(b) At low temperatures, the density of states is given by $A(\omega)/N(0) = (|\omega|/\Delta)$, so that the thermally averaged density of states

$$\overline{\left(\frac{A(\omega)}{N(0)}\right)} = \frac{k_B T}{\Delta} 2 \int_0^\infty \frac{x}{(e^x + 1)(e^{-x} + 1)} = \frac{k_B T}{\Delta} \ln 4 \tag{15.92}$$

grows linearly with temperature. Thus in a d-wave superconductor the inverse penetration depth $\frac{1}{\lambda_L^2} \propto Q(T)$ will exhibit a linear dependence on temperature at low temperatures, rather than the exponential dependence expected from a fully gapped s-wave superconductor:

$$1 - \frac{\lambda_L^2(0)}{\lambda_L^2(T)} \sim \frac{k_B T}{\Delta} \qquad (k_B T << \Delta).$$

(Note that in a dirty d-wave superconductor the density of states is constant at low temperatures, which leads to a quadratic temperature dependence of the inverse penetration depth at the lowest temperatures.)

15.5 Superfluid ^{3}He

15.5.1 Early history: theorists predict a new superfluid

As our second example of anisotropic pairing, we discuss the remarkable case of superfluid ^{3}He. As the 1950s came to an end and the wider significance of the BCS pairing instability was appreciated, the condensed-matter community began to realize that ^{3}He might form a BCS superfluid condensate, avoiding the mutual repulsion of the atoms by pairing in a higher angular momentum channel. Four independent groups (Lev Pitaevksii [6] at the Kapitza institute in Moscow; David Thouless at the Lawrence Radiation Laboratory, University of California, Berkeley [7]; Victor Emery and Andrew Sessler at the University of California, Berkeley [8]; and the Gang of Four, Keith Brueckner and Toshio Soda at the University of California, La Jolla, and Philip W. Anderson with Pierre Morel at Bell Laboratories, New Jersey [4] [9, 10]) came up with the idea of anisotropic pairing. Although these early papers examined both p- and d-wave pairs, each of them used *bare* nuclear interaction parameters as input to the BCS theory, and on the basis of these calculations came to the conclusion that the leading attractive channel was the $l = 2$, d-wave channel, predicting a d-wave superfluid condensate would develop in ^{3}He around $T_c = 50{-}150$ mK. The theory community would later be vindicated in their prediction of anistropic superfluidity in ^{3}He, but at a much lower temperature and with a p-wave rather than a d-wave symmetry.

During the 1960s the theory of anisotropic superfluidity developed rapidly, providing the framework for p-wave pairing that would ultimately be used to understand ^{3}He. In 1961 Morel and Anderson [10] introduced the ground state of what would later be identified

[4] Pierre Morel was officially a scientific attache at the French Embassy in New York City.

as the "A" phase, while in 1963 Roger Balian at the Centre d'Etude Nucléaires, Saclay, and Richard Werthamer at Bell Laboratories [11] discovered, an isotropic triplet paired ground state that would later be identified as the "B" phase. Gradually, towards the end of the 1960s, it became clear that the use of a bare interaction parameter as an input to BCS theory needed to be corrected for many-body effects, particularly with ladder diagram corrections to the pair scattering amplitude [12]. In a pioneering work, Walter Kohn at the University of California, San Diego, and Joaquin Luttinger at Columbia University, New York, [13] showed that, when many-body corrections to the Cooper channel interaction are considered, the sharpness of the Fermi surface guarantees that Fermi liquids are inevitably unstable to anisotropic pairing in some higher angular momentum channel. Using an input delta-function potential, Kohn and Luttinger derived an approximate asymptotic formula for T_c as a function of angular momentum l in ^{3}He, given by

$$T_c(l) \sim \epsilon_F \exp\left\{-\frac{\pi^2}{(k_F a)^2} l^4\right\}, \tag{15.93}$$

where l is the angular momentum of the pair, ϵ_F and k_F are the Fermi energy and momentum, respectively, and a is the diameter of the ^{3}He atom. Curiously, Kohn and Luttinger chose to illustrate this equation for $l = 2$, d-wave pairing, which for $k_F a \sim 2$ gives $T_c \sim 10^{-17}\epsilon_F$. Had they made the bold but uncontrolled insertion of $l = 1$, they would have obtained $T_c \sim 0.05\epsilon_F \sim 50$ mK, surely an indication that p-wave pairing is a stronger candidate than d-wave! Then in 1967 D. Fay and A. Layzer, working at the Stevens Institute of Technology, New Jersey, made the critical observation [14] that in dilute neutral fluids many-body effects, which tend to ferromagnetically enhance interactions, will also generally lead to p-wave pairing.

It was not until 1972 that Douglas Osheroff, Robert Richardson, and David Lee at Cornell University finally discovered superfluidity in ^{3}He, developing at 2.65 mK [15] (see Figure 15.7). From the anomalies in the NMR response, this team was able to identify two phases: a high-temperature A phase and a low-temperature B phase in which most of the magnetic response disappeared. By carefully analyzing the detailed NMR measurements

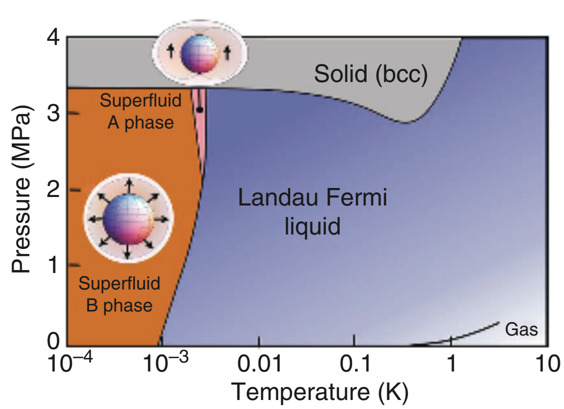


Fig. 15.7 Phase diagram of ^{3}He, showing the superfluid A and B phases, [22] with icons representing the gap anisotropy. Note that 0.1MPa = 1 atm. Adapted with permission from D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3*, Dover, 2013. Copyright 2013 by Dieter Vollhardt and Peter Wölfle.

carried out on these phases, Anthony Leggett, working at Sussex University [16], was able to show [17–19] that the pair symmetry of the A phase is triplet and probably corresponds to the Anderson–Morel state (now called the Anderson–Brinkman–Morel state). The pair symmetry of the B phase was later identified with the isotropic and fully gapped Balian–Werthamer state [20].

Curiously, although the early ^{3}He theorists predicted the wrong pair symmetry for ^{3}He, their efforts were not in vain, for d-wave pairing was realized seven years later in superconductors, with the discovery of the first anisotropic superconductor, $CeCuSi_2$, by Frank Steglich at Cologne University [21]. We now know many examples of d-wave superconductors, including the high-temperature cuprate superconductors.

15.5.2 Formulation of a model

The beauty of ^{3}He is that its isotropy provides us with a model system. The Fermi surface is perfectly spherical and in this case the pairing interaction between the quasiparticles depends only on the relative angle between the initial and final pair momenta $\mathbf{k}$ and $\mathbf{k}'$, i.e. $V_{\mathbf{k},\mathbf{k}'} = V(\cos\theta_{\mathbf{k},\mathbf{k}'})$. This implies that the pairing interaction can be decomposed as a multipole expansion involving Legendre polynomials:

$$V_{\mathbf{k},\mathbf{k}'} = \sum_l (2l+1) V_l P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'). \tag{15.94}$$

This is reminiscent of the multipole expansion of Fermi liquid interactions (6.38). Using the orthogonality relation $\int \frac{dc}{2} P_l(c) P_{l'}(c) = \delta_{l,l'}/(2l+1)$, the parameters V_l are given by

$$V_l = \int_{-1}^{1} \frac{d\cos\theta}{2} P_l(\cos\theta) V(\cos\theta) \qquad l \in \begin{cases} \text{even} & \text{(singlet)} \\ \text{odd} & \text{(triplet).} \end{cases} \tag{15.95}$$

These are the higher angular momentum analogues of the BCS s-wave interaction parameter. Now the parity of the Legendre polynomials alternates with l, $P_l = (-1)^l$ $(P_l(-x) = (-1)^l P_l(x))$, so the even l define singlet pair potentials while the odd l define triplet $(S=1)$ pair potentials.

