INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2024

Answers to Questions II. Sept 29

Here is an outline of the solutions.

1. (i) Expanding $|\psi\rangle = |1111100\ldots\rangle = c^\dagger{}_5 c^\dagger{}_4 c^\dagger{}_3 c^\dagger{}_2 c^\dagger{}_1 |0\rangle$ we obtain (in gory detail)

$$
\begin{aligned}
c^\dagger{}_3 c_6 c_4 c_6{}^\dagger c_3 |\psi\rangle &= c^\dagger{}_3 c_6 c_4 c_6{}^\dagger \cancel{c_3} c^\dagger{}_5 c^\dagger{}_4 \cancel{c^\dagger{}_3} c^\dagger{}_2 c^\dagger{}_1 |0\rangle \\
&= c^\dagger{}_3 \cancel{c_6} c_4 \cancel{c_6{}^\dagger} c^\dagger{}_5 c^\dagger{}_4 c^\dagger{}_2 c^\dagger{}_1 |0\rangle \\
&= -c^\dagger{}_3 \cancel{c_4} c^\dagger{}_5 \cancel{c^\dagger{}_4} c^\dagger{}_2 c^\dagger{}_1 |0\rangle \\
&= +c^\dagger{}_3 c^\dagger{}_5 c^\dagger{}_2 c^\dagger{}_1 |0\rangle \\
&= -c^\dagger{}_5 c^\dagger{}_3 c^\dagger{}_2 c^\dagger{}_1 |0\rangle = -|1110100\ldots\rangle .
\end{aligned}
$$

(ii) We may write

$$
\begin{aligned}
|11010011\ldots &= c^\dagger{}_8 |11010010\ldots\rangle = c^\dagger{}_8 c^\dagger{}_7 |11010000\ldots\rangle \\
&= c^\dagger{}_8 c^\dagger{}_7 c_5 |11011000\ldots\rangle \\
&= c^\dagger{}_8 c^\dagger{}_7 c_5 c_3 |11111000\ldots\rangle
\end{aligned} \qquad (1)
$$

This state can be interpreted as the creation of an electron in states 8 and 7 and a "hole" in states 3 and 5.

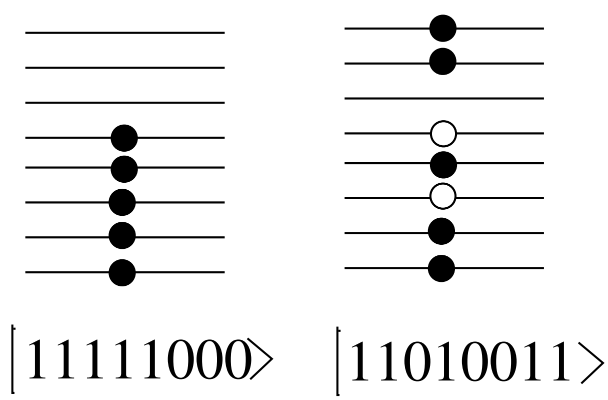

Figure 1:

(iii) To calculate $\langle\psi|\hat{N}|\psi\rangle$, where $\psi = A|100\ldots\rangle + B|111000\ldots\rangle$, note that

$$\hat{N}|\psi\rangle = A|100\ldots\rangle + 3B|111000\ldots\rangle, \qquad (2)$$

so that $\langle\psi|\hat{N}|\psi\rangle[|A|^2 + 3|B|^2]$.

2. (i) We need to confirm that $\{c_1,\ c^\dagger{}_1\} = \{c_2,\ c^\dagger{}_2\} = 1$ and also $\{c_1,\ c_2\} = \{c_2{}^\dagger,\ c_1{}^\dagger\} = 0$. Substituting for c_1 and c_2, we obtain

$$\{c_1,\ c_2\} = \{ua_1 + va^\dagger{}_2, -va^\dagger{}_1 + ua_2\} = -uv\{a_1, a_1{}^\dagger\} + vu\{a^\dagger{}_2, a_2\} = 0, \tag{3}$$

and

$$\{c_1,\ c^\dagger{}_1\} = \{ua_1 + va^\dagger{}_2, u^* a^\dagger{}_1 + v^* a_2\} = |u|^2\{a_1, a_1{}^\dagger\} + |v|^2\{a^\dagger{}_2, a_2\} = 1, \tag{4}$$

provided $|u|^2 + |v|^2 = 1$.

