INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2024

Answers to Questions 4. Monday, Nov 18th

1. (a) If we take the expression for the Free energy

$$F(\lambda) = -T \ln Z(\lambda) = -T \ln \mathrm{Tr}[e^{-\beta[H_o+\lambda V]}] \tag{1}$$

and differentiate it, we obtain

$$\frac{\partial F}{\partial \lambda} = -\frac{T}{Z}\frac{\partial Z}{\partial \lambda}. \tag{2}$$

Now

$$\frac{\partial Z}{\partial \lambda} = \mathrm{Tr}[\frac{\partial e^{-\beta[H_o+\lambda V]}}{\partial \lambda}] = -\beta \mathrm{Tr}[V e^{-\beta[H_o+\lambda V]}]] \tag{3}$$

so that

$$\frac{\partial F}{\partial \lambda} = \frac{\mathrm{Tr}[V e^{-\beta[H_o+\lambda V]}]]}{Z} = \langle V \rangle = \langle V_{int} \rangle / \lambda. \tag{4}$$

(b) By integrating the result of part (a) over λ, we obtain

$$\Delta F = \int_0^1 d\lambda \frac{\partial F}{\partial \lambda} = \int_0^1 \frac{d\lambda}{\lambda} \langle V_{int}(\lambda) \rangle \tag{5}$$

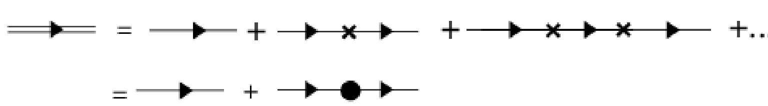
(c) If the interaction energy has an expansion $\langle V_{int}(\lambda)\rangle = \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3 + \ldots$, then

$$\Delta E = \int_0^1 \frac{d\lambda}{\lambda} \langle \phi | V_{int}(\lambda) | \phi \rangle = V_1 + \frac{1}{2} V_2 + \frac{1}{3} V_3 + \ldots \tag{6}$$

(d) When we turn on the interaction, the change in the ground-state energy involves the contributions from both the change in the Hamiltonian <u>and</u> the change in the ground-state. The factors of $\frac{1}{n}$ appearing in front of the n-th order terms reflect the fact that the ground-state relaxes in response to the change in hamiltonian, so that the change in the ground-state energy from each term is less than the corresponding change in the expectation value of the interaction.

2. (a) The crosses represent the scattering amplitude $V_{k,k'}$ and the lines represent the propagators.

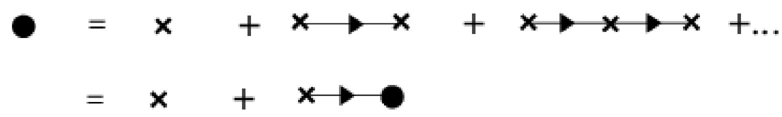
(b) Diagrammatically, we have:



or

$$G_{\mathbf{k},\mathbf{k}'}(E) = G^{(0)}_{\mathbf{k}}(E)\delta_{\mathbf{k},\mathbf{k}'} + G^{(0)}_{\mathbf{k}}(E) t_{\mathbf{k},\ \mathbf{k}'}(E) G^{(0)}_{\mathbf{k}'}(E) \tag{7}$$

where the "blob" is the t-matrix, represented by the following sum of diagrams



Written algebraically, this becomes

$$t_{\mathbf{k},\mathbf{k}'}(E) = U(\mathbf{k}-\mathbf{k}') + \int {\frac{d^dq}{2\pi}}^d \frac{U(\mathbf{k}-\mathbf{q})}{E - E(q) + i\delta} t_{\mathbf{q},\mathbf{k}'}(E) \tag{8}$$

where we assume that the Fermi surface is empty (i.e $\mu = 0$, so that $\delta_{\mathbf{k}} = \delta$ for all states.)