Using the relationship $(2l+1)P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') = 4\pi \sum_{m=-l}^{l} Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{k}}')$, we can factorize the anisotropic BCS interaction in the form

$$V_{\mathbf{k},\mathbf{k}'} = \sum_{l,m} V_l \, y^*_{lm}(\hat{\mathbf{k}}) y_{lm}(\hat{\mathbf{k}}'), \tag{15.96}$$

where we have used the notation $y_{lm} = \sqrt{4\pi} Y_{lm}$ to denote spherical harmonics normalized to give unit norm when averaged over the sphere $\int \frac{d\Omega}{4\pi} y^*_{lm} y_{lm} = \delta_{l,l'}\delta_{m,m'}$. This is the same kind of factorized interaction encountered in the previous section, and we can treat it in the same way. For ^{3}He, the hard-core repulsion between the atoms rules out an s-wave instability[5] and it is the p-wave $(l=1)$ triplet $(S=1)$ channel that takes over. Approximating $V_1 = -g/V$ and ignoring all other channels, then

5 Curiously, in optical atom traps in which the atomic interactions among highly dilute fermions can be tuned through a Feshbach resonance, it is possible to produce an attractive s-wave interaction, so a conventional BCS instability does occur.

$$V_{\mathbf{k},\mathbf{k}'} = -\frac{g}{V} 3\cos(\mathbf{k}\cdot\mathbf{k}') = -\frac{3g}{V}(\hat{k}_a \hat{k}'_a), \tag{15.97}$$

where $\hat{k}_a = k_a/k_F$ and the sum over the repeated index $a = 1, 2, 3$ is implied. The BCS Hamiltonian for a triplet superfluid is then [11]

$$\begin{aligned} H_{BCS} &= \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \frac{3g_0}{V} \sum_{\mathbf{k},\mathbf{k}'\in\frac{1}{2}\mathrm{BZ}} (\vec{\Psi}^{\dagger}_{\mathbf{k}} \hat{k}_\ell) \cdot (\hat{k}'_\ell \vec{\Psi}_{\mathbf{k}'}) \\ \vec{\Psi}_{\mathbf{k}} &= c_{-\mathbf{k}\alpha} \left(-i\sigma_2 \vec{\sigma}\right)_{\alpha\beta} c_{\mathbf{k}\beta} \\ \vec{\Psi}^{\dagger}_{\mathbf{k}} &= c^{\dagger}_{\mathbf{k}\alpha} \left(\vec{\sigma}\, i\sigma_2\right)_{\alpha\beta} c^{\dagger}_{-\mathbf{k}\beta}. \end{aligned} \tag{15.98}$$

Notice that there are now three *triplet* channels ($\vec{\Psi}_{\mathbf{k}} \equiv \Psi^a_{\mathbf{k}}$, $a = 1, 2, 3$) and three *orbital* channels ($\hat{k}_l$, $l = x, y, z$) in which the pairing takes place. The summation over momentum in the interaction takes place over one-half the Brillouin zone.

15.5.3 Gap equation

If we carry out a Hubbard–Stratonovich transformation, we get

$$H_{MFT} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\in\frac{1}{2}BZ} \left[\vec{\Psi}^{\dagger}_{\mathbf{k}} \cdot (\vec{\Delta}_l)\, \hat{k}_l + \text{H.c.}\right] + \frac{V}{3g_0}(\vec{\Delta}^*_l \cdot \vec{\Delta}_l). \tag{15.99}$$

The three vectors $\vec{\Delta}_l$ ($l = x, y, z$) define a three-dimensional matrix $\Delta^a_l \equiv (\vec{\Delta}_l)^a$ which links the spin and orbital degrees of freedom. If we denote $\vec{\Delta}_{\mathbf{k}} = \sum_{l=x,y,z} \vec{\Delta}_l \hat{k}_l$, then, since $\int \frac{d\Omega_{\hat{k}}}{4\pi} \hat{k}_l \hat{k}_m = \frac{1}{3}\delta_{lm}$, it follows that

$$\Delta^a_l = (\vec{\Delta}_l)^a = 3 \int \frac{d\Omega_{\hat{k}}}{4\pi} (\vec{\Delta}_{\mathbf{k}})^a \hat{k}_l. \tag{15.100}$$

Thus we can write

$$\frac{V}{3g_0}(\vec{\Delta}^*_l \cdot \vec{\Delta}_l) = \frac{3V}{g_0} \int \frac{d\Omega_{\hat{k}}}{4\pi} \frac{d\Omega_{\hat{k}'}}{4\pi} \vec{\Delta}_{\mathbf{k}} \cdot \vec{\Delta}_{\mathbf{k}'} (\hat{k}\cdot\hat{k}') \equiv -\int_{\hat{k},\hat{k}'} \vec{\Delta}^*_{\mathbf{k}} V^{-1}_{\mathbf{k},\mathbf{k}'} \vec{\Delta}_{\mathbf{k}'}, \tag{15.101}$$

where we have identified $V^{-1}_{\mathbf{k},\mathbf{k}'} \equiv -\frac{V}{g_0}(3\hat{k}\cdot\hat{k}')$ and denoted $\int_{\hat{k}} = \int \frac{d\Omega_{\hat{k}}}{4\pi}$. The mean-field Hamiltonian is then

$$H_{MFT} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\in\frac{1}{2}BZ} \left(\vec{\Psi}^{\dagger}_{\mathbf{k}} \cdot \vec{\Delta}_{\mathbf{k}} + \text{H.c.}\right) + \frac{3V}{g_0} \int \frac{d\Omega_{\hat{k}}}{4\pi} \frac{d\Omega_{\hat{k}'}}{4\pi} \vec{\Delta}_{\mathbf{k}} \cdot \vec{\Delta}_{\mathbf{k}'} (\hat{k}\cdot\hat{k}'). \tag{15.102}$$

Now, to diagonalize this mean-field theory we need to cast it into spinors. Triplet pairing mixes up and down electrons, which obliges us to use a four-component spinor called a *Balian–Werthamer spinor* [11] after its inventors:

$$\psi_{\mathbf{k}} \equiv \begin{pmatrix} c_{\mathbf{k}} \\ i\sigma_2 c^{\dagger}_{-\mathbf{k}} \end{pmatrix} \equiv \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{\mathbf{k}\downarrow} \\ c^{\dagger}_{-\mathbf{k}\downarrow} \\ -c^{\dagger}_{-\mathbf{k}\uparrow} \end{pmatrix}. \qquad \text{Balian–Werthamer spinor} \tag{15.103}$$

