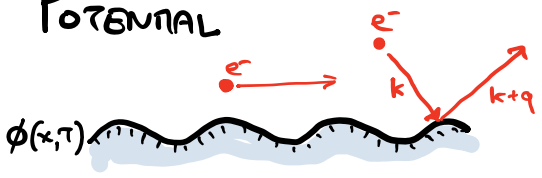
Feynman Diagrams: Fermions in a Varying Potential



$$F = -T\,\mathrm{Tr}\,\ln\left[-G_0^{-1} + \phi\right]$$

$$= -T\,\mathrm{Tr}\,\ln\left(-G_0^{-1}\right) - T\,\mathrm{Tr}\left[1 - G_0\phi\right]$$

$$= F_0 + T\,\mathrm{Tr}\left[G_0\phi + \frac{1}{2}(G_0\phi)^2 + \frac{1}{3}(G_0\phi)^3 + \dots\right]$$

$$= F_0 - T\left[-\int d^4x\; G_0(x,x)\phi(x) - \frac{1}{2}\int d^4x\, d^4y\, \phi(x) G_0(x-y)\phi(y) G_0(y-x) - \dots\right]$$

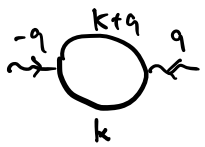
Fermion loop (NOTE DOUBLE NEGATIVE).

$$S = -\beta F_0 - \left[\;\circlearrowright + \;\circlearrowright_{k+q,\,k} + \;\circlearrowright\right] \qquad Z = e^{-S} = \sum\left(\begin{matrix}\text{closed}\\ \text{unlinked}\\ \text{diagrams}\end{matrix}\right)$$

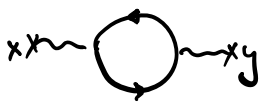
Real Space	Momentum Space
$x' \to x$ $\quad G_0(x-x') = -\langle T\,\psi(x)\psi^\dagger(x')\rangle_{H_0}$	$G_0(k) = \dfrac{1}{i\omega_n - \epsilon_k}$
$\phi(x)$	$\phi(q)$ $\quad \phi_{-q} = \phi_q^*$
$\prod_{i=1}^{N}\int d^4x_i$ all intermediate co-ordinates	$\prod^{\text{loops}} \frac{1}{V\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}$ All loop momenta.
$(-1)^F$ F = # Fermion loops.	$(-1)^F$ Fermion loops.
$\frac{1}{n}$ symmetry factor	$\frac{1}{n}$ symmetry factor.
$-V(x-y)$ interaction	$-V(q)$

$$\frac{\Delta S}{\beta V} = \frac{\Delta F}{V} = -\frac{1}{\beta V}\left[\sum \text{closed link diagrams in real space}\right]$$

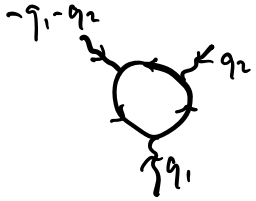
$$= -\sum\left[\text{closed link diagrams in momentum space.}\right]$$



$$= (-)^1 \frac{N}{2\beta V} \sum \phi_{-q}\phi_q G(k+q) G(k)$$

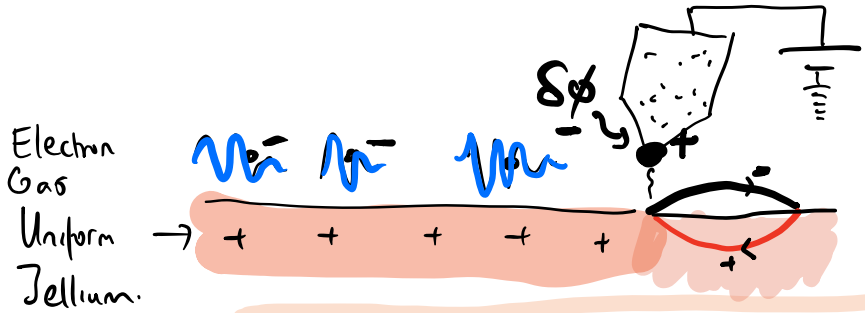


$$= -\frac{N}{2\beta V} \int d^4x\, d^4y \;\; \phi(x)\phi(y) G(x-y) G(y-x)$$



$$= -\frac{N}{3\beta V} \sum \phi_{-q_1-q_2}\phi_{q_1}\phi_{q_2} G(k+q_2) G(k+q_1) G(k)$$

$$= -\frac{N}{3\beta V} \int d^3x\, d^4y\, d^4z \;\; \phi(x)\phi(y)\phi(z) G(x-z) G(z-y) G(y-x)$$



