INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2024

Solution to Problems 5.

1. (a) Suppose we write the interaction in the form

$$\hat{V} = 1/2 \sum_{ijkl} V_{ijkl} c^\dagger{}_i c^\dagger{}_j c_l c_k,$$

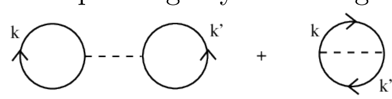
then using Wick's theorem

$$\begin{aligned} \langle\phi|\hat{V}|\phi\rangle &= 1/2 \sum_{ijkl} V_{ijkl} \langle\phi|c^\dagger{}_i c^\dagger{}_j c_l c_k|\phi\rangle \\ &= 1/2 \sum_{ijkl} V_{ijkl} \Big\{ \langle\phi|c^\dagger{}_i c^\dagger{}_j c_l c_k|\phi\rangle + \langle\phi|c^\dagger{}_i c^\dagger{}_j c_l c_k|\phi\rangle \Big\} \\ &= 1/2 \sum_{ijkl} n_i n_j \Big(V_{ijji} - V_{ijij} \Big) \end{aligned} \tag{1}$$

where $n_i = \langle\phi|c^\dagger{}_i c_i|\phi\rangle$. For the case of a translationally invariant system, this becomes

$$\langle\phi|\hat{V}|\phi\rangle = 1/2 \sum_{\vec{k}\sigma\vec{k}'\sigma'} n_{\vec{k}\sigma} n_{\vec{k}\sigma'} \Big(V(\vec{q}=0) - V(\vec{k}-\vec{k}')\delta_{\sigma\sigma'} \Big) \tag{2}$$

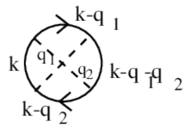
The first term corresponds to the Classical interaction energy in a uniform gas of particles. The second term describes the exchange energy between particles in the same spin-state. Particles in the same spin state must have a spatially asymmetric mutual wavefunction, which lowers the repulsive energy between them. The corresponding Feynman diagrams are

$\frac{\Delta E}{V} = -i \times \Big[$ 

$\Big]$

(b,c) Here are the five connected diagrams appearing in second-order perturbation theory:

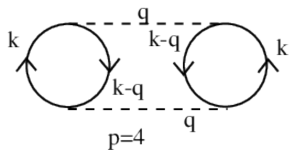
$$(\Delta E^{(2)}/Volume) = i\times$$



p=4

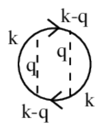
$$\frac{-1.(i^2).(2S+1)}{4}\int\frac{d^4k d^4q_1 d^4q_2}{(2\pi)^{12}}V(q_1)V(q_2)G(k)G(k-q_1)$$

$$\times G(k-q_1-q_2)G(k-q_2)+$$



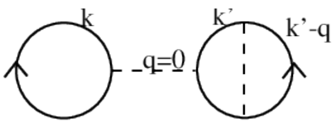
p=4

$$\frac{(-1)^2.(i^2).(2S+1)^2}{4}\int\frac{d^4q}{(2\pi)^4}V(q)^2\left[\int\frac{d^4k}{(2\pi)^4}G(k)G(k+q)\right]^2+$$



p=2

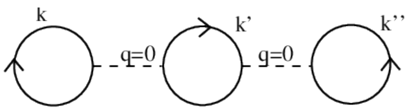
$$\frac{-1.(i^2).(2S+1)}{2}\int\frac{d^4k}{(2\pi)^4}G(k)^2\left[\int\frac{d^4q}{(2\pi)^4}V(q)G(k-q)\right]^2+$$



p=1

$$(-1)^2.(i^2).(2S+1)^2V(0)\int\frac{d^4k}{(2\pi)^4}G(k)e^{i\omega 0^+}$$

$$\times\int\frac{d^4k}{(2\pi)^4}G^2(k)G(k-q)V(q)e^{i\omega 0^+}+$$



p=2

$$\frac{-1.(i^2).(2S+1)^3}{2}V(0)^2\left[\int\frac{d^4k}{(2\pi)^4}G[k]e^{i\omega 0^+}\right]^2\left[\int\frac{d^4k}{(2\pi)^4}G[k]^2e^{i\omega 0^+}\right]+$$

Actually, the last diagram vanishes, because the integral $\int\frac{d^4k}{(2\pi)^4}G(k)^2 = 0$, a result you can understand by completing the contour of integration on the side opposite to the double pole of $G(k)^2$.