The upper two entries are the destruction operators for particles of momentum **k**, while the lower two,

$$\begin{pmatrix} a_{\mathbf{k}\uparrow} \\ a_{\mathbf{k}\downarrow} \end{pmatrix} \equiv \begin{pmatrix} c^{\dagger}_{-\mathbf{k}\downarrow} \\ -c^{\dagger}_{-\mathbf{k}\uparrow} \end{pmatrix}, \tag{15.104}$$

are the destruction operators for *holes* of momentum **k**. Hole-destruction operators are the time reversal (denoted by the operator θ) of the corresponding particle-creation operators, and the minus sign in the lower entry appears on time reversal of a down-spin state, $a^{\dagger}_{\mathbf{k}\downarrow} = \theta c_{\mathbf{k}\downarrow}\theta^{-1} = -c_{-\mathbf{k}\uparrow}$.[6] Notice how the $i\sigma_2$ that appears in the triplet pair operators is now neatly absorbed into the spinor. Moreoever, the BW spinor obeys canonical anticommutation rules:

$$\{\psi_{\mathbf{k}\alpha}, \psi^{\dagger}_{\mathbf{k}'\beta}\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha\beta}.$$

Of course, we have doubled the number of components in the spinor, so we must now restrict the momentum to one-half of momentum space, $\mathbf{k} \in \frac{1}{2}BZ$. The payoff is that we now have a rotationally invariant representation in which the spin operator is defined in terms of block-diagonal Pauli matrices:

$$\vec{\sigma}_4 \equiv \underline{1} \otimes \vec{\sigma} = \left(\begin{array}{c|c} \vec{\sigma} & \\ \hline & \vec{\sigma} \end{array}\right), \tag{15.105}$$

while the Nambu matrices are now block matrices:

$$\vec{\tau}_4 \equiv \vec{\tau} \otimes \underline{1} = \left\{ \left(\begin{array}{c|c} & \underline{1} \\ \hline \underline{1} & \end{array}\right), \left(\begin{array}{c|c} & -i\underline{1} \\ \hline i\underline{1} & \end{array}\right), \left(\begin{array}{c|c} \underline{1} & \\ \hline & -\underline{1} \end{array}\right) \right\}. \tag{15.106}$$

In this notation, the BCS Hamiltonian can be succinctly rewritten as

$$H_{MFT} = \sum_{\mathbf{k}\in\frac{1}{2}BZ} \psi^{\dagger}_{\mathbf{k}} h_{\mathbf{k}} \psi_{\mathbf{k}} + \frac{3V}{g_0} \int \frac{d\Omega_{\hat{k}}}{4\pi} \frac{d\Omega_{\hat{k}'}}{4\pi} \vec{\Delta}_{\mathbf{k}} \cdot \vec{\Delta}_{\mathbf{k}'} (\hat{k} \cdot \hat{k}')$$

$$h_{\mathbf{k}} = \left(\begin{array}{c|c} \epsilon_{\mathbf{k}} & \vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma} \\ \hline \vec{\Delta}^{*}_{\mathbf{k}} \cdot \vec{\sigma} & -\epsilon_{\mathbf{k}} \end{array}\right) \equiv \epsilon_{\mathbf{k}}\tau_3 + (\vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma})\tau_{+} + (\vec{\Delta}^{*}_{\mathbf{k}} \cdot \vec{\sigma})\tau_{-}, \tag{15.107}$$

where $\tau_{\pm} = \frac{1}{2}(\tau_1 \pm i\tau_2)$. It is common to denote the direction of the gap function in spin space by the complex *d-vector* $\vec{d}_{\mathbf{k}}$,

$$\vec{\Delta}_{\mathbf{k}} = \Delta\vec{d}_{\mathbf{k}}, \tag{15.108}$$

which is normalized so that its angular average over the Fermi surface is unity:

$$\int \frac{d\Omega_{\mathbf{k}}}{4\pi} |\vec{d}_{\mathbf{k}}|^2 = 1. \tag{15.109}$$

[6] You can also verify that the diagonal and off-diagonal matrix elements of the spin operator are the same for particles and for holes, so that $h^{\dagger}_{\mathbf{k}}\vec{\sigma}h_{\mathbf{k}} = c_{-\mathbf{k}}(i\sigma_2)\vec{\sigma}(-i\sigma_2)c_{-\mathbf{k}} = c^{\dagger}_{-\mathbf{k}}\vec{\sigma}c_{-\mathbf{k}}$, where the last step follows because $\vec{\sigma}^T = -\sigma_2\vec{\sigma}\sigma_2$.

The d-vector is an emergent property of the Fermi surface, and the textures it gives rise to in momentum space define the state of the condensate.

If we take the determinant of $\omega - h_{\mathbf{k}}$ by multiplying out its two-dimensional block diagonals, we find

$$\begin{aligned}\det(\omega - h_{\mathbf{k}}) &= \det\left[(\omega^2 - \epsilon_{\mathbf{k}}^2)\underline{1} - (\vec{\Delta}_{\mathbf{k}}^* \cdot \vec{\sigma})(\vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma})\right] \\ &= \det\left[(\omega^2 - \epsilon_{\mathbf{k}}^2)\underline{1} - \Delta^2(|\vec{d}_{\mathbf{k}}|^2 + i\vec{d}_{\mathbf{k}}^* \times \vec{d}_{\mathbf{k}} \cdot \vec{\sigma})\right] \\ &= \det\left[(\omega^2 - \epsilon_{\mathbf{k}}^2)\underline{1} - \Delta^2(|\vec{d}_{\mathbf{k}}|^2 + 2\vec{d}_{1\mathbf{k}} \times \vec{d}_{2\mathbf{k}} \cdot \vec{\sigma})\right], \end{aligned} \qquad (15.110)$$

where we have used the identity $\sigma^a\sigma^b = \delta^{ab} + i\epsilon^{abc}\sigma^c$ on the second line and decomposed $\vec{d}_{\mathbf{k}} = \vec{d}_{1\mathbf{k}} - i\vec{d}_{2\mathbf{k}}$ into its real and imaginary parts on the last line. The quasiparticle energies determined by pairing matrix $h_{\mathbf{k}}$ are then

$$E_{\mathbf{k}\pm} = \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta^2(|\vec{d}_{\mathbf{k}}|^2 \pm 2|\vec{d}_{1\mathbf{k}} \times \vec{d}_{2\mathbf{k}}|)}.$$

There are in fact two superfluid phases of ^{3}He, and, in both, $\vec{d}_{1\mathbf{k}}$ and $\vec{d}_{2\mathbf{k}}$ are parallel, and the gap functions take the form

$$\vec{\Delta}_{\mathbf{k}} = \Delta \times \begin{cases} \hat{k}_x\hat{\mathbf{x}} + \hat{k}_y\hat{\mathbf{y}} + \hat{k}_z\hat{\mathbf{z}} & \text{BW or B phase} \\ \sqrt{\frac{3}{2}}(\hat{k}_x + i k_y)\hat{\mathbf{z}}\,. & \text{ABM or A phase} \end{cases} \qquad (15.111)$$

The BW or B phase is named after Balian and Werhammer. In this phase the d-vector points radially outwards from the Fermi sea, forming a topological "hedgehog" configuration (see Figure 15.8(a)) with a uniform gap and quasiparticle energy given simply by

$$E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta^2}. \qquad \text{B phase}$$

The B phase, with a full gap, dominates the phase diagram. The ABM or A-phase, named after its discoverers, Anderson, Brinkman, and Morel, develops in a small sliver of the phase diagram under pressures of about 2 MPa (see Figure 15.7(b)). This phase involves pairing in a single triplet orbital channel with a uniform ("z") direction of the d-vector; now the magnitude of the gap is momentum-dependent:

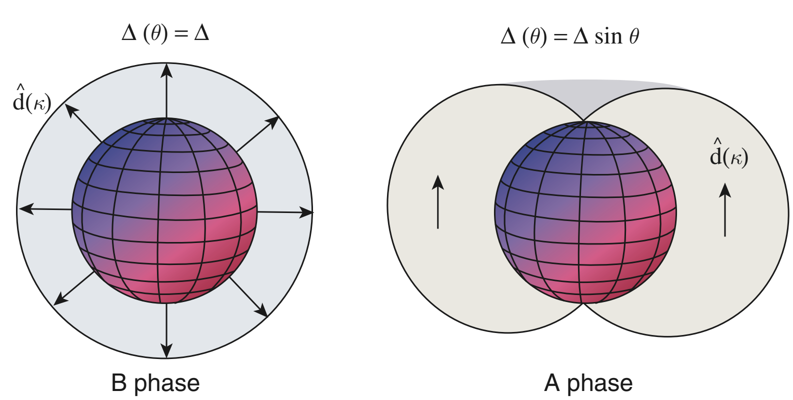


Fig. 15.8 Showing the gap structure and d-vector orientation for the B and A phases of superfluid ^{3}He.

$$\vec{\Delta}_{\mathbf{k}} = \sqrt{\frac{3}{2}}\Delta \sin\theta e^{i\phi}\hat{\mathbf{z}}. \qquad \text{A phase}$$

This function vanishes at the poles, giving rise to a quasiparticle excitation spectrum

$$E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + \frac{3}{2}\Delta^2 \sin^2\theta}. \qquad \text{A phase}$$

The derivation of the mean-field equations for these two solutions is simplified by the observation that, for both of them, the potential energy term is

$$\frac{3V}{g_0}\int \frac{d\Omega_{\hat{k}}}{4\pi}\frac{d\Omega_{\hat{k}'}}{4\pi}\vec{\Delta}_{\mathbf{k}}\cdot\vec{\Delta}_{\mathbf{k}'}(\hat{k}\cdot\hat{k}') = \frac{V}{g_0}\Delta^2.$$

The free energy of the mean-field theory then takes precisely the same form as in BCS theory:

$$F_{MFT} = -2T\sum_{\mathbf{k}}\ln\left(2\cosh\frac{\beta E_{\mathbf{k}}}{2}\right) + \frac{V}{g_0}\Delta^2.$$

If we differentiate with respect to Δ^2 we obtain the gap equation:

$$\frac{1}{g_0 N(0)} = \int_{-1}^{1}\frac{d\cos\theta}{2}\int_{-\omega_D}^{\omega_D} d\epsilon \frac{\Delta(\theta)^2/\Delta^2}{\sqrt{\epsilon^2+\Delta(\theta)^2}}\tanh\left[\frac{\sqrt{\epsilon^2+\Delta(\theta)^2}}{2}\right].$$

According to this analysis, the A and B phases have identical mean-field transition temperatures. However, at lower temperatures the B phase wins out because its fully gapped Fermi surface gives rise to a lower free energy.

Example 15.6 Consider a single triplet Cooper pair described by the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(\hat{d}\cdot\vec{\Psi}^{\dagger}_{\mathbf{k}}\right)|0\rangle = \frac{1}{\sqrt{2}}\hat{d}\cdot\left(c^{\dagger}_{\mathbf{k}}\vec{\sigma}i\sigma_2 c^{\dagger}_{-\mathbf{k}}\right)|0\rangle,$$

where $\hat{d}$ is a real unit vector.

(a) Show that

$$\vec{S}|\Psi\rangle = \frac{i}{\sqrt{2}}(\hat{d}\times\vec{\Psi}^{\dagger}_{\mathbf{k}})|0\rangle$$

and use this to prove that the spin of the state is $S=1$, i.e.

$$S^2|\Psi\rangle = 2|\Psi\rangle, \tag{15.112}$$

while the component of the spin in the direction of the d-vector vanishes:

$$(\hat{\mathbf{d}}\cdot\vec{S})|\Psi\rangle = 0 \tag{15.113}$$

and the expectation value of the magnetic moment is zero, i.e. $\langle\Psi|\vec{S}|\Psi\rangle = 0$.

(b) Show that the expectation value is

$$\langle\Psi|S^a S^b|\Psi\rangle = \delta^{ab} - \hat{d}^a\hat{d}^b, \tag{15.114}$$

so that $\langle S^2\rangle = S(S+1) = 2$, corresponding to a spin-quadrupole with a fluctuating moment in the plane perpendicular to the d-vector.

Solution

(a) The effective spin operator for this state only involves momenta $\pm\mathbf{k}$, so we may use $\vec{S} = \frac{1}{2}[c^\dagger_{\mathbf{k}}\vec{\sigma}c_{\mathbf{k}} + c^\dagger_{-\mathbf{k}}\vec{\sigma}c_{-\mathbf{k}}]$. To determine the action of the spin operator on the triplet pair, we need to commute it past the triplet pair operator onto the vacuum. The commutator is

$$\begin{aligned}\left[S^a, (\vec{\Psi}^\dagger_{\mathbf{k}})^b\right] &= \left[\left(c^\dagger_{\mathbf{k}}\sigma^a c_{\mathbf{k}} + c^\dagger_{-\mathbf{k}}\sigma^a c_{-\mathbf{k}}\right), \left(c^\dagger_{\mathbf{k}}\sigma^b i\sigma_2 c^\dagger_{-\mathbf{k}}\right)\right] \\ &= c^\dagger_{\mathbf{k}}\left(\sigma^a\sigma^b i\sigma_2 + \sigma^b i\sigma_2(\sigma^a)^T\right)c^\dagger_{-\mathbf{k}},\end{aligned}$$

where the first and second terms derive from the positive and negative momentum components of the spin operator. Using $\sigma_2(\sigma^a)^T = -\sigma^a\sigma_2$ we obtain

$$\left[S^a, (\vec{\Psi}^\dagger_{\mathbf{k}})^b\right] = \frac{1}{2}c^\dagger_{\mathbf{k}}\left[\sigma^a, \sigma^b\right]i\sigma_2 c^\dagger_{-\mathbf{k}} = i\epsilon_{abc}c^\dagger_{\mathbf{k}}\sigma^c i\sigma_2 c^\dagger_{-\mathbf{k}} \tag{15.115}$$

or

$$\left[S^a, \hat{d}\cdot\left(c^\dagger_{\mathbf{k}}\vec{\sigma}i\sigma_2 c^\dagger_{-\mathbf{k}}\right)\right] = i\epsilon_{abc}d^b\left(c^\dagger_{\mathbf{k}}\sigma^c i\sigma_2 c^\dagger_{-\mathbf{k}}\right) = i(\hat{d}\times\vec{\Psi}^\dagger_{\mathbf{k}})^a, \tag{15.116}$$

and hence

$$\vec{S}|\Psi\rangle = \frac{1}{\sqrt{2}}[\vec{S}, (\hat{d}\cdot\vec{\Psi}^\dagger_{\mathbf{k}})]|0\rangle = \frac{i}{\sqrt{2}}\left(\hat{d}\times\vec{\Psi}^\dagger_{\mathbf{k}}\right)|0\rangle. \tag{15.117}$$

Using (15.115), we have

$$S^a(\vec{\Psi}^\dagger_{\mathbf{k}})^b|0\rangle = i\epsilon_{abc}(\vec{\Psi}^\dagger_{\mathbf{k}})^c|0\rangle, \tag{15.118}$$

so that

$$\begin{aligned}S^2(\vec{\Psi}^\dagger_{\mathbf{k}})^b|0\rangle &= S^a S^a(\vec{\Psi}^\dagger_{\mathbf{k}})^b|0\rangle \\ &= i\epsilon_{abc}S_a(\vec{\Psi}^\dagger_{\mathbf{k}})^c|0\rangle \\ &= \overbrace{i\epsilon_{abc}i\epsilon_{acd}}^{2\delta_{bd}}(\vec{\Psi}^\dagger_{\mathbf{k}})^d|0\rangle = 2(\vec{\Psi}^\dagger_{\mathbf{k}})^b|0\rangle,\end{aligned} \tag{15.119}$$

so, writing this in vector notation,

$$S^2\vec{\Psi}^\dagger_{\mathbf{k}}|0\rangle = 2\vec{\Psi}^\dagger_{\mathbf{k}}|0\rangle. \tag{15.120}$$

Hence $S^2|\Psi\rangle = S^2(\hat{d}\cdot\vec{\Psi}^\dagger_{\mathbf{k}})|0\rangle = 2|\Psi\rangle$, corresponding to a spin of 1.