"Dolphins leaping out of the fermi sea"

$$\langle \psi^{+}(x)\psi(x)\rangle = \rho_0$$

$$S = \int d^4x \left\{ \bar{\psi}_\sigma \left(-\gamma_0^{-1}\right)\psi_\sigma + \phi\left(\bar{\psi}_\sigma\psi_\sigma - \rho_0\right) - \frac{\epsilon_0}{2e^2}(\nabla\phi)^2 \right\}$$

Make saddle point exact $\sigma = 1, \dots N$ $\quad \epsilon_0 = \tilde{\epsilon}_0$

Repulsive interaction $\left(\frac{e^2}{\epsilon_0 q^2}\right)$

$$S = \int d^4x \left\{ \bar{\psi}_\sigma \left(-\gamma_0^{-1}\right)\psi_\sigma + \phi\left(\bar{\psi}_\sigma\psi_\sigma - \rho_0\right) - \underbrace{\frac{\epsilon_0}{2e^2}(\nabla\phi)^2}_{-\frac{\epsilon_0 q^2}{2e^2}|\phi_q|^2} \right\}$$

" $|\delta\phi_q|^2 = \frac{-e^2}{\epsilon_0 q^2}$ "

Repulsive

$$\frac{e^2}{\epsilon_0} = \frac{\tilde{e}^2}{\epsilon_0 N} \qquad \frac{e^2}{4\pi\epsilon_0 r} = \left(\frac{\tilde{e}^2}{4\pi\epsilon_0 r\, N}\right)$$

$$S = \int d^4x \left\{ \bar{\psi}_\sigma \left(-\gamma^{-1}\right)\psi_\sigma + \phi\left(\bar{\psi}_\sigma\psi_\sigma - \rho_0\right) - \frac{N\epsilon_0}{2\tilde{e}^2}(\nabla\phi)^2 \right\}$$

$$Z = \int \mathcal{D}[\phi]\; e^{\frac{N\epsilon_0}{2e^2}\int d\tau\, d^3x\, (\nabla\phi)^2} \int \mathcal{D}[\bar\psi,\psi]\; e^{-N\int d^4x \left[\bar\psi\left(-G_0^{-1} + \phi\right)\psi - \phi\rho_0\right]}$$

↑ +ve background

$$= \int \mathcal{D}[\phi]\; e^{-S_\phi N}\, e^{N\,\mathrm{Tr}\ln\left(-G_0^{-1} + \phi\right)}$$

$$= \int \mathcal{D}[\phi]\; e^{-\left[S_\phi - \mathrm{Tr}\ln\left(-G_0^{-1} + \phi\right)\right]N}$$

$$= \int \mathcal{D}[\phi]\; e^{-S_{EFF}[\phi]}$$

$$\frac{BF}{N} = \frac{S_{EFF}[\phi]}{N} = -\mathrm{Tr}\ln\left[-G_0^{-1} + \phi\right] - \int d^4x \left[\frac{\epsilon_0}{2e^2}(\nabla\phi)^2 + \phi\tilde\rho_0\right]$$

$$= -\mathrm{Tr}\ln\left[-G_0^{-1}\left(1 - G_0\phi\right)\right] - S_\phi$$

$$= -\mathrm{Tr}\ln\left(-G_0^{-1}\right) + \int d^4x\, \phi\left(G_0 - \rho_0\right) + \frac{1}{2}\mathrm{Tr}\left(G_0\phi G_0\phi\right) + O(\phi^3) - \int d^4x \left[\frac{\epsilon_0}{2e^2}(\nabla\phi)^2\right]N$$

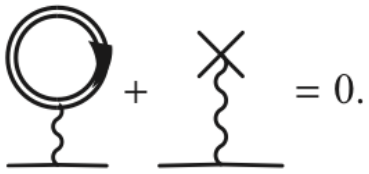
$$S_{EFF} = S_0 - \left[\;\bigcirc + \bigcirc\!\!\bigcirc + \bigcirc\!\!\!\!\bigcirc + \cdots\right] - \phi\rho_0$$

$$= S_0 - \int d^4x \left[\rho_0 + \underbrace{\bigcirc}_{-G(0^-)N}\right]\phi(x) - \int \frac{\phi(x)}{2}\left[x \sim\!\bigcirc\!\sim x' + \frac{\epsilon_0}{e^2}\nabla^2\right]\phi(x')\, d^3x\, d^3x' + O(\phi^3)$$

$$\simeq NS_0 - \sum_k \frac{1}{2}\left[\sim\!\bigcirc^{k+q}_{k}\!\sim^{q} + \frac{Nq^2}{e^2}\right]|\phi_q|^2 + O(\phi^3)$$

$$\frac{\delta S_{EFF}}{\delta \phi} = 0 \quad \Rightarrow \quad N G_0(\vec{x}^-, x) = \rho_0 = -\langle T \psi_\sigma(x^-) \psi_\sigma^\dagger(x) \rangle = \langle \psi_\sigma^\dagger(x) \psi_\sigma(x) \rangle$$