2. (a) The Fermi wavevector is given by

$$\rho = \frac{2}{(2\pi)^3}\left(\frac{4\pi}{3}k_F^3\right) = \frac{1}{3\pi^2}k_F^3$$

so

$$k_F = (3\pi^2\rho)^{\frac{1}{3}}.$$

Now since $\rho = \frac{3}{4\pi R_e^3}$, where $R_e = ar_s$ defines the separation R_e in terms of the dimensionless separation r_s, it follows that

$$k_F = \left(\frac{9\pi}{4R_e^3}\right)^{\frac{1}{3}} = \frac{1}{\left(\frac{4}{9\pi}\right)^{1/3} r_s a}$$

(b) The total Hartree-Fock energy is the sum of the kinetic energy E_0, the Hartree energy E_H and the Fock exchange energy E_F, $E = E_0 + E_H + E_F$. Charge neutrality guarantees that

$E_H = 0$, and the remaining terms are

$$\begin{aligned} E_0/V &= 2\int_0^{k_F} \frac{4\pi k^2 dk}{(2\pi)^3}\left(\frac{\hbar^2 k^2}{2m}\right) = \frac{\hbar^2}{2m}\frac{k_F^5}{5\pi^2 m} = \frac{3\hbar^2 k_F^2}{10m}\rho \\ E_F/V &= -2\times\frac{1}{2}\int_{k,k'} \frac{e^2}{\epsilon_0|\mathbf{k}-\mathbf{k}'|^2} f_{\mathbf{k}} f_{\mathbf{k}'} \end{aligned} \tag{3}$$

where the minus sign derives from the exchange. Substituting $k_F = 1/(\alpha r_s a_B)$ into E_0, we obtain

$$\frac{E_0}{\rho V} = \frac{3}{5}\overbrace{\left(\frac{\hbar^2}{2ma_B^2}\right)}^{R_Y}\frac{1}{(\alpha r_s)^2} = \frac{3}{5}\frac{R_Y}{(\alpha r_S)^2} = \frac{2.21 R_Y}{r_s^2} \tag{4}$$

Next, writing out the exchange energy, we have

$$\begin{aligned} \frac{E_F}{V} &= -\int_{\mathbf{k},\mathbf{k}'} \frac{e^2}{\epsilon_0(k^2 + k'^2 - 2kk'\cos\theta)} \\ &= -\frac{e^2}{\epsilon_0}\int_0^{k_F}\frac{k^2 dk}{2\pi^2}\int_0^{k_F}\frac{k'^2 dk'}{2\pi^2}\int_{-1}^{1}\frac{dc}{2}\frac{1}{k^2+k'^2-2kk'c} \\ &= -\frac{e^2}{8\pi^4\epsilon_0}\int_0^{k_F} kk'dk'dk \ln\left|\frac{k+k'}{k-k'}\right| \\ &= -\frac{e^2 k_F^4}{8\pi^4\epsilon_0}\overbrace{\int_0^1 xydxdy\ln\left|\frac{x+y}{x-y}\right|}^{\frac{1}{2}} = -\frac{e^2}{\epsilon_0}\times\frac{k_F^4}{(2\pi)^4} \\ &= -\frac{3e^2 k_F}{(4\pi)^2\epsilon_0}\rho \end{aligned} \tag{5}$$

so that

$$\frac{E_F}{\rho V} = -\overbrace{\frac{e^2}{4\pi\epsilon_0 a_B}}^{2R_Y}\left(\frac{3}{4\pi\alpha r_s}\right) = -\frac{3}{2\pi\alpha r_s}R_Y \approx -\frac{0.916}{r_S}R_Y \tag{6}$$

yielding our final answer

$$\begin{aligned} \frac{E}{\rho V} &= \frac{3}{5}\frac{R_Y}{\alpha^2 r_s^2} - \frac{3}{2\pi}\frac{R_Y}{\alpha r_s} \\ &= \left(\frac{2.21}{r_s^2} - \frac{0.916}{r_S}\right) R_Y. \end{aligned} \tag{7}$$