If we evaluate the expectation value of the moment, we get

$$\langle\Psi|\vec{S}|\Psi\rangle = \frac{1}{2}\langle 0|(\hat{d}^*\cdot\vec{\Psi}_{\mathbf{k}})(\hat{d}\times\vec{\Psi}^\dagger_{\mathbf{k}})|0\rangle = i\hat{d}\times\hat{d}^*. \tag{15.121}$$

In our case $\hat{d}$ is real, so that $\langle\vec{S}\rangle = 0$. Note, however, that if $\hat{d} = \hat{d}_1 + i\hat{d}_2$ is complex, then $\langle\vec{S}\rangle = 2\hat{d}_1 \times \hat{d}_2$, so that if $\hat{d}_1$ and $\hat{d}_2$ are not parallel, the Cooper pair state carries a net magnetic moment.

(b) Taking the result (15.117), we have

$$\begin{aligned}
\langle\Psi|S^a S^b|\Psi\rangle &= \frac{1}{2}\epsilon_{apq}\epsilon_{brs}d^p d^r \overbrace{\langle 0|\left(\vec{\Psi}_{\mathbf{k}}\right)^q \left(\vec{\Psi}^\dagger_{\mathbf{k}}\right)^s |0\rangle}^{2\delta^{qs}} = \epsilon_{apq}\epsilon_{brq} \\
&= \left(\delta^{ab}\delta^{pr} - \delta^{ar}\delta^{pb}\right) d^p d^r \\
&= \delta^{ab} - d^a d^b, \qquad (15.122)
\end{aligned}$$

so the moment fluctuations of the pair lie in the plane perpendicular to the d-vector.

Example 15.7 Derive the BCS pair wavefunction for the B phase of ^{3}He.

Solution

By analogy with the case of singlet pairing, we expect the ground state to be a coherent state of a triplet pair,

$$|\Psi\rangle = \exp\left[\Lambda^\dagger_T\right]|0\rangle, \qquad (15.123)$$

where

$$\Lambda^\dagger_T = \frac{1}{2}\sum_{\mathbf{k}} \phi_{\mathbf{k}}(\hat{k}\cdot\vec{\Psi}^\dagger_{\mathbf{k}}) \qquad (15.124)$$

creates a triplet pair and $\phi_{\mathbf{k}} = \phi_{-\mathbf{k}}$ is an even function of momentum. The factor of $\frac{1}{2}$ is included as a normalization that takes account of the fact that $\Psi^\dagger_{\mathbf{k}}$ is only independent in one-half of momentum space.

Now the ground state is annihilated by the quasiparticle destruction operators. For the triplet B phase we write the quasiparticle-creation operators as

$$a^\dagger_{\mathbf{k}} = \psi^\dagger_{\mathbf{k}} \cdot \begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \end{pmatrix} = c^\dagger_{\mathbf{k}\sigma} u_{\mathbf{k}\sigma} + \tilde{c}_{\mathbf{k}\sigma} v_{\mathbf{k}\sigma}, \qquad (15.125)$$

where $\tilde{c}_{\mathbf{k}\alpha} = c_{-\mathbf{k}\beta}[-i\sigma_2]_{\beta\alpha}$ and the $u_{\mathbf{k}\sigma}$ and $v_{\mathbf{k}\sigma}$ are two-component spinors. For the B phase, we can take the mean-field Hamilonian to be

$$H_{MFT} = \sum_{\mathbf{k}\in\frac{1}{2}BZ} \psi^\dagger_{\mathbf{k}} h_{\mathbf{k}} \psi_{\mathbf{k}}, \qquad h_{\mathbf{k}} = \left(\begin{array}{c|c} \epsilon_{\mathbf{k}} & \Delta(\hat{k}\cdot\vec{\sigma}) \\ \hline \Delta(\hat{k}\cdot\vec{\sigma}) & -\epsilon_{\mathbf{k}} \end{array}\right). \qquad (15.126)$$

Now since $[H, a_{\mathbf{k}}] = E_{\mathbf{k}} a_{\mathbf{k}}$, it follows that

$$\left(\begin{array}{c|c} \epsilon_{\mathbf{k}} & \Delta(\hat{k}\cdot\vec{\sigma}) \\ \hline \Delta(\hat{k}\cdot\vec{\sigma}) & -\epsilon_{\mathbf{k}} \end{array}\right)\begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \end{pmatrix} = E_{\mathbf{k}}\begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \end{pmatrix}. \qquad (15.127)$$

(Notice that, if we choose a spin quantization axis parallel to $\hat{k}$, then this eigenvalue equation is identical to singlet pairing.)

Now we must find the condensate that is annihilated by the quasiparticle operators:

$$a_{\mathbf{k}} = (u^\dagger_{\mathbf{k}}, v^\dagger_{\mathbf{k}}) \cdot \psi_{\mathbf{k}} = u^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + v^\dagger_{\mathbf{k}\sigma} \tilde{c}^\dagger_{\mathbf{k}\sigma}. \tag{15.128}$$

To commute the quasiparticle operator with the pair creation operator, we note that

$$[a_{\mathbf{k}}, c^\dagger_{\mathbf{k}'\sigma}] = u^\dagger_{\mathbf{k}\sigma} \delta_{\mathbf{k},\mathbf{k}'}, \tag{15.129}$$

so that

$$\begin{aligned}
[a_{\mathbf{k}}, \Lambda^\dagger_T] &= \frac{1}{2}\left[a_{\mathbf{k}}, \sum_{\mathbf{k}'} \phi_{\mathbf{k}'} c^\dagger_{\mathbf{k}'} (\hat{k}' \cdot \vec{\sigma}) i\sigma_2 c^\dagger_{-\mathbf{k}'} \right] \\
&= \frac{1}{2}\phi_{\mathbf{k}} \left[u^\dagger_{\mathbf{k}} (\hat{k}\cdot\vec{\sigma}) i\sigma_2 c^\dagger_{-\mathbf{k}} + c^\dagger_{-\mathbf{k}} (\hat{k}\cdot\vec{\sigma}) i\sigma_2 (u^\dagger_{\mathbf{k}})^T \right] \\
&= \frac{1}{2}\phi_{\mathbf{k}} \left[u^\dagger_{\mathbf{k}} (\hat{k}\cdot\vec{\sigma}) i\sigma_2 c^\dagger_{-\mathbf{k}} + u^\dagger_{\mathbf{k}} \overbrace{-i\sigma_2 (\hat{k}\cdot\vec{\sigma}^T)}^{(\hat{k}\cdot\vec{\sigma})i\sigma_2} c^\dagger_{-\mathbf{k}} \right] \\
&= \phi_{\mathbf{k}} \left[u^\dagger_{\mathbf{k}} (\hat{k}\cdot\vec{\sigma}) \tilde{c}^\dagger_{-\mathbf{k}} \right],
\end{aligned} \tag{15.130}$$

where we have used $\sigma_2 \vec{\sigma}^T = -\vec{\sigma}\sigma_2$ and the fact that $\phi_{\mathbf{k}} = \phi_{-\mathbf{k}}$. Now by (15.127),

$$u^\dagger_{\mathbf{k}} (\hat{k}\cdot\vec{\sigma}) = \frac{(E_{\mathbf{k}} + \epsilon_{\mathbf{k}})}{\Delta} v^\dagger_{\mathbf{k}}, \tag{15.131}$$

so that

$$\frac{|u_{\mathbf{k}}|}{|v_{\mathbf{k}}|} = \frac{(E_{\mathbf{k}} + \epsilon_{\mathbf{k}})}{\Delta}, \tag{15.132}$$

enabling the commutator of the quasiparticle operator with the pair creation operator to be written in the compact form

$$[\alpha_{\mathbf{k}}, \Lambda^\dagger_T] = \frac{|u_{\mathbf{k}}|}{|v_{\mathbf{k}}|} \phi_{\mathbf{k}} (v^\dagger_{\mathbf{k}} \tilde{c}^\dagger_{-\mathbf{k}}). \tag{15.133}$$