(ii) Consider $H = \omega[c^\dagger{}_1 c_1 - c_2 c^\dagger{}_2]$, then if

$$\begin{pmatrix} c_1 \\ c^\dagger{}_2 \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} a_1 \\ a^\dagger{}_2 \end{pmatrix} = U \begin{pmatrix} a_1 \\ a^\dagger{}_2 \end{pmatrix}, \tag{5}$$

where (assuming u, v real) U is a unitary transformation, we may re-write H as

$$H = (c_1{}^\dagger, c_2) \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2{}^\dagger \end{pmatrix}$$

so that using (5)

$$\begin{aligned} H &= (a_1{}^\dagger, a_2) U^\dagger \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} U \begin{pmatrix} a_1 \\ a_2{}^\dagger \end{pmatrix} \\ &= (a_1{}^\dagger, a_2) \begin{pmatrix} \epsilon & \Delta \\ \Delta & -\epsilon \end{pmatrix} \begin{pmatrix} a_1 \\ a_2{}^\dagger \end{pmatrix} \\ &= \epsilon[a^\dagger{}_1 a_1 - a_2 a^\dagger{}_2] + [\Delta a^\dagger{}_1 a^\dagger{}_2 + \text{H.c}] \end{aligned} \tag{6}$$

where

$$\begin{aligned} \epsilon &= \omega(u^2 - v^2), \\ \Delta &= \omega(2uv). \end{aligned} \tag{7}$$

Squaring both expressions and adding the results, we obtain $\omega = (\epsilon^2 + \Delta^2)^{\frac{1}{2}}$ and

$$u^2 = \frac{1}{2}\left(1 + \frac{\epsilon}{\omega}\right), \qquad v^2 = \frac{1}{2}\left(1 - \frac{\epsilon}{\omega}\right), \tag{8}$$

(iii) The ground-state is annihilated by both c_1 and c_2, so that if $H = \omega[c^\dagger{}_1 c_1 + c^\dagger{}_2 c_2 - 1]$, the ground-state energy is $E_o = -\omega = -(\epsilon^2 + \Delta^2)^{\frac{1}{2}}$.

3. Let us write our starting Hamiltonian in the form

$$H = -\sum_j (J_x S^x_{j+1} S^x_j + J_y S^y_{j+1} S^y_j)$$

$$= -\sum_j \left\{ \frac{J_x + J_y}{4} \left(S^+_{j+1} S^-_j + \text{H.c.} \right) + \frac{J_x - J_y}{4} \left(S^+_{j+1} S^+_j + \text{H.c.} \right) \right\}, \tag{9}$$

where $S^\pm = S^x \pm iS^y$. Using the Jordan Wigner transformation,

$$\begin{aligned} S^z_j &= ({c^\dagger}_j cj - \frac{1}{2}) \\ \mathbf{S}^+_j &= {c^\dagger}_j e^{i\pi \sum_{l<i} \hat{n}_l}, \end{aligned} \tag{10}$$

we have

$$\begin{aligned} S^+_{j+1} S^-_j &= {c^\dagger}_{j+1} c_j, \\ S^+_{j+1} S^+_j &= -{c^\dagger}_{j+1} {c^\dagger}_j, \end{aligned} \tag{11}$$

so that

$$H = -\sum_j t[{c^\dagger}_{j+1} c_j + \text{H.c}] - \sum_j \Delta[{c^\dagger}_{j+1} {c^\dagger}_j + \text{H.c}] \tag{12}$$

where $t = \frac{J_x + J_y}{4}$, $\Delta = \frac{J_y - J_x}{4}$.

(ii) Transforming to a momentum basis, ${c^\dagger}_j = \frac{1}{\sqrt{N}} \sum_q {d^\dagger}_q e^{-ix_j q}$, the Hamiltonian takes the form

$$H = -\sum_{q>0} 2t\cos(qa)[{d^\dagger}_q d_q - d_{-q} {d^\dagger}_{-q}] - \sum_q \Delta[e^{-iqa} {d^\dagger}_q {d^\dagger}_{-q} + \text{H.c}]. \tag{13}$$

