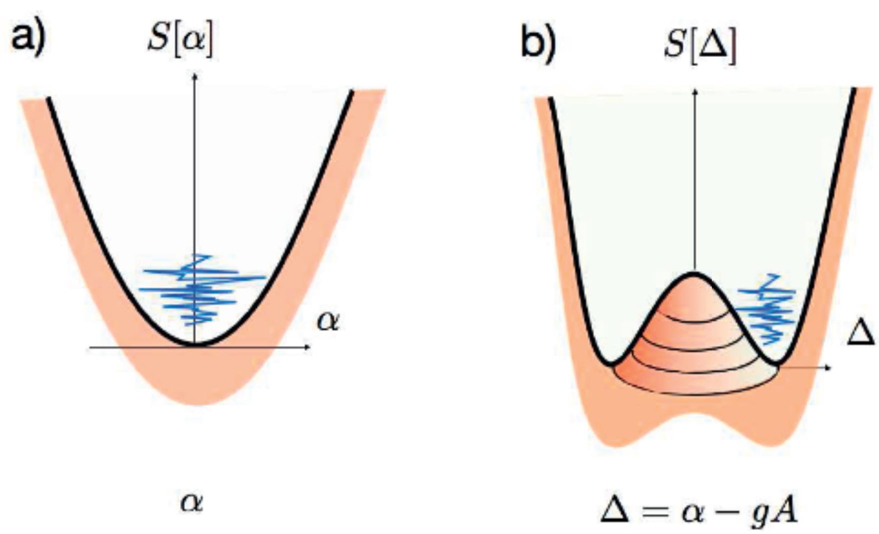

Fig. 12.6 (a) Action for initial white-noise variable α. (b) Action for shifted variable Δ is shifted off-center when the related quantity A has a predisposition towards developing an expectation value.

$$Z = \sum_{\{\Delta\}} \exp\left(-\sum_j \int d\tau \frac{|\Delta_j|^2}{g}\right) \times \left[\text{path integral of fermions moving in field } \Delta\right], \tag{12.177}$$

where the summation represents a sum over all possible configurations $\{\Delta\}$ of the auxiliary field Δ. The transformed field

$$\Delta_j = \alpha_j - gA_j$$

is a combination of a white-noise field α_j and the physical field $-gA_j$, so its fluctuations now acquire the correlations associated with the electron fluid. Indeed, when the associated variable A is prone to the development of a broken-symmetry expectation value, the distribution function for Δ becomes concentrated around a non-zero value (Figure 12.6). We call Δ_j a *Weiss field* after Weiss, who first introduced such a field in the context of magnetism.

12.5.3 Effective action

Since the fermionic action inside the path integral is actually Gaussian, we can formally integrate out the fermions as follows:

$$e^{-S_\psi[\bar{\Delta},\Delta]} = \int \mathcal{D}[\bar{c}, c] e^{-\tilde{S}} = \det[\partial_\tau + \underline{h}_E[\bar{\Delta}, \Delta]], \tag{12.178}$$

where $\underline{h}_E$ is the matrix representation of H_E. The full path integral may thus be written

$$Z = \int \mathcal{D}[\bar{\Delta}, \Delta] e^{-S_E[\bar{\Delta},\Delta]},$$

where

$$S_E[\bar{\Delta}, \Delta] = \sum_j \int d\tau \frac{\bar{\Delta}_j \Delta_j}{g} - \ln\det[\partial_\tau + \underline{h}_E[\bar{\Delta}, \Delta]]$$
$$= \sum_j \int d\tau \frac{\bar{\Delta}_j \Delta_j}{g} - \mathrm{Tr}\ln[\partial_\tau + \underline{h}_E[\bar{\Delta}, \Delta]]. \tag{12.179}$$

effective action

Here we have made the replacement $\ln\det \to \mathrm{Tr}\ln$. This quantity is called the *effective action* of the field Δ. The additional fermionic contribution to this action can profoundly change the distribution of the field Δ. For example, if S_E develops a minima around $\Delta = \Delta_o \neq 0$, then $\Delta = -A/g$ will acquire a vacuum expectation value. This makes the Hubbard–Stratonovich transformation an invaluable tool for studying the development of broken symmetry in interacting Fermi systems.

