INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2024

Questions 6. Finite Temperature and superfluidity (Due Mon, 16th Dec.)

1. Use the method of complex contour integration to carry out the Matsubara sums in the following:

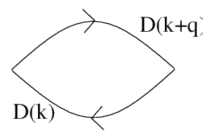
   (i) Derive the density of a spinless Bose Gas at finite temperature from the boson propagator $D(k) \equiv D(\mathbf{k}, \mathbf{i}\nu_\mathbf{n}) = [\mathbf{i}\nu_\mathbf{n} - \omega_\mathbf{k}]^{-\mathbf{1}}$, where $\omega_\mathbf{k} = E_\mathbf{k} - \mu$ is the energy of a boson, measured relative to the chemical potential.

$$\rho(T) = \frac{N}{V} = V^{-1} \sum_{\mathbf{k}} \langle T b_\mathbf{k}(0^-) b^\dagger{}_\mathbf{k}(0) \rangle = -(\beta V)^{-1} \sum_{i\nu_n, \mathbf{k}} D(k) e^{i\nu_n 0^+}. \qquad (1)$$

   How do you need to modify your answer to take account of Bose Einstein condensation?

   (ii) The dynamic charge-susceptibility of a free Bose gas, i.e

$$\chi_c(q, i\nu_n) =$$

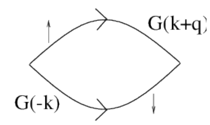


$$= T \sum_{i\nu_r} \int \frac{d^3k}{(2\pi)^3} D(q+k) D(k). \qquad (2)$$

   where ν_r is the Bose Matsubara frequency of the internal loop. Please analyticeally extend your final answer to real frequencies ($i\nu_n \to \nu$).

   (iii) The "pair-susceptibility" of a spin-1/2 free Fermi gas, i.e.

$$\chi_P(q, i\nu_n) =$$



$$= T \sum_{i\omega_r} \int \frac{d^3k}{(2\pi)^3} G(q+k) G(-k) \qquad (3)$$

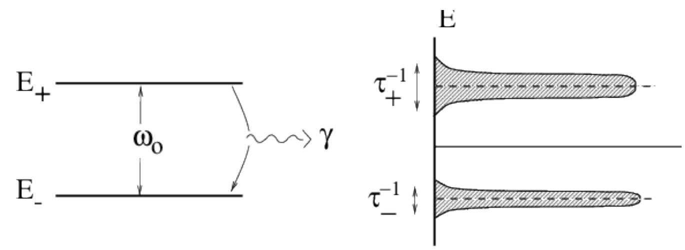
   where $G(k) \equiv G(\mathbf{k}, \mathbf{i}\omega_\mathbf{n}) = [\mathbf{i}\omega_\mathbf{n} - \epsilon_\mathbf{k}]^{-\mathbf{1}}$. (Note the direction of the arrows: why is there no minus sign for the Fermion loop?) Show that the static pair susceptibility, $\chi_P(0)$is given by

$$\chi_P = \int \frac{d^3k}{(2\pi)^3} \frac{\tanh[\beta\epsilon_\mathbf{k}/2]}{2\epsilon_\mathbf{k}} \qquad (4)$$

   Can you see that this quantity diverges at low temperatures? How does it diverge, and why ?

2. A simple model an atom with two atomic levels coupled to a radiation field is described by the Hamiltonian

$$H = H_o + H_I + H_{\text{photon}}, \tag{5}$$



where

$$H_o = \tilde{E}_- c^\dagger{}_- c_- + \tilde{E}_+ c^\dagger{}_+ c_+ \tag{6}$$

describes the atom, treating it as a *fermion*

$$H_I = V^{-1/2} \sum_{\vec{q}} g(\omega_{\vec{q}}) \left(c^\dagger{}_+ c_- + c^\dagger{}_- c_+ \right) \left[a^\dagger{}_{\vec{q}} + a_{-\vec{q}} \right] \tag{7}$$

describes the coupling to the radiation field (V is the volume of the box enclosing the radiation) and

$$H_{photon} = \sum_{\vec{q}} \omega_{\vec{q}} a^\dagger{}_{\vec{q}} a_{\vec{q}}, \qquad (\omega_q = cq) \tag{8}$$

is the Hamiltonian for the electromagnetic field. The "dipole" matrix element $g(\omega)$ is weak enough to be treated by second order perturbation theory and the polarization of the photon is ignored.

(i) Calculate the self-energy $\Sigma_+(\omega)$ and $\Sigma_-(\omega)$ for an atom in the + and − states.

(ii) Use the self-energy obtained above to calculate the life-times $\tau_\pm$ of the atomic states, i.e.

$$\tau_\pm^{-1} = 2\text{Im}\Sigma_\pm(\tilde{E}_\pm - i\delta). \tag{9}$$

If the gas of atoms is non-degenerate, i.e the Fermi functions are all small compared with unity, $f(E_\pm) \sim 0$ show that

$$\begin{aligned} \tau_+^{-1} &= 2\pi |g(\omega_o)|^2 F(\omega_o)[1 + n(\omega_o)] \\ \tau_-^{-1} &= 2\pi |g(\omega_o)|^2 F(\omega_o) n(\omega_o), \end{aligned} \tag{10}$$

where $\omega_o = \tilde{E}_+ - \tilde{E}_-$ is the separation of the atomic levels and

$$F(\omega) = \int \frac{d^3 q}{(\mathbf{2}\pi)^3} \delta(\omega - \omega_q) = \frac{\omega^2}{2\pi c^3} \tag{11}$$

is the density of state of the photons at energy ω. What do these results have to do with stimulated emission? Do your final results depend on the initial assumption that the atoms were fermions?

(iii)Why is the decay rate of the upper state larger than the decay rate of the lower state by the factor $[1 + n(\omega_0)]/n(\omega_0)$?

3. We can construct a state of bosons in which the bosonic field operator has a definite expectation value using a coherent state as follows

$$|\Psi\rangle = \exp\left[\int d^3x \Psi(x)\hat{\psi}^\dagger(x)\right]|0\rangle,$$

where $\Psi(x)$ is a complex function of position. The Hermitian conjugate of this state is $\langle\bar{\psi}| = \langle 0|e^{\int d^3x \hat{\psi}(x)\Psi^*(x)}$.

(a) Show that this coherent state is an eigenstate of the field destruction operator: $\hat{\psi}(x)|\Psi\rangle = \Psi(x)|\Psi\rangle$.

(b) Show that overlap of the coherent state with itself is given by $\langle\bar{\Psi}|\Psi\rangle = e^N$, where $N = \int d^3x|\Psi(x)|^2$ is the number of particles in the condensate.

(c) If

$$H = \int d^3x \left[\hat{\psi}^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi(x) + U : (\psi^\dagger(x)\psi(x))^2 :\right]$$

is the (normal ordered) energy density, calculate the expectation value of H, i.e $\langle\bar{\Psi}|\hat{H}|\Psi\rangle/\langle\bar{\Psi}|\Psi\rangle$ in the coherent state.