MANY BODY PHYSICS: 621. Spring 2024

Exercise 2. Anisotropic Superconductivity. (Due Apr 10th. Pdf solutions by email welcome.)

1. Using Nambu notation, calculate the total density of states for a 2 dimensional d-wave superconductor with a BCS Hamiltonian given by

$$\begin{aligned}\hat{H} &= \sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \mathcal{H}(\mathbf{k}) \psi_{\mathbf{k}}, \\ \mathcal{H}(\mathbf{k}) &= (\epsilon_{\mathbf{k}} \tau_3 + \Delta_{\mathbf{k}} \tau_1),\end{aligned} \tag{1}$$

where $\psi^\dagger_{\mathbf{k}} = (c^\dagger_{\mathbf{k}\uparrow}, c_{-\mathbf{k}\downarrow})$, $\epsilon_{\mathbf{k}} = (k^2/2m) - \mu$ is the kinetic energy, $\Delta_{\mathbf{k}} = \Delta_D \cos 2\phi$, where ϕ is the angular polar co-ordinate in 2D (i.e. $\mathbf{k} = k(\cos\phi, \sin\phi)$).

(a) Write down an explicit expression for the Nambu propagator $G(\mathbf{k}, E) = (E - \mathcal{H}(\mathbf{k}))^{-1}$.

(b) Calculate the density of states

$$N(E) = \frac{1}{\pi} \sum_{\mathbf{k}} \mathrm{Im}\,(\mathrm{Tr}[G(\mathbf{k}, E - i\delta))\,. \tag{2}$$

By sending the limits of energy integration to infinity, show that a good approximation to the density of states is $N(E) = \mathrm{Re}\mathcal{N}(z)|_{z=E-i\delta}$, where

$$\mathcal{N}(z) = \frac{2N(0)}{\pi} \mathrm{Re}\left[K\left(\frac{\Delta_D^2}{z^2}\right)\right] \tag{3}$$

is an analytic function and

$$K(z) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - z^2 \sin^2 \phi}} \tag{4}$$

is the complete elliptic integral of the first kind.

(c) What is the branch-cut structure of $\mathcal{N}(z)$? What is the equivalent expression for an s-wave superconductor with an isotropic gap Δ_S?

(d) The specific heat capacity $C_V(T)$ at temperature T, of a system with a density of state $N(E)$ is given by the first moment of the energy fluctuations, divided by the temperature,

$$C_V(T) = \frac{1}{T} \overline{N(E) E^2} \tag{5}$$

where

$$\overline{A(E)} = \int dE \left(-\frac{\partial f(E)}{\partial E}\right) A(E)$$

denotes the thermal average, where $f(E) = [1 + e^{\beta E}]^{-1}$ is the Fermi function. (This approximation ignores the temperature dependence of the gap). Contrast the *low-temperature* (i.e $T \ll \Delta$) dependence of the specific heat of a clean d- and s-wave superconductor.

(e) What do you expect for the low temperature dependence of the specific heat in a "dirty" d-wave superconductor, in which electrons scatter off non-magnetic impurities? To answer this question, note that the effect of disorder on a d-wave superconductor is to introduce a finite scattering rate to the electrons. Qualitatively, the density of states of a dirty d-wave superconductor can be approximated by replacing the infinitesimal imaginary part used in calculating $\mathcal{N}(E - i\delta)$, by a finite imaginary part representing the scattering rate $\tau^{-1} = 2\Gamma$, i.e $\delta \to \Gamma$. To answer this question, first plot $\text{Re}\mathcal{N}(E - i\Gamma)$ for a set of Γ/Δ.

(f) The trick of replacing $E - i\delta \to E - i\Gamma$ does not work for an s-wave superconductor under the influence of non-magnetic disorder. Why not?

2. In strontium titanate, superconductivity persists at carrier densities where the Fermi temperature becomes less than the Debye energy ω_D. Modify the Anderson-Morel model for retardation to take this into account by assuming that the range of energies in the conduction band run from

$$-\epsilon_F < \epsilon_k < \Lambda \tag{6}$$

where $\epsilon_F < \omega_D$ is the Fermi energy, and $\Lambda \gg \omega_D$ is the upper band-width. Assume, as in the Anderson-Morel model, that the effective interaction between the electrons has the form

$$V_{eff}(\mathbf{k}, \mathbf{k}') = N(0)^{-1} \begin{cases} \mu - \lambda & (|\epsilon_{\mathbf{k}}|, |\epsilon_{\mathbf{k}'}| < \omega_D) \\ \mu & \text{otherwise.} \end{cases} \tag{7}$$

Assume a gap equation of the form

$$\Delta(\epsilon) = -N(0) \int_{-\epsilon_F}^{\Lambda} d\epsilon' V(\epsilon, \epsilon') \frac{\Delta(\epsilon')}{2E(\epsilon')} \tag{8}$$

where $E(\epsilon) = \sqrt{\epsilon^2 + \Delta(\epsilon)^2}$. Notice that the low carrier concentration suppresses the contribution of holes ($\epsilon' < 0$) in the gap equation.