As in the case of singlet pairing, if we choose

$$\phi_{\mathbf{k}} = -\frac{|v_{\mathbf{k}}|}{|u_{\mathbf{k}}|} \tag{15.134}$$

then

$$[\alpha_{\mathbf{k}}, \Lambda^\dagger_T] = -v^\dagger_{\mathbf{k}} \cdot \tilde{c}^\dagger_{-\mathbf{k}}, \tag{15.135}$$

and since $\tilde{c}_{\mathbf{k}}$ commutes with $\Lambda^\dagger_T$, it follows that

$$[\alpha_{\mathbf{k}}, (\Lambda^\dagger_T)^n] = -n(\Lambda^\dagger_T)^{n-1} v^\dagger_{\mathbf{k}} \tilde{c}_{\mathbf{k}}, \tag{15.136}$$

so that

$$[\alpha_{\mathbf{k}}, \exp[\Lambda^\dagger_T]] = -\exp[\Lambda^\dagger_T] v^\dagger_{\mathbf{k}} \tilde{c}_{\mathbf{k}}. \tag{15.137}$$

This means that

$$\alpha_{\mathbf{k}} \exp[\Lambda^\dagger_T] = \exp[\Lambda^\dagger_T]\alpha_{\mathbf{k}} - \exp[\Lambda^\dagger_T] v^\dagger_{\mathbf{k}} \tilde{c}_{\mathbf{k}} = \exp[\Lambda^\dagger_T] u^\dagger_{\mathbf{k}} \cdot c_{\mathbf{k}}, \tag{15.138}$$

so that $\alpha_{\mathbf{k}}$ annihilates the coherent state:

$$\alpha_{\mathbf{k}} \exp[\Lambda_T^\dagger]|0\rangle = \exp[\Lambda_T^\dagger] u_{\mathbf{k}}^\dagger \cdot c_{\mathbf{k}}|0\rangle = 0, \tag{15.139}$$

proving that

$$|\Psi\rangle = \exp\left[-\frac{1}{2}\sum \frac{|v_{\mathbf{k}}|}{|u_{\mathbf{k}}|}(\hat{k}\cdot\vec{\Psi}_{\mathbf{k}}^\dagger)\right]|0\rangle \tag{15.140}$$

is the ground state.

Note that we have to be careful in reducing this to the usual multiplicative BCS form, for the square of the triplet pair operator is *not* zero. If one splits the sum over momentum space into two parts, $k_z > 0$ and $k_z < 0$, then the Cooper pair operator can be written as

$$\begin{aligned}\Lambda_T^\dagger &= \sum_{k_z>0} \phi_{\mathbf{k}} c_{\mathbf{k}}^\dagger \left(\frac{\hat{k}\cdot\vec{\sigma}+1}{2}\right)\tilde{c}_{-\mathbf{k}} + \sum_{k_z<0} \phi_{\mathbf{k}} c_{\mathbf{k}}^\dagger \left(\frac{\hat{k}\cdot\vec{\sigma}-1}{2}\right)\tilde{c}_{-\mathbf{k}} \\ &= \sum_{\mathbf{k}} \phi_{\mathbf{k}} c_{\mathbf{k}}^\dagger \left(\frac{\hat{k}\cdot\vec{\sigma}+\mathrm{sgn}(k_z)}{2}\right)\tilde{c}_{-\mathbf{k}}.\end{aligned} \tag{15.141}$$

The additional singlet term that has been added and subtracted from the upper and lower halves of momentum space cancel with each other. Now the terms inside the pair operators are projection operators, and the squares of these operators do vanish. We can now expand the coherent triplet paired state as a BCS product, as follows:

$$|\Psi\rangle = \prod_{\mathbf{k}} \left(|u_{\mathbf{k}}| - |v_{\mathbf{k}}| c_{\mathbf{k}}^\dagger \left(\frac{\hat{k}\cdot\vec{\sigma}+\mathrm{sgn}(k_z)}{2}\right)\tilde{c}_{-\mathbf{k}}\right)|0\rangle. \tag{15.142}$$

Example 15.8

(a) Show that the Nambu Green's function for ^{3}HeB is given by

$$\mathcal{G}(k) = [i\omega_n - \epsilon_{\mathbf{k}}\tau_3 - (\vec{\Delta}_{\mathbf{k}}\cdot\vec{\sigma})\tau_1]^{-1} = \frac{i\omega_n + \epsilon_{\mathbf{k}}\tau_3 + (\vec{\Delta}_{\mathbf{k}}\cdot\vec{\sigma})\tau_1}{(i\omega_n)^2 - E_{\mathbf{k}}^2}.$$

(b) Calculate the magnetic susceptibility of the B phase of ^{3}He. Show that the ground-state condensate has a finite Pauli susceptibility equal to 2/3 of the normal state.

Solution

(a) As in the case of singlet pairing, we can write the propagator as $G(\mathbf{k}) = -\frac{1}{\partial_\tau + h_{\mathbf{k}}}$. Let us start with the imaginary-time propagator, which we will write

$$\mathcal{G}(\mathbf{k},\tau) = -\langle T\psi_{\mathbf{k}}(\tau)\psi_{\mathbf{k}}^\dagger(0)\rangle \tag{15.143}$$

or, written out explicitly, $\mathcal{G}_{\alpha\beta}(\mathbf{k},\tau) = -\langle T\psi_{\mathbf{k}\alpha}(\tau)\psi_{\mathbf{k}\beta}^\dagger(0)\rangle$, where $\psi_{\mathbf{k}\alpha}$ is a Balian–Werthamer spinor. The expectation values are to be evaluated with the mean-field Hamiltonian $H = \sum_{\mathbf{k}\in\frac{1}{2}BZ} \psi_{\mathbf{k}}^\dagger h_{\mathbf{k}} \psi_{\mathbf{k}}$, where

$$h_{\mathbf{k}} = \epsilon_{\mathbf{k}}\tau_3 + (\vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma})\tau_1. \tag{15.144}$$

