INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2024

Questions 4. (Due Thurs, Nov 7)

1. Suppose we turn on the interaction adiabatically, writing $V_{int}(\lambda) = \lambda \hat{V}$. It is possible to relate the change in the Free energy ΔF to the interaction energy averaged over interaction strength with weight $1/\lambda$. Prove this using the following outline.

   (a) Consider the Free energy of the system, which is given by $F(\lambda) = -T \ln \mathrm{Tr}[e^{-\beta[H_o + \lambda V]}]$. First show by differentiating this result that

   $$\frac{dF(\lambda)}{d\lambda} = \frac{1}{\lambda}\langle V_{int}(\lambda)\rangle \tag{1}$$

   (b) Use this result to reason that the change in the free energy as a result of turning on the interaction is given by

   $$\Delta F = \int_0^1 \frac{d\lambda}{\lambda}\langle V_{int}(\lambda)\rangle \tag{2}$$

   By taking the zero temperature limit of this expression, we can reason that

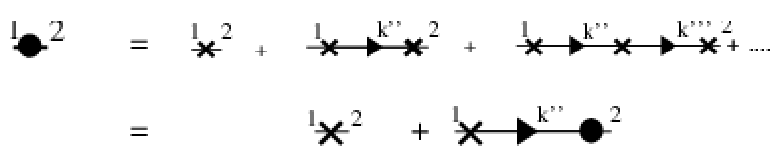
   $$\Delta E = \int_0^1 \frac{d\lambda}{\lambda}\langle \phi(\lambda)|V_{int}(\lambda)|\phi(\lambda)\rangle \tag{3}$$

   where $|\phi(\lambda)\rangle$ is the ground-state for $H(\lambda) = H_0 + \lambda \hat{V}$. In otherwords, the change in energy is equal to the interaction energy, averaged over the interaction strength, with weight $1/\lambda$.

   (c) If the interaction energy has an expansion $\langle V_{int}(\lambda)\rangle = \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3 + \ldots$, what is the corresponding expression for the change in the ground-state energy in increasing λ to full strength $\lambda = 1$?

   (d) Why is the change in the ground-state energy not equal to the change in the interaction energy? (Hint- as the interaction increases- what is changing, other than the Hamiltonian?)

2. Construct the diagram technique for the one-particle scattering amplitude on the potential $U(x)$.

   (a) What is denoted by the crosses and the solid line?

   (b) Show that the electron Green's function has the form

   $$G_{\vec{k},\vec{k}'}(E) = G^{(0)}_{\vec{k}}(E)\delta_{\vec{k},\vec{k}'} + G^{(0)}_{\vec{k}}(E)t_{\vec{k},\,\vec{k}'}(E)G^{(0)}_{\vec{k}}(E) \tag{4}$$

   where the scattering amplitude satisfies the integral equation

   $$t_{\vec{k},\vec{k}'}(E) = U(\vec{k}-\vec{k}') + \int \frac{d^d q}{(2\pi)^d}\frac{U(\vec{k}-\vec{q})}{E - E(q) + i\delta} t_{\vec{q},\vec{k}'}(E) \tag{5}$$

(c) Evaluate the t-matrix for a delta-function potential $U(x) = U\delta^{(d)}(x)$. (Hint: the t-matrix is momentum independent).

(d) Show that in dimensions $d \leq 2$, a bound-state forms beneath the bottom of the conduction band for arbitrarily weak attractive scattering potential. (Hint: this means checking to see if there are negative energy poles in the t-matrix)