NEUTRALITY



$$\frac{\delta^2 S}{e^2 \delta\phi(x) \delta\phi(x')} = \left[N \, G(x-x') G(x-x') - \frac{\epsilon_0 N \nabla^2}{e^2} \right]$$

$$G(x) = \frac{T}{V} \sum G(k) e^{ikx}$$

$$\phi(x) = \frac{1}{\sqrt{\beta V}} \sum \phi(k) e^{ikx}$$

$$\phi(x) \, G(x-x') \, G(x'-x) \, \phi(x')$$

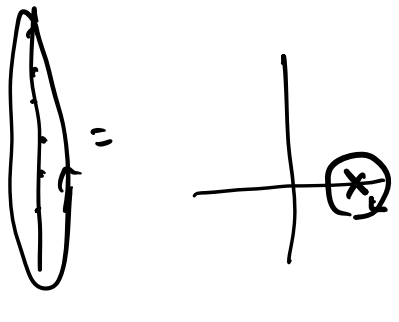
$$S_{QUAN} = \frac{N}{2} \sum \left[G(k+q) G(k) + \frac{\epsilon_0}{e^2} q^2 \right] |\phi_q|^2$$

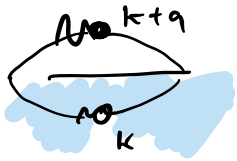
$$-\frac{1}{N\beta V} \frac{\delta^2 S}{\delta \phi_q^* \delta \phi_q} = \tilde{\chi}(q) = -\frac{1}{V\beta} \sum G(k+q) G(k)$$

$$\tilde{\chi}(\vec{q}, i\nu_n) = -\frac{1}{V} \sum_k T \sum \frac{1}{i\omega_n + i\nu_n - \epsilon_{k+q}} \frac{1}{i\omega_n - \epsilon_k}$$

$$= \frac{1}{V} \sum_k \oint \frac{f(z) dz}{2\pi i} \frac{1}{z + i\nu_n - \epsilon_{k+q}} \frac{1}{z - \epsilon_k}$$

poles on imaginary axis





$$= -\frac{1}{V}\sum \oint \frac{f(z)\,dz}{2\pi i} \quad \frac{1}{z-(\epsilon_{k+q}-i\nu_n)} \quad \frac{1}{z-\epsilon_k}$$

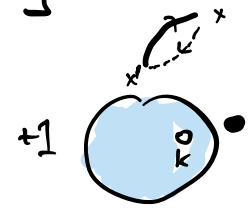
Poles at $z = \epsilon_k,\ \epsilon_{k+q}-i\nu_n$

$$= -\frac{1}{V}\sum_{\vec{k}} \left[\frac{f(\epsilon_{k+q}-i\nu_n)}{\epsilon_{k+q}-\epsilon_k - i\nu_n} + \frac{f(\epsilon_k)}{\epsilon_k - \epsilon_{k+q} + i\nu_n} \right]$$

$$\tilde{\chi}(\vec{q}, i\nu_n) = \frac{1}{V}\sum_{k} \left[\frac{f_{k+q} - f_k}{i\nu_n - (\epsilon_{k+q} - \epsilon_k)} \right] = \int \frac{d^3k}{(2\pi)^3} \left(\frac{f_{k+q}-f_k}{i\nu_n - (\epsilon_{k+q}-\epsilon_k)} \right)$$

$$\chi(q,z) = N_e \int \frac{d^3k}{(2\pi)^3} \frac{f_{k+q}-f_k}{z-(\epsilon_{k+q}-\epsilon_k)}$$

Lindhardt Function "Charge susceptibility"

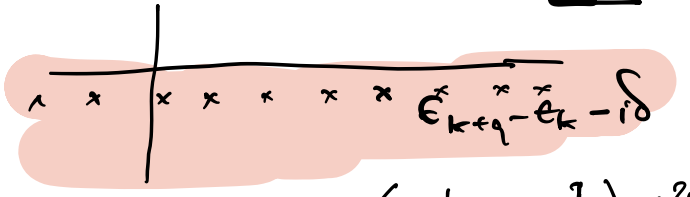


$$\frac{1}{\omega - \omega_q}$$

Retarded response function.
Particle-hole excitation.

