MANY BODY PHYSICS: 621. Spring 2023

Exercise 1. Path Integrals. (Due Fri 17 Feb.)

1. In this problem consider $\hbar = 1$. Suppose $|0\rangle$ is the ground-state of a harmonic oscillator problem, where $b|0\rangle = 0$. Consider the state formed by simultaneously translating this state in momentum and position space as follows:

$$|p, x\rangle = \exp\left[-i(x\hat{p} - p\hat{x})\right]|0\rangle.$$

By rewriting $\hat{b} = (\hat{x} + i\hat{p})/\sqrt{2}$, $z = (x + ip)/\sqrt{2}$, show that this state can be rewritting as

$$|p, x\rangle = e^{b^\dagger z - \bar{z} b}|0\rangle$$

Using the relation $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$, provided $[A, [A, B]] = [B, [A, B]] = 0$, show that $|p, x\rangle$ is equal to a normalized coherent state

$$|p, x\rangle \equiv |z\rangle e^{-\bar{z}z/2} = e^{b^\dagger z}|0\rangle e^{-\frac{1}{2}\bar{z}z}$$

showing that the coherent state $|z\rangle$ represents a minimum uncertainty wavepacket centered at (q, p) in phase space.

2. (a) Suppose $H = \epsilon c^\dagger c$ the Hamiltonian for an energy level ϵ that can be occupied by a single fermion. Consider the approximation to the partition function obtained by first dividing up the period $\tau \in [0, \beta]$ into N equal time-slices,

$$Z_N = \mathrm{Tr}[(e^{-\Delta\tau H})^N], \tag{1}$$

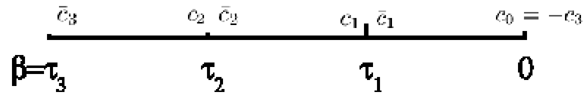
which is given by

$$\begin{aligned} Z_N &= \int \prod_{j=1}^{N} d\bar{c}_j dc_j \exp\left[-S_N\right] \\ S_N &= \sum_{j=1}^{N} \Delta\tau \left[\bar{c}_j(c_j - c_{j-1})/\Delta\tau + \epsilon\bar{c}_j c_{j-1}\right]. \end{aligned} \tag{2}$$

(a) Show that Z_3 can be written as a "toy functional integral",

$$Z_3 = \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \exp\left\{-(\bar{c}_3, \bar{c}_2, \bar{c}_1)\begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ \alpha & 0 & 1 \end{pmatrix}\begin{pmatrix} c_3 \\ c_2 \\ c_1 \end{pmatrix}\right\}, \tag{3}$$

where $\alpha = 1 - \Delta\tau\epsilon$. In this formula, the discrete time-line is labelled as follows,



(4)

where $(\bar{c}_j, c_j))$ are the conjugate Grassman variables at each discrete time $\tau_j = j\Delta\tau$.

(b) Evaluate Z_3.

(c) Generalize the result to N time slices and obtain an expression for Z_N. What is the limiting value of your result as $N \to \infty$?

(d) Repeat the calculation for a boson $H = \epsilon b^\dagger b$

3. A system of weakly interacting superfluid bosons is described by the action

$$S = \int_0^\beta d\tau d^3x \left[\bar{\psi} \left(\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right) \psi + \frac{g}{2}|\psi|^4 \right]. \tag{5}$$

(a) Ignoring fluctuations of the Bose Field, sketch the Free energy ($F = S/\beta$) as a function of the magnitude of a uniform Bose field, for $\mu > 0$ $\mu = 0$ and $\mu < 0$.

(b) Assuming you can ignore fluctuations of the Bose field, when $\mu > 0$, what is the uniform equilibrium value of the Bose field $\psi(x)$?

(c) What is the corresponding coherent state?

(d) Assume that the superfluid lives on a torus. Suppose the phase of the wavefunction winds through l turns as one goes around the torus in the x direction. Write down the wavefunction for this state. What is the superfluid velocity of this state and why is the superflow persistent?

(e) (Harder.) In class we considered the fluctuations in a "cartesian" basis, expanding the action to Gaussian order in terms of $\delta\psi(x)$ and $\delta\bar{\psi}(x)$. Carry out the calculation in the "radial" basis - i.e write the action in terms $\psi(x) = r(x)e^{i\phi(x)}$ and expand it to Gaussian order in $\delta r(x)$ and $\delta\phi(x)$. (Hint, choose $\phi_0 = 0$, so that $\delta\psi = \delta r(x) + ir_0\delta\phi(x) + O(\delta\psi^2)$. Note also that total derivatives, like $\int_0^\beta r\partial_\tau r = \frac{1}{2}\int_0^\beta \partial_\tau(r^2) = 0$ vanish). Confirm that you obtain the same results for the low-lying excitation spectrum as we obtained in class.