Since ${d^\dagger}_q {d^\dagger}_{-q} = -{d^\dagger}_{-q} {d^\dagger}_q$ is an odd function of q, we can replace $\Delta e^{-iqa} \longrightarrow -i\Delta \sin qa$, to get

$$\begin{aligned} H &= \sum_{q>0} \epsilon_q [{d^\dagger}_q d_q - d_{-q} {d^\dagger}_{-q}] + \sum_{q>0} i\Delta_q [{d^\dagger}_q {d^\dagger}_{-q} - \text{H.c}] \\ &= \sum_{q>0} \left({d^\dagger}_q, d_{-q} \right) \begin{pmatrix} \epsilon_q & i\Delta_q \\ -i\Delta_q & -\epsilon_q \end{pmatrix} \begin{pmatrix} d_q \\ {d^\dagger}_{-q} \end{pmatrix} \\ &= \sum_{q>0} {d^\dagger}_q (\epsilon_q \tau_3 - \Delta_q \tau_2) d_q, \end{aligned} \tag{14}$$

where $\epsilon_q = -2t\cos(qa)$, $\Delta_q = 2\Delta \sin qa$ and we have introduced the Nambu notation

$$d_q = \begin{pmatrix} d_q \\ {d^\dagger}_{-q}. \end{pmatrix} \tag{15}$$

Notice how the sum over $q > 0$ is needed so that d_q and d_{-q} are independent. Carrying out the Boguilubov transformation ${a^\dagger}_q = {d^\dagger}_q U_q$, or

$$\left({a^\dagger}_q, a_{-q} \right) = \left({d^\dagger}_q, d_{-q} \right) \cdot \overbrace{\begin{pmatrix} u_q & -v^*_q \\ v_q & u^*_q \end{pmatrix}}^{U_q}, \tag{16}$$

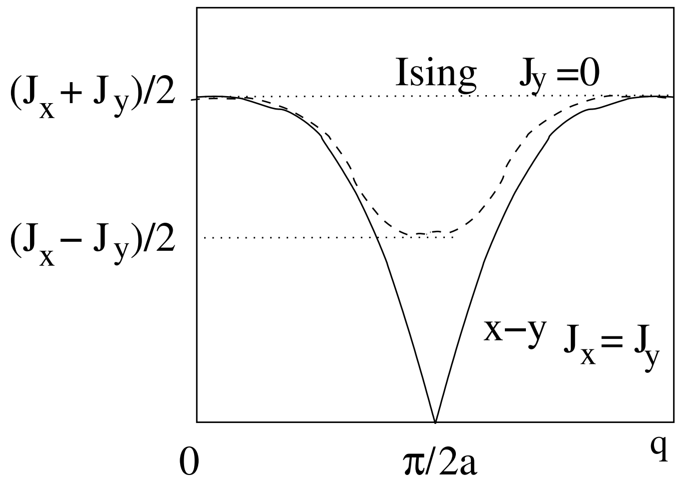

Figure 2: Showing dispersion for x-y, anisotropic x-y and Ising limits of model.

where U_q is the unitary matrix composed of the two column eigenvectors of $\mathcal{H} = \epsilon_{\mathbf{q}}\tau_3 - \Delta_q\tau_2$. Following the results of the last section, the Hamiltonian takes the form

$$H = \sum_{q>0} \omega_{\vec{q}} [a^\dagger{}_q a_q - a_{-q} a^\dagger{}_{-q}], \tag{17}$$

where

$$\begin{aligned} \omega_q &= \left(\epsilon_q^2 + \Delta_q^2\right)^{\frac{1}{2}} = \frac{1}{2}\Big[J_x^2 + J_y^2 + 2J_xJ_y\cos(2qa)\Big]^{\frac{1}{2}} \\ \begin{pmatrix} u_q \\ v_q \end{pmatrix} &= \begin{pmatrix} \Big[1 + \frac{\epsilon_q}{\omega_q}\Big]^{\frac{1}{2}} \\ i\Big[1 - \frac{\epsilon_q}{\omega_q}\Big]^{\frac{1}{2}} \end{pmatrix}. \end{aligned} \tag{18}$$

The spectrum of spin-excitations is shown above. For the case $J_y = J_x$, the excitation spectrum is gapless, corresponding to the continuous rotational symmetry (Goldstone mode). For the case J_y, or $J_x = 0$, the excitation spectrum is flat $\omega = J_x/2$ as expected for the 1d Ising model. We can interpret $J_x/2$ as the energy to create a "domain wall" in the Ising Ferromagnet. If the ground-state is $|\phi\rangle = |\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rangle$ then the flat band of fermions corresponds to domain wall excitations. If we use open boundary conditions, then energy to create a single domain wall, i.e $|\rightarrow\rightarrow\rightarrow\rightarrow\leftarrow\leftarrow\leftarrow\rangle$ is $J_x/2$. In periodic boundary conditions, such domain walls can only be created in pairs.