$$\tilde{\chi}_R(\vec{q},\nu) = \chi(q,z)\big|_{z=\nu+i\delta}$$

$$\chi_A = \chi(q,t)\big|_{z=\nu-i\delta}$$

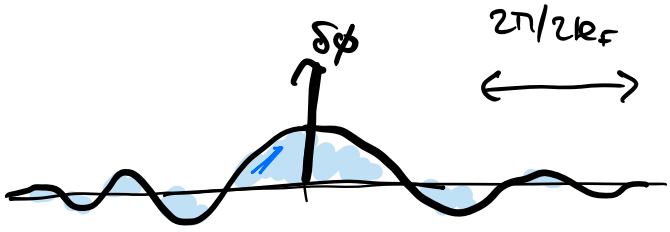


$$\left(\frac{1}{\omega-\omega_q} - \frac{1}{\omega+\omega_{-q}}\right) = \left(\frac{2\omega_q}{\omega^2-\omega_q^2}\right)$$

$$\delta\rho(q) = \chi(q)\,\phi(q). \qquad \chi(x) = \int \frac{d^4q}{(2\pi)^4} \chi(q) e^{iqx}$$

$$\delta\rho(x,t) = \int d^4x'\, \chi(x-x')\,\phi(x') \qquad \delta\phi = \phi\,\delta^3(\vec{x})$$

$$\delta\rho(\vec{x}) = \int d^3x' \chi(\vec{x}\ , \omega=0)\,\phi_0$$



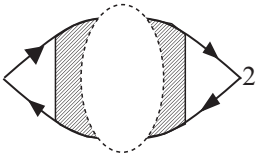
Friedel oscillations of charge.

so that

$$i^2 \frac{\delta \ln S[\phi]}{\delta\phi(2)\delta\phi(1)} = \frac{\langle\psi_0|T\{S\rho(1)\rho(2)\}|\psi_0\rangle}{S[\phi]} - \frac{\langle\psi_0|S\rho(1)|\psi_0\rangle}{S[\phi]}\frac{\langle\psi_0|S\rho(2)|\psi_0\rangle}{S[\phi]}$$
$$= \langle T\rho(1)\rho(2)\rangle - \langle\rho(1)\rangle\langle\rho(2)\rangle$$
$$= \langle T(\rho(1) - \langle\rho(1)\rangle)(\rho(2) - \langle\rho(2)\rangle) = \langle T\delta\rho(1)\delta\rho(2)\rangle\rangle. \tag{7.192}$$

With this result and (7.82), we can now identify

$$\frac{\delta^2 \ln S[\phi]}{\delta\phi(2)\delta\phi(1)} = -\langle T\delta\rho(1)\delta\rho(2)\rangle = 1$$

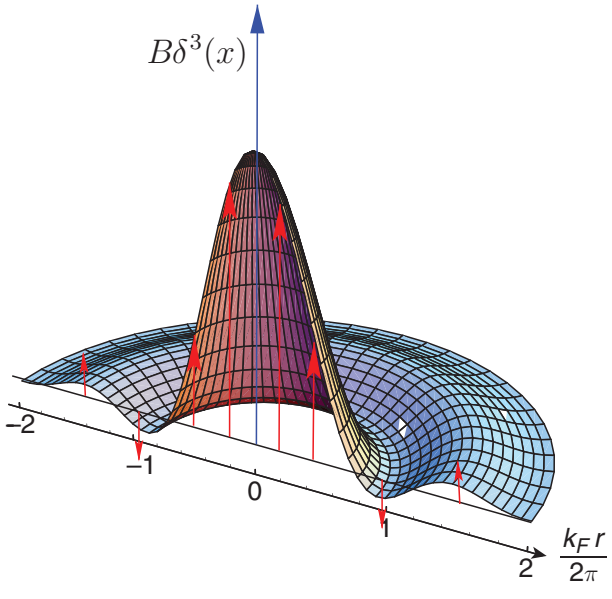


$$2\,. \tag{7.193}$$

7.6.1 Magnetic susceptibility of non-interacting electron gas

One of the fundamental qualities of a Fermi liquid is its non-local response to an applied field. Suppose, for example, that one introduces a localized delta-function disturbance in the magnetic field, $\delta B_z(x) = B\delta^3(x)$. Since the fermions have a characteristic wavevector of order k_F, this local disturbance will "heal" over a length scale of order $l \sim 1/k_F$. Indeed, since the maximum wavevector for low-energy particle–hole excitations is sharply cut off at $2k_F$, the response produces oscillations in the spin density with a wavelength $\lambda = 2\pi/k_F$ that decay gradually from the site of the disturbance. These oscillations are called *Friedel oscillations* (Figure 7.5). In the case of the example just cited, the change in the spin density in response to the shift in the chemical potential is given by

$$\delta M(\vec{x}) = \chi_s(\vec{x})B, \tag{7.194}$$



Friedel oscillations in the spin density, in response to a delta-function disturbance in the magnetic field at the origin. These oscillations may be calculated from the Fourier transform of the Lindhard function. Fig. 7.5

where

$$\chi_s(\vec{x}) = \int_{\mathbf{q}} \chi(\mathbf{q}, \omega = 0) e^{i\vec{q}\cdot\vec{x}} \tag{7.195}$$

is the Fourier transform of the dynamical spin susceptibility. We shall now calculate this quantity as an example of the application of Feynman diagrams.