(a) Seeking a "two gap" solution

$$\Delta(\epsilon) = \begin{cases} \Delta_1 & (|\epsilon| < \omega_D) \\ \Delta_2 & (\omega_D < \epsilon < \Lambda) \end{cases} \tag{9}$$

show that the gap equation becomes

$$\begin{aligned} \Delta_1 &= (\lambda - \mu) \int_{-\epsilon_F}^{\omega_D} d\epsilon \frac{\Delta_1}{2\sqrt{\epsilon^2 + \Delta^2}} - \mu \int_{\omega_D}^{\Lambda} d\epsilon \frac{\Delta_2}{\sqrt{\epsilon^2 + \Delta_2^2}} \\ \Delta_2 &= -\mu \int_{-\epsilon_F}^{\omega_D} d\epsilon \frac{\Delta_1}{2\sqrt{\epsilon^2 + \Delta_1^2}} - \mu \int_{\omega_D}^{\Lambda} d\epsilon \frac{\Delta_2}{\sqrt{\epsilon^2 + \Delta_2^2}}. \end{aligned} \tag{10}$$

(b) Show that in this situation, the renormalization of the Coulomb interaction is halved (there are no virtual hole pairs)

$$\mu^* = \frac{\mu}{1 + \frac{\mu}{2}\ln(\frac{\Lambda}{\omega_D})},$$

and show that provided $\lambda - \mu^* > 0$, a solution of the form

$$\begin{aligned} \Delta_1 &= 2\sqrt{\omega_D \epsilon_F} \exp\left[-\frac{1}{\lambda - \mu^*}\right] \\ \Delta_2 &= -\frac{\mu^*}{\lambda - \mu^*}\Delta_1 \end{aligned} \tag{11}$$

is obtained. We see that the retardation effects are weaker. If we put in representative values $\mu = 1$, $\Lambda/\omega_D = 10^2$, in a large carrier density metal, we would get $\mu^* = 0.13$, but in the low carrier density metal we get $\mu^* = 0.23$.

3. Consider the effect of Zeeman splitting by a magnetic field on the A-phase of superfluid ^{3}He with a d-vector that points along the z-axis. The BCS Hamiltonian for this situation is given by

$$H = \frac{1}{2}\sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}}(\epsilon_{\mathbf{k}}\tau_3 + \Delta_{\mathbf{k}}\sigma_3\tau_1 - \vec{\sigma}\cdot\vec{B})\psi_{\mathbf{k}} + \frac{\Delta^2}{g}V \tag{12}$$

where $\psi^\dagger_{\mathbf{k}} = (c^\dagger_{\mathbf{k}\uparrow}, c^\dagger_{\mathbf{k}\downarrow}, -c_{-\mathbf{k}\downarrow}, c_{-\mathbf{k}\uparrow})$ is a Balian Werthammer spinor and $\Delta_{\mathbf{k}} = \Delta(\hat{k}_x + i\hat{k}_y) = \Delta e^{i\phi}\cos(\theta)$ where θ and ϕ are the polar directions of the momentum vector $\mathbf{k}$ and $\epsilon_{\mathbf{k}} = \frac{k^2}{2M} - \mu$ is the kinetic energy. By considering the gap equation for this superfluid, show that a magnetic field along the z-direction suppresses T_c (Pauli limiting), but that a magnetic field in the basal plane, perpendicular to the d-vector has no effect.

(a) Show that if the field is perpendicular to the d-vector $\vec{B} = B\hat{x}$, the energy eigenvalues are

$$\pm E_{\mathbf{k}\sigma} = \pm\sqrt{(\epsilon_{\mathbf{k}} - \sigma B)^2 + |\Delta_{\mathbf{k}}|^2}$$

so that in the gap equations, the application of a field perpendicular to the d-vector can be absorbed into a shift of the conduction electron energies.

(b) Show that if the field is parallel to the d-vector ($\vec{B} = B\hat{z}$), then the energy eigenvalues are

$$\pm E_{\mathbf{k}\sigma} = -\sigma B \pm \sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2} = -\sigma B \pm E_{\mathbf{k}}, \qquad (\sigma = \pm) \tag{13}$$

Use this result to construct the gap equation, demonstrating that at zero temperature, the critical field is about $B_c \sim 1.75 k_B T_c$.