(0,1)$$

$$\sqrt{3}\,k_\theta=\frac{4\pi}{3a}$$

$$k_\theta=\frac{4\pi}{3\sqrt{3}a}$$

$$k_{1,2}=\frac{4\pi}{3\sqrt{3}a}(\pm 1,0)$$

$$\left.\begin{array}{l} c_i \rightarrow z_i c_i \\ u_{ij} \rightarrow z_i z_j u_{ij} \end{array}\right\} \quad z_2 \text{ g. invariance}$$



Ancillary Qubit Method.

$$\Phi^S_j,\ \Phi^T_j \qquad b^a_i=\Phi^T_j\,\sigma^a_j$$

$$\{b^a_l, b^b_j\}=\delta_{ab}\delta_{ij}$$