From the interaction in (7.168), the magnetization is given by

$$\vec{M}(x) = \int d^4x' \underline{\chi}(x - x')\vec{B}(x'), \tag{7.196}$$

where

$$\underline{\chi}_{ab}(x) = i\langle\phi|[\sigma^a(x), \sigma^b(0)]|\phi\rangle\theta(t). \tag{7.197}$$

The electron fluid mediates this non-local response. If we Fourier transform this expression, then $\vec{M}(q) = \underline{\chi}(q)\vec{B}(q)$, where (in a relativistic shorthand)

$$\chi_{ab}(q) = i\mu_B^2 \int d^4x \langle\phi|[\sigma^a(x), \sigma^b(0)]|\phi\rangle\theta(t)e^{-iq\cdot x}. \tag{7.198}$$

We can relate $\chi_{ab}(\vec{q}, \nu) = -i\chi^T_{ab}(\vec{q}, \nu + i\delta)$, where the time-ordered Green's function is given by

$$\begin{aligned}\chi^T_{ab}(q) &= \mu_B^2 \; \sigma^b \; \overset{k+q}{\bigcirc} \; \sigma^a \\ &= -\mu_B^2 \int_{\mathbf{k}} \frac{d\omega^k}{2\pi} \overbrace{\mathrm{Tr}\left[\sigma^a G(k+q)\sigma^b G(k)\right]}^{2\delta^{ab}G(k+q)G(k)} = \delta_{ab}\chi^T(q).\end{aligned} \tag{7.199}$$

The susceptibility $\chi^T(q)$ is then

$$\chi^T(q) = -2\mu_B^2 \int_{\mathbf{k}} \frac{d\omega}{2\pi} \left[\frac{1}{\omega + \nu - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}}} \frac{1}{\omega - \tilde{\epsilon}_{\mathbf{k}}}\right], \tag{7.200}$$

where we have invoked the notation $\tilde{\epsilon}_{\mathbf{k}} = \epsilon_{\mathbf{k}} - i\delta\,\mathrm{sgn}(\epsilon_{\mathbf{k}})$. The term inside the square brackets has two poles, at $\omega = \tilde{\epsilon}_{\mathbf{k}}$ and at $\omega = \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}} - \nu$:

$$\int_\omega = \int \frac{d\omega}{2\pi} \frac{1}{(\tilde{\epsilon}_{\mathbf{k}+\mathbf{q}} - \tilde{\epsilon}_{\mathbf{k}}) - \nu} \left[\frac{1}{\omega + \nu - \epsilon_{\mathbf{k}+\mathbf{q}} + i\delta_{\mathbf{k}+\mathbf{q}}} - \frac{1}{\omega - \epsilon_{\mathbf{k}} + i\delta_{\mathbf{k}}}\right].$$

We may carry out the frequency integral by completing the contour in the upper half-plane. Each Green's function gives a contribution $2\pi i \times$ Fermi function, so that

$$\chi^T(q) = -2i\mu_B^2 \int_{\mathbf{k}} \frac{f_{\mathbf{k}+\mathbf{q}} - f_{\mathbf{k}}}{(\tilde{\epsilon}_{\mathbf{k}+\mathbf{q}} - \tilde{\epsilon}_{\mathbf{k}}) - \nu}, \tag{7.201}$$

so that the dynamical susceptibility $\chi(\mathbf{q}, \nu) = -i\chi^T(\mathbf{q}, \nu + i\delta)$ is given by

$$\chi(\mathbf{q}, \nu + i\delta) = 2\mu_B^2 \int_{\mathbf{k}} \frac{f_{\mathbf{k}+\mathbf{q}} - f_{\mathbf{k}}}{\nu - (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}) + i\delta}. \tag{7.202}$$

dynamical spin susceptibility

There are a number of important pieces of physics encoded in the above expression that deserve special discussion:

- Spin conservation. The total spin of the system is conserved, so that the application of a strictly uniform magnetic field to the fluid cannot change the total magnetization. Indeed, in keeping with this expectation, if we take $\vec{q} \to 0$ we find $\lim_{\vec{q}\to 0} \chi(\vec{q}, \nu) = 0$.
- Static susceptibility. When we take the limit $\nu \to 0$, we obtain the magnetization response to a spatially varying magnetic field. The static susceptibility is given by

$$\chi(\mathbf{q}) = 2\mu_B^2 \int_{\mathbf{k}} \frac{f_{\mathbf{k}} - f_{\mathbf{k}+\mathbf{q}}}{(\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}})}. \tag{7.203}$$

  This response is finite, because the spins can always redistribute themselves in response to a non-uniform field. When we take the wavelength of the applied field to infinity, i.e. $q \to 0$, we recover the Pauli susceptibililty:

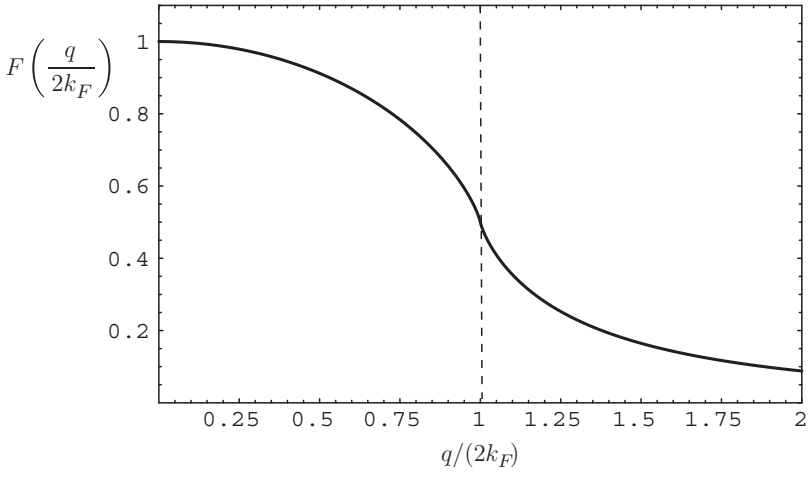
$$\chi \to 2\mu_B^2 \int_{\mathbf{k}} \left(-\frac{df(\epsilon)}{d\epsilon}\right) = 2\mu_B^2 \int_{\mathbf{k}} \delta(\epsilon_{\mathbf{k}}) = 2\mu_B^2 N(0), \tag{7.204}$$

  where $N(0) = \frac{mk_F}{2\pi^2}$ is the density of states per spin. The detailed momentum-dependent static susceptibility can be calculated (see Section 7.6.2), and is given by

$$\begin{aligned} \chi(\mathbf{q}) &= 2\mu_B^2 N(0) F(\frac{q}{2k_F}) \\ F(x) &= \frac{1}{4x}(1-x^2);\ \ln\left|\frac{1+x}{1-x}\right| + \frac{1}{2}. \end{aligned} \tag{7.205}$$

  The function $F(x)$ is known as the *Lindhard function* [5]; see Figure 7.6. It has the property that $F(0) = 1$, while $F'(x)$ has a weak logarithmic singularity at $|x| = 1$.
- Dissipation and the imaginary part of the susceptibility. The full dynamical spin susceptibility has both a real and an imaginary part, given by

$$\chi(\mathbf{q}, \nu) = \chi'(\mathbf{q}, \nu) + i\chi''(\mathbf{q}, \nu),$$



The Lindhard function. The Fourier transform of this function governs the magnetic response of a non-interacting metal to an applied field. Notice the weak singularity around $q/(2k_F) = 1$ that results from the match between the Fermi surface and the wavevector of the magnetic response. **Fig. 7.6**

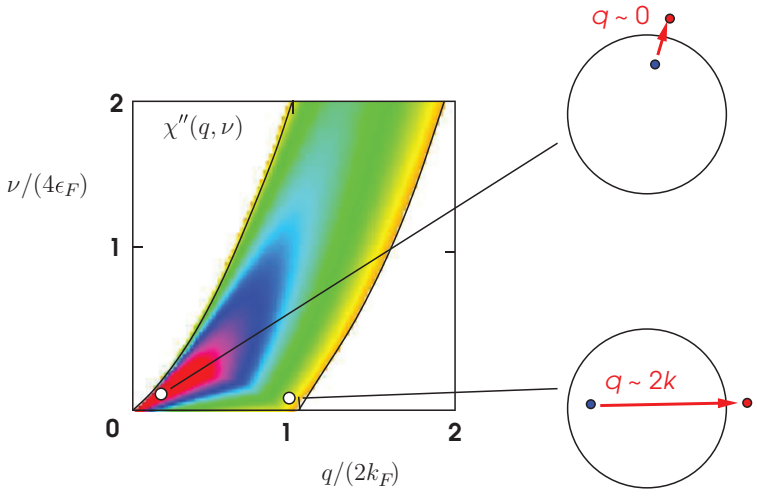


Fig. 7.7 Density plot of the imaginary part of the dynamical spin susceptibility calculated from (7.212), showing the band of width $2k_F$ that spreads up to higher energies. Excitations on the left side of the band correspond to low-momentum-transfer excitations of electrons from just beneath the Fermi surface to just above the Fermi surface. Excitations on the right-hand side of the band correspond to high-momentum-transfer processes, right across the Fermi surface.