(c) If $U(x) = U\delta^{(d)}(x)$, then $U(q) = U$ and the t-matrix is now momentum independent. We may immediately solve (8) to obtain

$$t(\omega) = \frac{U}{1 - UF(\omega)} \tag{9}$$

where

$$F(\omega) = \int \frac{d^dk}{(2\pi)^d} G^{(0)}(k,\omega)$$

(d) Let us examine how the integral in the denominator of the t-matrix scales with energy at low energies. We take the case of a "drained Fermi sea", in which the chemical potential $\mu = 0$, so that for $\omega < 0$,

$$\begin{aligned} F(\omega) = \int \frac{d^dk}{(2\pi)^d} G^{(0)}(k,\omega) \quad &\propto \quad \int d\epsilon \epsilon^{(\frac{d}{2}-1)} \frac{1}{\omega - \epsilon + i\delta} \\ &\propto \quad -(-\omega)^{(\frac{d}{2}-1)} \\ &\propto \quad -\ln\left(\frac{\Lambda}{-\omega}\right), \qquad (d = 2). \end{aligned} \tag{10}$$

Thus in dimensions $d \leq 2$, if $U < 0$, the denominator of the t-matrix

$$1 - UF(\omega) = 1 + |U|F(\omega) \sim 1 - \frac{|U|}{(-\omega)^{(2-d)/2}}$$

will pass through zero at some small $\omega = -\omega^* \propto |U|^{2/(2-d)}$, for arbitrarily small $|U|$, giving rise to a pole in the t-matrix. To see that this means the development of a bound-state, consider the density of one-particle states

$$\rho(\omega) = \sum_\lambda \delta(\omega - E_\lambda) \tag{11}$$

where E_λ is the energy of the eigenstate $|\lambda\rangle$. We may rewrite this in the form

$$\rho(\omega) \quad = \quad -\frac{1}{\pi} \mathrm{Im} \sum_\lambda \frac{1}{(\omega - E_\lambda + i\delta)}$$

$$
\begin{aligned}
&= & -\frac{1}{\pi}\mathrm{Im}\sum_{\lambda}\langle\lambda|\hat{G}(\omega)|\lambda\rangle \\
= &= & -\frac{1}{\pi}\mathrm{Im}\mathrm{Tr}\left[\hat{G}(\omega)\right],
\end{aligned}
\tag{12}
$$

where $\hat{G}(\omega) = (\omega - H + i\delta)^{-1}$. We may also take the trace by summing over the momentum eigenstates, rather than energy eigenstates, so that

$$
\begin{aligned}
\rho(\omega) &= -\frac{1}{\pi}\sum_{\mathbf{k}}\mathrm{Im}\langle\mathbf{k}|\hat{G}(\omega)|\mathbf{k}\rangle \\
&= -\frac{1}{\pi}\sum_{\mathbf{k}}\mathrm{Im}G_{\mathbf{k},\mathbf{k}}(\omega)
\end{aligned}
\tag{13}
$$

Writing the Green-function in terms of the t-matrix, the change in the density of states due to scattering is then

$$
\Delta\rho(\omega) = -\frac{1}{\pi}\mathrm{Im}\left[\left(\int\frac{d^dk}{(2\pi)^d}G^{(0)}(k,\omega)\right)^2 t(\omega)\right]
\tag{14}
$$

Near the pole at negative energies, we may write this in the form

$$
\Delta\rho(\omega) = -\frac{1}{\pi}\mathrm{Im}[F(\omega)^2 t(\omega)] = -\frac{1}{\pi}\mathrm{Im}\left[\left(\frac{1}{U} - \frac{1}{t(\omega)}\right)^2 t(\omega)\right] \approx -\frac{1}{U^2\pi}\mathrm{Im}\left[t(\omega)\right]
\tag{15}
$$

Thus a pole in $t(\omega)$ implies a pole at negative energies in the density of states, indicating a bound-state.