12.5.4 Generalizations to real variables and repulsive interactions

The method outlined in the previous section can also be applied to real fields. If we have an interaction between real fields, we can introduce a real white-noise field as follows:

$$H_I = -\frac{g}{2}\sum_j A_j^2 \to \sum_j \left\{ -\frac{g}{2}A_j^2 + \frac{q_j^2}{2g} \right\}. \tag{12.180}$$

Then, by redefining $q_j = Q_j + gA_j$, one obtains

$$-\frac{g}{2}\sum_j A_j^2 \to \sum_j \left\{ Q_j A_j + \frac{Q_j^2}{g} \right\}. \tag{12.181}$$

For example, we can use the Hubbard–Stratonovich transformation to replace an attractive interaction between fermions by a white-noise potential with variance g:

$$H_I = -\frac{g}{2}\sum_j (n_j)^2 \to \sum_{j\sigma} V_j n_j + \frac{V_j^2}{2g},$$

where $n_j = n_{j\uparrow} + n_{j\downarrow}$.

But what about repulsive interactions? These require a little more care, because we can't just change the sign of g in (12.181), for the integral over the white-noise fields will no longer be convergent. Instead, after introducing the dummy white-noise fields as before,

$$H_I = \frac{g}{2}A_j^2 \to \sum_j \left\{ \frac{g}{2}A_j^2 + \frac{q_j^2}{2g} \right\}, \tag{12.182}$$

we shift each variable in the path integral $q_j(\tau)$ by an imaginary amount, $q_j(\tau) = Q_j(\tau) + igA_j(\tau)$, to obtain [3]

$$\frac{g}{2}\sum_j A_j^2 \to \sum_j \left\{ iQ_jA_j + \frac{Q_j^2}{2g} \right\}. \tag{12.183}$$

Note finally that, if one replaces $Q_j = -i\tilde{Q}_j$, this takes the form

$$\frac{g}{2}\sum A_j^2 \to \sum_j \left\{ \tilde{Q}_jA_j - \frac{\tilde{Q}_j^2}{2g} \right\}. \tag{12.184}$$

At first sight, this looks like the generalization of (12.181) to negative g, except that now the integrals over each $Q_j(\tau)$ traverse the imaginary rather than the real axis.

Example 12.7 Using the Hubbard–Stratonovich transformation, show that the Coulomb interaction can be decoupled in terms of a fluctuating potential as follows:

$$H_I = \frac{1}{2}\int_{\mathbf{x},\,\mathbf{x}'} \rho(\mathbf{x})\rho(\mathbf{x}')\frac{e^2}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}'|} \to \int_{\mathbf{x}} \left[e\rho(\mathbf{x})\phi(\mathbf{x}) - \epsilon_0\frac{(\nabla\phi)^2}{2} \right]. \tag{12.185}$$

What is the interpretation of the new term, quadratic in the potential field (and why is the sign negative)?

Solution

Because of the non-local nature of the Coulomb interaction, it is more transparent to make this transformation in momentum space. Writing

$$\rho(x) = \int_{\mathbf{q}} \rho_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}, \qquad \frac{1}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}'|} = \int_{\mathbf{q}} \frac{1}{\epsilon_0 q^2} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{12.186}$$

where $\int_{\mathbf{q}} \equiv \int \frac{d^3q}{(2\pi)^3}$, the interaction becomes

$$H_I = \frac{1}{2}\int_{\mathbf{q}} \frac{(e\rho_{\mathbf{q}})(e\rho_{-\mathbf{q}})}{\epsilon_0 q^2}.$$