where the imaginary part determines the dissipative part of the magnetic response. The dissipation arises because an applied magnetic field generates a cloud of electron–hole pairs which carry away the energy. If we use the Cauchy–Dirac relation $1/(x+i\delta) = P(1/x) - i\pi\delta(x)$ in (7.202), we obtain

$$\chi''(\mathbf{q},\nu) = 2\mu_B^2 \int_{\mathbf{k}} \pi\delta[\nu - (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}})](f_{\mathbf{k}} - f_{\mathbf{k}+\mathbf{q}}). \tag{7.206}$$

This quantity defines the density of states of particle–hole excitations. The excitation energy of a particle–hole pair is given by

$$\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} = \frac{q^2}{2m} + \frac{qk}{m}\cos\theta,$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{q}$. This quantity is largest when $\theta = 0$, $k = k_F$, and smallest when $\theta = \pi$, $k = k_F$, so that

$$\frac{q^2}{2m} + \frac{qk_F}{m} > \nu > \frac{q^2}{2m} - \frac{qk_F}{m}$$

defines a band of allowed wavevectors where the particle–hole density of states is finite, as shown in Figure 7.7. Outside this region, $\chi_0(\mathbf{q},\nu)$ is purely real.

7.6.2 Derivation of the Lindhard function

The dynamical spin susceptibility

$$\chi(\mathbf{q},\nu) = 2\mu_B^2 \int_{\mathbf{k}} \frac{f_{\mathbf{k}} - f_{\mathbf{k}+\mathbf{q}}}{(\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} - \nu)} \tag{7.207}$$

can be rewritten as

$$\chi(\mathbf{q},\nu) = 2\mu_B^2 \int_{\mathbf{k}} f_{\mathbf{k}} \left[\frac{1}{(\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} - \nu)} + \frac{1}{(\epsilon_{\mathbf{k}-\mathbf{q}} - \epsilon_{\mathbf{k}} + \nu)} \right]. \tag{7.208}$$

Written out explicity, this is

$$\chi(\mathbf{q},\nu) = 2\mu_B^2 \int_0^{k_F} \frac{k^2 dk}{2\pi^2} \int_{-1}^{1} \frac{d\cos\theta}{2} \left[\frac{1}{(\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} - \nu)} + ((\nu,\ \mathbf{q}) \to -(\nu,\ \mathbf{q})) \right].$$

By replacing $\epsilon_{\mathbf{k}} \to \frac{k^2}{2m} - \mu$ and rescaling $x = k/k_F$, $\tilde{q} = q/(2k_F)$, and $\tilde{\nu} = \nu/(4\epsilon_F)$, we obtain $\chi(\mathbf{q},\nu) = 2\mu_B^2 N(0)\mathcal{F}(\tilde{q},\tilde{\nu})$, where

$$\mathcal{F}(\tilde{q},\tilde{\nu}) = \frac{1}{4\tilde{q}} \int_0^1 x^2 dx \int_{-1}^{1} dc \left[\frac{1}{xc + \tilde{q} - \frac{\tilde{\nu}}{\tilde{q}}} + (\nu \to -\nu) \right] \tag{7.209}$$

is the Lindhard function. Carrying out the integral over angle, we obtain

$$\begin{aligned} \mathcal{F}(\tilde{q},\tilde{\nu}) &= \frac{1}{4\tilde{q}} \int_0^1 x\, dx \left(\ln\left[\frac{\tilde{q} - \frac{\tilde{\nu}}{\tilde{q}} + x}{\tilde{q} - \frac{\tilde{\nu}}{\tilde{q}} - x} \right] + (\tilde{\nu} \to -\tilde{\nu}) \right) \\ &= \frac{1}{8\tilde{q}} \left(\left[1 - \left(\tilde{q} - \frac{\tilde{\nu}}{\tilde{q}} \right)^2 \right] \ln\left[\frac{\tilde{q} - \frac{\tilde{\nu}}{\tilde{q}} + 1}{\tilde{q} - \frac{\tilde{\nu}}{\tilde{q}} - 1} \right] + (\tilde{\nu} \to -\tilde{\nu}) \right) + \frac{1}{2}. \end{aligned} \tag{7.210}$$