When we take account of the time-ordering, the equation of motion for $\mathcal{G}$ is

$$\begin{aligned}\partial_\tau \mathcal{G}(\mathbf{k}, \tau) &= -\delta(\tau)\langle\{\psi_{\mathbf{k}}, \psi^\dagger_{\mathbf{k}}\}\rangle - \langle T(\partial_\tau \psi_{\mathbf{k}}(\tau))\psi^\dagger_{\mathbf{k}}(0)\rangle \\ &= -\delta(\tau)\underline{1} - \langle T[H, \psi_{\mathbf{k}}(\tau)]\psi^\dagger_{\mathbf{k}}(0)\rangle \\ &= -\delta(\tau)\underline{1} - h_{\mathbf{k}}\mathcal{G}(\mathbf{k}, \tau),\end{aligned} \tag{15.145}$$

where we have used $\psi_{\mathbf{k}}(\tau) = e^{H\tau}\psi_{\mathbf{k}}e^{-H\tau}$ and $\partial_\tau\psi_{\mathbf{k}}(\tau) = [H, \psi_{\mathbf{k}}(\tau)] = -h_{\mathbf{k}}\psi_{\mathbf{k}}$. It follows that

$$(\partial_\tau + h_{\mathbf{k}})\mathcal{G}(\mathbf{k}, \tau) = -\delta(\tau)\underline{1} \tag{15.146}$$

or $\mathcal{G}(\mathbf{k}, \tau) = -1/[\partial_\tau + h_{\mathbf{k}}]$. Fourier transforming this expression in time ($\mathcal{G}(\mathbf{k}, \tau) \to \mathcal{G}(\mathbf{k}, i\omega_n)$, $\partial_\tau \to -i\omega_n$), it follows that $(-i\omega_n + h_{\mathbf{k}})\mathcal{G}(k) = -1$, or

$$G(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon_{\mathbf{k}}\tau_3 - (\vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma})\tau_1} = \frac{i\omega_n + \epsilon_{\mathbf{k}}\tau_3 + (\vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma})\tau_1}{(i\omega_n)^2 - E^2_{\mathbf{k}}}, \tag{15.147}$$

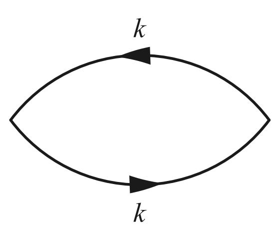
where, for ^{3}He-B, we can take $\Delta_{\mathbf{k}} = \Delta\hat{k}$, so that $(\vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma})^2 = \Delta^2$.

(b) In a magnetic field, the free energy becomes

$$F = -\frac{T}{2}\sum_k \mathrm{Tr}\ln[-\mathcal{G}^{-1}(k) - \mu_N\vec{\sigma} \cdot \vec{B}] + \text{field-independent terms}, \tag{15.148}$$

where the factor of $\frac{1}{2}$ derives from expanding the summation over one-half the Brillouin zone to the entire momentum space and μ_N is the nuclear moment of the ^{3}He-atom. We can either differentiate this twice with respect to the field or write the spin susceptibility as a mean-field polarization bubble, to obtain

$$\chi^{ab} = -\frac{\partial^2 F}{\partial B_a \partial B_b} = a$$



$$b \quad = -\frac{T\mu_N^2}{2}\sum_k \mathrm{Tr}\left[\sigma^a \mathcal{G}(k)\sigma^b\mathcal{G}(k)\right]. \tag{15.149}$$

Inserting (15.147), we obtain

$$\chi^{ab} = -\frac{T\mu_N^2}{2}\sum_k \mathrm{Tr}\left[\sigma^a \frac{i\omega_n + \epsilon_{\mathbf{k}}\tau_3 + (\vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma})\tau_1}{(i\omega_n)^2 - E^2_{\mathbf{k}}}\sigma^b \frac{i\omega_n + \epsilon_{\mathbf{k}}\tau_3 + (\vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma})\tau_1}{(i\omega_n)^2 - E^2_{\mathbf{k}}}\right]. \tag{15.150}$$

Now we can carry out the traces over the Nambu and Pauli matrices separately. Carrying out the trace over the Nambu components, we obtain

$$\chi^{ab} = -T\mu_N^2 \sum_k \frac{1}{[(i\omega_n)^2 - E^2_{\mathbf{k}}]^2}\left([(i\omega_n)^2 + \epsilon^2_{\mathbf{k}}]\mathrm{Tr}[\sigma^a\sigma^b] + \left[\sigma^a(\vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma})\sigma^b(\vec{\Delta}_{\mathbf{k}} \cdot \vec{\sigma})\right]\right).$$

Now $\mathrm{Tr}[\sigma^a\sigma^b] = 2\delta^{ab}$. To calculate $\mathrm{Tr}[\sigma^a\sigma^b\sigma^c\sigma^d]$, one can cyclically anticommute σ^a around the trace (using $\sigma^a\sigma^b = 2\delta^{ab} - \sigma^b\sigma^a$), picking up the remainders, to obtain

$$\mathrm{Tr}[\sigma^a\sigma^b\sigma^c\sigma^d] = 2\left(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right),$$

so that

$$\mathrm{Tr}[\sigma^a(\vec{\Delta}_{\mathbf{k}}\cdot\vec{\sigma})\sigma^b(\vec{\Delta}_{\mathbf{k}}\cdot\vec{\sigma})] = 2[2\Delta^a_{\mathbf{k}}\Delta^b_{\mathbf{k}} - \delta^{ab}\Delta_{\mathbf{k}}\cdot\Delta_{\mathbf{k}}] = 2\Delta^2[2\hat{k}^a\hat{k}^b - \delta^{ab}], \quad (15.151)$$

so the susceptibility can be rewritten

$$\begin{aligned}\chi^{ab} &= -2T\mu_N^2\sum_k \frac{1}{[(i\omega_n)^2 - E^2_{\mathbf{k}}]^2}\left([(i\omega_n)^2 + \epsilon^2_{\mathbf{k}}]\delta^{ab} + \Delta^2[2\hat{k}^a\hat{k}^b - \delta^{ab}]\right)\\ &= -2T\mu_N^2\sum_k \frac{1}{[(i\omega_n)^2 - E^2_{\mathbf{k}}]^2}\left([(i\omega_n)^2 + \epsilon^2_{\mathbf{k}} + \Delta^2]\delta^{ab} + 2\Delta^2[\hat{k}^a\hat{k}^b - \delta^{ab}]\right).\end{aligned} \quad (15.152)$$

After the momentum sums, $\hat{k}^a\hat{k}^b \to \frac{1}{3}\delta^{ab}$ so the susceptibility is isotropic, $\chi^{ab} = \chi(T)\delta^{ab}$, where

$$\chi = -2T\mu_N^2\sum_k \frac{1}{[(i\omega_n)^2 - E^2_{\mathbf{k}}]^2}\left([(i\omega_n)^2 + E^2_{\mathbf{k}}] - \frac{4}{3}\Delta^2\right). \quad (15.153)$$

The first term is recognized as the Pauli susceptibility of a singlet BCS superconductor, which drops exponentially to zero as $T \to 0$, while the second term must be interpreted as an additional contribution derived from the polarizability of the triplet condensate. The evaluation of the Matsubara sums follows the same lines as for a singlet superconductor. We obtain

$$\begin{aligned}\chi &= -2\mu_N^2\sum_{\mathbf{k}}\oint_{z=\pm E_{\mathbf{k}}}\frac{dz}{2\pi i}f(z)\frac{1}{(z-E_{\mathbf{k}})^2(z+E_{\mathbf{k}})^2}\left(z^2 + E^2_{\mathbf{k}} - \frac{4}{3}\Delta^2\right)\\ &= -2\mu_N^2\sum_{\mathbf{k}}\left\{\frac{\partial}{\partial z}\left[f(z)\frac{1}{(z+E_{\mathbf{k}})^2}\left(z^2 + E^2_{\mathbf{k}} - \frac{4}{3}\Delta^2\right)\right]_{z=E_{\mathbf{k}}} + (E_{\mathbf{k}} \to -E_{\mathbf{k}})\right\}\\ &= 2\mu_N^2\sum_{\mathbf{k}}\left\{-f'(E_{\mathbf{k}})\left(1 - \frac{2\Delta^2}{3E^2_{\mathbf{k}}}\right) + (1 - 2f(E_{\mathbf{k}}))\frac{\Delta^2}{3E^3_{\mathbf{k}}}\right\}.\end{aligned} \quad (15.154)$$

At zero temperature, the first term vanishes. The second term becomes

$$\begin{aligned}\chi(T=0) &= 2\mu_N^2 N(0)\int_{-\infty}^{\infty} d\epsilon\left(\frac{\Delta^2}{3[\epsilon^2 + \Delta^2]^{3/2}}\right)\\ &= 2\mu_N^2 N(0)\left[\frac{\epsilon}{3\sqrt{\epsilon^2 + \Delta^2}}\right]_{-\infty}^{\infty} = \frac{2}{3}\times 2\mu_N^2 N(0),\end{aligned} \quad (15.155)$$

so the zero-temperature susceptibility is 2/3 of the normal-state Pauli susceptibility. This intrinsic susceptibility of the condensate is present because the triplet pairs become slightly spin-polarized in a magnetic field.