We now add in a dummy white-noise term:

$$H_I \to H_I' = \frac{1}{2}\int_{\mathbf{q}} \left[\frac{(e\rho_{\mathbf{q}})(e\rho_{-\mathbf{q}})}{\epsilon_0 q^2} - \epsilon_0 q^2 \phi_{\mathbf{q}}\phi_{-\mathbf{q}} \right],$$

[3] One might be worried about the legitimacy of shifting a real field by an imaginary quantity. However, just as the integral

$$\int_{-\infty}^{\infty} dQ e^{-Q^2/2} = \int_{-\infty+iA}^{\infty+iA} dQ e^{-Q^2/2}$$

is unaffected by a constant shift of the variable Q by an imaginary amount, $Q \to Q + iA$, a multi-variable path integral

$$\int D[Q] e^{-\int d\tau Q(\tau)^2/2}$$

is similarly unaffected by shifting the integration variable $Q(\tau)$ by an amount $iA(\tau)$, $Q(\tau) \to Q(\tau) + iA(\tau)$.

with the understanding that, in the path integral, the $\phi_{\mathbf{q}}$ field is to be integrated along the imaginary axis $\phi_{\mathbf{q}} = i\tilde{\phi}_{\mathbf{q}}$. Now if we shift $\phi_{\mathbf{q}} \to \phi_{\mathbf{q}} - \frac{e\rho_{\mathbf{q}}}{\epsilon_0 q^2}$, we obtain

$$H_I' = \int_{\mathbf{q}} \left[(e\rho_{\mathbf{q}})\phi_{-\mathbf{q}} - \frac{\epsilon_0}{2} q^2 \phi_{\mathbf{q}}\phi_{-\mathbf{q}} \right].$$

Finally, Fourier transforming back into real space ($q^2 \to -\nabla^2$), we obtain

$$H_I' = \int_{\mathbf{x}} \left[e\rho(x)\phi(x) + \frac{\epsilon_0}{2}\phi\nabla^2\phi \right]. \tag{12.187}$$

Integrating the last term by parts gives

$$H_I' = \int_{\mathbf{x}} \left[e\rho(x)\phi(x) \overbrace{-\frac{\epsilon_0}{2}(\nabla\phi)^2}^{-\epsilon_0 E^2/2} \right]. \tag{12.188}$$

We can identify the last term in this expression as $-\epsilon_0 E^2/2$, which is the electrostatic contribution to the action. The minus sign can be traced back to the fact that, inside the electromagnetic (Maxwell) action

$$S_{EM} = \int d^3x d\tau \left[\frac{B^2}{2\mu_0} - \frac{\epsilon_0 E^2}{2} \right], \tag{12.189}$$

the electrostatic contribution to the action enters with the opposite sign to the magnetic part. The complete path integral for interacting electrons in this representation is then

$$Z = \int \mathcal{D}[\bar{\psi}, \psi, \phi] \exp\left[-\int_0^\beta d\tau \int d^3x \left(\bar{\psi}\big(-\frac{1}{2m}\nabla^2 + e\phi(x) - \mu\big)\psi - \frac{\epsilon_0}{2}(\nabla\phi)^2 \right) \right].$$

Thus, by carrying out a Hubbard–Stratonovich transformation, the action becomes local. This formulation is ideal for the development of RPA approximations to the electron gas, while mean-field solutions of this path integral can be used to explore the formation of Wigner crystals.

Appendix 12A Derivation of key properties of bosonic coherent states

Here we derive the matrix elements and the completeness properties of bosonic coherent states.

Matrix elements

Matrix elements of normal-ordered operators $O[\hat{b}^\dagger, \hat{b}]$ between two coherent states are obtained simply by replacing the operators $\hat{b}$ and $\hat{b}^\dagger$ by the c-numbers b and $\bar{b}$, respectively:

$$\langle \bar{b}_1 | \hat{O}[\hat{b}^\dagger, \hat{b}] | b_2 \rangle = O[\bar{b}_1, b_2] \times \langle \bar{b}_1 | b_2 \rangle = O[\bar{b}_1, b_2] \times e^{\bar{b}_1 | b_2}. \tag{12.190}$$