Its static limit, $F(\tilde{q}) = \mathcal{F}(\tilde{q},\tilde{\nu}=0)$,

$$F(\tilde{q}) = \frac{1}{4\tilde{q}} \left(\left[1 - \tilde{q}^2 \right] \ln\left| \frac{\tilde{q}+1}{\tilde{q}-1} \right| \right) + \frac{1}{2}, \tag{7.211}$$

has the properties that $F(0) = 1$ and dF/dx is singular at $x = 1$, as shown in Figure 7.6. The imaginary part of $\chi(\mathbf{q},\nu + i\delta)$ is given by

$$\chi''(\mathbf{q},\nu) = 2\mu_B^2 N(0) \times \frac{\pi}{8\tilde{q}} \left\{ \left(1 - \left[\tilde{q} - \frac{\tilde{\nu}}{\tilde{q}} \right]^2 \right) \theta \left[1 - \left[\tilde{q} - \frac{\tilde{\nu}}{\tilde{q}} \right]^2 \right] - (\nu \to -\nu) \right\}, \tag{7.212}$$

and is plotted in Figure 7.7.

7.7 The RPA (large-N) electron gas

Although the Feynman diagram approach gives us a way to generate all perturbative corrections, we still need a way to select the physically important diagrams. In general, as we have seen from the previous examples, it is important to re-sum particular classes of diagrams to obtain a physical result. What principles can be used to select classes of diagrams?

Frequently, however, there is no obvious choice of small parameter, in which case one needs an alternative strategy. For example, in the electron gas we could select diagrams according to the power of r_s entering the diagram. This would give us a high-density expansion of the properties – but what if we would like to examine a low-density electron gas in a controlled way?

One way to select Feynman diagrams in a system with no natural small parameter is to take the so-called *large-N* limit. This involves generalizing some internal degree of freedom so that it has N components. Examples include:

- the hydrogen atom in N dimensions
- an electron gas with $N = 2S + 1$ spin components
- spin systems with spin S in the limit that S becomes large
- quantum chromodynamics with N rather than three colours.

In each of these cases, the limit $N \to \infty$ corresponds to a new kind of semiclassical limit, where certain variables cease to undergo quantum fluctuations. The parameter $1/N$ plays the role of an effective $\hbar$:

$$\frac{1}{N} \sim \hbar. \tag{7.213}$$

This does not, however, mean that quantum effects have been lost, merely that their macroscopic consequences can be lumped into certain semiclassical variables.

We shall now examine the second of these examples. The idea is to take an interacting Fermi gas where each fermion has $N = 2S + 1$ possible spin components. The interacting Hamiltonian is still written

$$H = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \frac{1}{2} \sum V_{\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} c_{\mathbf{k}'\sigma'} c_{\mathbf{k}\sigma}, \tag{7.214}$$

but now the spin summations run over $N = 2S + 1$ values rather than just two. As N is made very large, it is important that both the kinetic energy and the interaction energy scale extensively with N. For this reason, the original interaction $V_{\mathbf{q}}$ is rescaled, writing

$$V_{\mathbf{q}} = \frac{1}{N} \mathcal{V}_{\mathbf{q}}, \tag{7.215}$$

where it is understood that, as $N \to \infty$, V is to be kept fixed. The idea is to now calculate quantities as an expansion in powers of $1/N$, and at the end of the calculation to give N the value of specific interest, in our case $N = 2$. For example, if we are interested in a Coulomb gas of spin–$\frac{1}{2}$ electrons, then we study the family of problems where

$$V_{\mathbf{q}} = \frac{1}{N} \frac{\tilde{e}^2}{q^2} = \frac{\mathcal{V}_{\mathbf{q}}}{N} \tag{7.216}$$

and $\tilde{e}^2 = 2e^2/\epsilon_0$. At the end, we set $N = 2$, boldly hoping that the key features of the solution around $N = 2$ will be shared by the entire family of models. In practice, this only holds true if the density of the electron gas is large enough to avoid instabilities such as the formation of Wigner crystal. For historical reasons, the approxation that appears in the large-N limit is called the *random phase approximation* (RPA), a method developed during the 1950s. The early version of the RPA was developed by David Bohm and David Pines [6], while its reformulation in a diagrammatic language was later given by Hubbard [7].[2] The large-N treatment of the electron gas recovers the RPA electron gas in a controlled approximation.

With the above substitution, the Feynman rules are unchanged, except that now we associate a factor $1/N$ with each interaction vertex. Before we start, however, there are a few preliminaries; in particular, we need to know how to handle long-range Coulomb

[2] A more detailed discussion of this early history can be found in the book by Nozières and Pines [8].