We can actually do a little better than this, however, by noticing that, at a finite temperature (denoting $E = \sqrt{\epsilon^2 + \Delta^2}$),

$$\begin{aligned}\frac{2}{3} &= \frac{1}{3}\int_{-\infty}^{\infty} d\epsilon \frac{d}{d\epsilon}\left[\frac{\epsilon}{\sqrt{\epsilon^2+\Delta^2}}[1-2f(E)]\right] \\ &= \frac{1}{3}\int_{-\infty}^{\infty} d\epsilon \frac{d}{d\epsilon}\left[\frac{\Delta^3}{E^2}[1-2f(E)] - 2f'(E)\left(1-\frac{\Delta^2}{E^2}\right)\right],\end{aligned} \tag{15.156}$$

which we recognize as the argument of the second part of the integral in (15.154). We can thus rewrite the susceptibility as

$$\chi(T) = \frac{1}{3}\chi_S(T) + \frac{2}{3}\chi_P,$$

where $\chi_P = 2\mu_N^2 N(0)$ is the Pauli susceptibility of the normal state and

$$\chi_S(T) = 2\mu_N^2 N(0)\int_{-\infty}^{\infty} d\epsilon[-f'(\sqrt{\epsilon^2+\Delta^2})] = 2\mu_N^2 N(0) Y\left[\frac{\Delta}{2T}\right], \tag{15.157}$$

where

$$Y[x] = \frac{1}{2}\int_{-\infty}^{\infty} \frac{du}{\cosh^2[\sqrt{u^2+x^2}]} \tag{15.158}$$

is called the *Yoshida function*, after its inventor, Kei Yoshida. The final expression for the susceptibility of the B phase is then

$$\chi_B(T) = \chi_P\left[\frac{2}{3} + \frac{1}{3}Y[\Delta/2T]\right]. \tag{15.159}$$

Exercises

Exercise 15.1 The standard two-component Nambu spinor approach does not allow a rotationally invariant treatment of the electron spin and the Zeeman coupling of fermions to a magnetic field. This drawback can be overcome by switching to a four-component Balian–Werthamer spinor, denoted by

$$\psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}}^\dagger \\ -i\sigma_2(c_{\mathbf{k}}^\dagger)^T \end{pmatrix} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ -c_{\mathbf{k}\downarrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \\ c_{-\mathbf{k}\uparrow}^\dagger \end{pmatrix}. \tag{15.160}$$

(a) Show, using this notation, that the total electron spin can be written

$$\vec{S} = \frac{1}{4}\sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger \vec{\sigma}_{(4)}\psi_{\mathbf{k}}, \tag{15.161}$$

where

$$\vec{\sigma}_4 = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{15.162}$$

is the four-component Pauli matrix. (You may find it useful to use the relationship $\vec{\sigma}^T = i\sigma_2\vec{\sigma}i\sigma_2$.) In practical usage, the subscript 4 is normally dropped.

(b) Show that, in a Zeeman field, the BCS Hamiltonian

$$H_{MFT} = \sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\alpha}[\epsilon_{\mathbf{k}}\delta_{\alpha\beta} - \vec{\sigma}_{\alpha\beta}\cdot\vec{B}]c_{\mathbf{k}\beta} + \sum_{\mathbf{k}}\left[\bar{\Delta}c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow} + c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}\Delta\right] + \frac{V}{g_0}\bar{\Delta}\Delta \tag{15.163}$$

can be rewritten using Balian–Werthamer spinors in the compact form

$$H_{MFT} = \frac{1}{2}\sum_{\mathbf{k}}\psi^\dagger_{\mathbf{k}}\left[\underline{h}_{\mathbf{k}} - \vec{\sigma}_4\cdot\vec{B}\right]\psi_{\mathbf{k}} + \frac{V}{g_0}\bar{\Delta}\Delta, \tag{15.164}$$

where $\underline{h}_{\mathbf{k}} = \epsilon_{\mathbf{k}}\tau_1 + \Delta_1\tau_1 + \Delta_2\tau_2$ as before, but the $\vec{\tau}$ now refer to the four-dimensional Nambu matrices

$$\vec{\tau} = \left(\begin{bmatrix}0 & \underline{1}\\ \underline{1} & 0\end{bmatrix}, \begin{bmatrix}0 & -i\underline{1}\\ i\underline{1} & 0\end{bmatrix}, \begin{bmatrix}\underline{1} & 0\\ 0 & -\underline{1}\end{bmatrix}\right). \tag{15.165}$$

(c) Show that the quasiparticle energies in a field are given by $\pm E_{\mathbf{k}} - \sigma B$.

Exercise 15.2 Pauli limited type II superconductors.

The BCS Hamiltonian introduced in describes a *Pauli limited superconductor*, in which the Zeeman coupling of the paired electrons with the magnetic field dominates over the orbital coupling to the magnetic field. In the flux lattice of a Pauli limited type II superconductor, the magnetic field penetrates the condensate and can be considered to be approximately uniform.

(a) Assuming that the orbital coupling of the electron to the magnetic field is negligible, use the Balian–Werthamer approach developed in the previous problem to formulate BCS theory in a uniform Zeeman field, as a path integral. Show that the free energy can be written

$$\begin{aligned} F &= -\frac{T}{2}\sum_{\mathbf{k}}\mathrm{Tr}\ln[\partial_\tau + h_{\mathbf{k}} - \vec{\sigma}_4\cdot\vec{B}] + \frac{V}{g_0}\bar{\Delta}\Delta \\ &= -\frac{T}{2}\sum_{\mathbf{k},i\omega_n,\sigma}\ln\left[E^2_{\mathbf{k}} - (i\omega_n - \sigma B)^2\right] + \frac{V}{g_0}\bar{\Delta}\Delta \\ &= -T\sum_{\mathbf{k},\sigma}\ln\left[2\cosh\frac{\beta(E_{\mathbf{k}} - \sigma B)}{2}\right] + \frac{V}{g_0}\bar{\Delta}\Delta. \end{aligned} \tag{15.166}$$

(b) Show that the gap equation for a Pauli limited superconductor becomes

$$\frac{1}{g_0} = \frac{1}{2}\sum_{\mathbf{k},\sigma}\tanh\left(\frac{\beta(E_{\mathbf{k}} - \sigma B)}{2}\right)\frac{1}{2E_{\mathbf{k}}}.$$

Use this expression to show that the upper critical field is given by $g\mu_B B_{c2}/2 = \Delta/2$, where Δ is the zero-temperature value of the gap.

(c) Pauli limited superconductors usually undergo a first-order transition to the flux state at a higher field than the one just estimated. Why is this?

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