(c) The exchange energy provides a measure of the strength of interaction effects in the uniform quantum electron plasma. The ratio of the two terms is given by

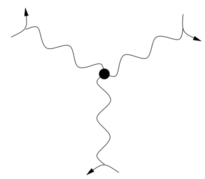
$$\frac{|E_F|}{E_0} \sim r_S$$

so that interaction effects become small at small r_S, or high density.

3. (a) The three-body delta-function potential can be written

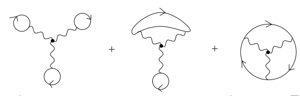
$$V(r_i, r_j, r_k) = \frac{\beta}{3!} \int d^d x : \rho(x)^3 : \tag{8}$$

(b) Let us denote a three-body vertex by the diagram



with which we associate the amplitude $\beta(-i)(i^3) = -\beta$. Here the (i^3) is derived from the three propagators per interaction, and the $(-i)$ is associated with the expansion of the time-ordered exponential. The leading-order contributions to the energy are then

$$\left(\frac{E_{int}^{(3)}}{V}\right) = i \sum (\text{linked cluster diagrams})$$

$$= i \Big\{$$  $$\Big\}$$

$$= -i\beta \left\{ -\frac{n_s^3}{6} + \frac{n_s^2}{2} - \frac{n_s}{3} \right\} \times \left[\int \frac{d^d p}{(2\pi)^d} G(p) \right]^3$$

$$= (-i)(i)^3 \beta \tilde{\rho}^3 [-n_s^3 + 3n_s^2 - 2n_s]/6 = \frac{\beta \tilde{\rho}^3}{6} n_s(n_s - 1)(n_2 - 2) \tag{9}$$

Note that this vanishes for $n_s = 1$ and $n_s = 2$. This is because three identical fermions can not come together at a point unless they all have different spin components, which requires a spin degeneracy in excess of two. The three-body interaction is identically zero unless $n_s > 2$, and discussions of stability completely mimic the above section. For $n_s > 2$, the ground-state energy per particle is then

$$\epsilon(n_s, \tilde{\rho}) = \left[a\tilde{\rho}^{2/d} - \alpha b \tilde{\rho} + \beta c \tilde{\rho}^2 \right] \tag{10}$$

where

$$\begin{aligned} a &= \frac{2\pi d}{d+2} \left(\frac{\hbar^2}{m}\right) [(\frac{d}{2})!]^{2/d} \\ b &= \frac{1}{2}(n_s - 1) \\ c &= \frac{1}{6}(n_s - 1)(n_2 - 2) \end{aligned} \tag{11}$$

The effect of the three-body term is to introduce an additional "hard-core" interaction that stabilizes the nuclear matter in two and three dimensions. Provided $\alpha > \alpha_c = \frac{8\beta c \tilde{\rho}}{b}$, the fluid forms a stable high density configuration with density

$$\tilde{\rho} \sim \frac{\alpha}{\beta}. \tag{12}$$

(c) In nuclear matter, where Coulomb forces are a small perturbation on the strong interaction between nucleons, the proton and neutron are essentially indistinguishable, forming part of an "isospin" doublet. Thus each nucleon is described by the tensor product of its spin and isospin quantum numbers, giving rise to an effective "spin" degeneracy $n_s = 2 \times 2 = 4$.

$$\begin{pmatrix} p \\ n \end{pmatrix} \otimes \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \equiv \begin{pmatrix} |p\uparrow\rangle \\ |p\downarrow\rangle \\ |n\uparrow\rangle \\ |n\downarrow\rangle \end{pmatrix} \tag{13}$$

Because $n_s > 2$, point three-body interactions become active inside the nucleon and play an important role in providing the hard-core repulsive interaction which stabilizes the nucleus at high densities.