INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2024

Questions 2. (Due Thursday Oct 3)

1. In this question ${c_i}^\dagger$ and c_i are fermion creation and annihilation operators and the states are fermion states. Use the convention

   $|11111000\ldots\rangle = {c_5}^\dagger {c_4}^\dagger {c_3}^\dagger {c_2}^\dagger {c_1}^\dagger |\text{vacuum}\rangle$.

   (i) Evaluate ${c_3}^\dagger c_6 c_4 {c_6}^\dagger c_3 |111110000\ldots\rangle$.

   (ii) Write $|1101001100\ldots\rangle$ in terms of excitations about the "filled Fermi sea" $|1111100000\ldots\rangle$. Interpret your answer in terms of electron and hole excitations.

   (iii) Find $\langle\psi|\hat{N}|\psi\rangle$ where $|\psi\rangle = A|100\rangle + B|111000\rangle$, $\hat{N} = \sum_i {c_i}^\dagger c_i$.

2. (a) Consider two fermions, a_1 and a_2. Show that the Bogoliubov transformation

$$\begin{aligned} c_1 &= u a_1 + v {a^\dagger}_2 \\ {c^\dagger}_2 &= -v a_1 + u {a^\dagger}_2 \end{aligned} \tag{1}$$

   where u and v are real, preserves the canonical anti-commutation relations if $u^2 + v^2 = 1$.

   (b) Use this result to show that the Hamiltonian

$$H = \epsilon({a^\dagger}_1 a_1 - a_2 {a_2}^\dagger) + \Delta({a^\dagger}_1 {a^\dagger}_2 + \text{H.c.}) \tag{2}$$

   can be diagonalized in the form

$$H = \sqrt{\epsilon^2 + \Delta^2}({c^\dagger}_1 c_1 + {c^\dagger}_2 c_2 - 1) \tag{3}$$

   (c) What is the ground-state energy of this Hamiltonian?

3. (i)Use the Jordan Wigner transformation to show that the one dimensional anisotropic XY model

$$H = -\sum_i [J_1 S_x(i) S_x(i+1) + J_2 S_y(i) S_y(i+1)] \tag{4}$$

   can be written as

$$H = -\sum_i [t({d^\dagger}_{i+1} d_i + \text{H.c}) + \Delta({d^\dagger}_{i+1} {d^\dagger}_i + \text{H.c})] \tag{5}$$

   where $t = \frac{1}{4}(J_1 + J_2)$ and $\Delta = \frac{1}{4}(J_2 - J_1)$.

   (ii)Calculate the excitation spectrum for this model and sketch your results. Comment specifically on the two cases $J_1 = J_2$ and $J_2 = 0$. (Hint: when you Fourier transform the pairing terms, since fermion operators anticommute, the resulting term must be odd in momentum, eliminating the even $\cos k$ part of the exponent, leaving just an odd $\sin k$ function.)