INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2024

Questions I. (Due Thu, Sept 19th.)

These questions can *all* be done with a minimum of algebra. They will familiarize you with the method of second-quantization, as applied to free bosonic fields. Please choose one of the four harder questions. If you do more than one, I will grade the others as extra credit.

1. In 1906, in what is arguably the first paper in theoretical condensed matter physics[**?**] Albert Einstein postulated that vibrational excitations of a solid are quantized with energy $\hbar\omega$, just like the photons in the vacuum. Repeat his calculation for diamond: calculate the energy $E(T)$ of one mole of simple harmonic oscillators with characteristic frequency ω at temperature T and show that the specific heat capacity is

$$C_V(T) = \frac{dE}{dT} = RF\left(\frac{\hbar\omega}{k_B T}\right)$$

where

$$F(x) = \left(\frac{x/2}{\sinh(x/2)}\right)^2.$$

and $R = N_{AV} k_B$ the product of Avagadro's number N_{AV} and Boltzmann's constant k_B. Plot $C(T)$ and show that it deviates from Dulong and Petit's law $C_V = (R/2)$ per quadratic degree of freedom at temperatures $T << \hbar\omega/k_B$.

2. (a) Show that if a is a canonical bose operator, the canonical transformation

$$\begin{aligned} b &= ua + va^\dagger, \\ b^\dagger &= ua^\dagger + va, \end{aligned} \tag{1}$$

(where u and v are real), preserves the canonical commutation relations, provided $u^2 - v^2 = 1$.

(b) Using the results of (a), diagonalize the Hamiltonian

$$H = \omega(a^\dagger a + \frac{1}{2}) + \frac{1}{2}\Delta(a^\dagger a^\dagger + aa), \tag{2}$$

by transforming it into the form $H = \tilde{\omega}(b^\dagger b + \frac{1}{2})$. Find $\tilde{\omega}$, u and v in terms of ω and Δ. What happens when $\Delta = \omega$?

3. (Harder) According to the "Lindeman" criterion, a crystal melts when the rms displacement of its atoms exceeds a third of the average separation of the atoms. Consider a three dimensional crystal with separation a, atoms of mass m and a nearest neigbor quadratic interaction $V = \frac{m\omega^2}{2}(\vec{\Phi}_{\mathbf{R}} - \vec{\Phi}_{\mathbf{R}+\mathbf{a}})^2$.

(i) Estimate the amplitude of zero point fluctuations using the uncertainty principle, to show that if

$$\frac{\hbar}{m\omega a^2} > \zeta_c \tag{3}$$

where ζ_c is a dimensionless number of order one, the crystal will be unstable, even at absolute zero, and will melt due to zero-point fluctuations. (Hint... what would the answer be for a simple harmonic oscillator?)

(ii) Calculate ζ_c in the above model. If you like, to start out, imagine that the atoms only move in one direction, so that Φ is a scalar displacement at the site with equilibrium position $\mathbf{R}$. Calculate the rms zero-point displacement of an atom $\sqrt{\langle 0|\Phi(x)^2|0\rangle}$. Now generalize your result to take account of the fluctuations in three orthogonal directions.

(iii)Suppose $\hbar\omega/k_B = 300K$, and the atom is a Helium atom. Assuming that ω is independent of atom separation a, estimate the critical atomic separation a_c at which the solid becomes unstable to quantum fluctuations. Note that in practice ω is dependent on a, and rises rapidly at short distances, with $\omega \sim a^{-\alpha}$, where $\alpha > 2$. Is the solid stable for $a < a_c$ or for $a > a_c$?

4. (Harder) Find the transformation that diagonalizes the Hamiltonian

$$H = \sum_j \left\{ J_1(a^\dagger_{j+1}a_j + H.c) + J_2(a^\dagger_{j+1}a^\dagger_j + H.c)\right\} \tag{4}$$

where $a^\dagger_j$ creates a boson located at the jth site. The position of the jth-site is located at $R_j = aj$. You may find it helpful to (i) transform to momentum space, writing $a_j = \frac{1}{N^{1/2}}\sum_q e^{iqR_j}a_q$, using periodic boundary conditions on a ring of n-sites. After this, carry out a canonical transformation of the form $b_q = u_q a_q + v_q a^\dagger_{-q}$, where $u^2 - v^2 = 1$. What happens when $J_1 = J_2$?

5. (Harder) Find the classical normal mode frequencies and normal co-ordinates for the one dimensional chain with Hamiltonian

$$H = \sum_j \left[\frac{p_j^2}{2m_j} + \frac{k}{2}(x_j - x_{j-1})^2\right] \tag{5}$$

where at even sites $m_{2j} = m$ and at odd sites $m_{2j+1} = M$. Please sketch the dispersion curves.

(ii) What is the gap in the excitation spectrum?

(iii)Write the diagonalized Hamiltonian in second quantized form and discuss how you might arrive at your final answer. You will now need two types of creation operator.

6. (Harder) This problem sketches the proof that the displacement of the quantum Harmonic oscillator, originally in its ground-state (in the distant past), is given by

$$\langle x(t)\rangle = \int_0^t R(t-t')f(t')dt',$$

where

$$R(t-t') = \frac{i}{\hbar}\langle 0|[x(t), x(t')]|0\rangle$$

is the "response function" and $x(t)$ is the position operator in the Heisenberg representation of H_0. A more detailed discussion can be found in chapter 10.

An applied force $f(t)$ introduces an additional forcing term to the harmonic oscillator Hamiltonian

$$\hat{H}(t) = H_0 + V(t) = \hat{H}_0 - f(t)\hat{x},$$

where $H_0 = \hbar\omega(a^\dagger a + \frac{1}{2})$ is the unperturbed Hamiltonian. To compute the displacement of the Harmonic oscillator, it is convenient to work in the "interaction representation", which is the Heisenberg representation for H_0. In this representation, the time-evolution of the wavefunction is due to the force term. The wavefunction of the harmonic oscillator in the interation representation $|\psi_I(t)\rangle$ is related to the Schrodinger state $|\psi_S(t)\rangle$ by the relation $|\psi_I(t)\rangle = e^{iH_0 t/\hbar}|\psi_S(t)\rangle$.

(a) By using the equation of motion for the Schrodinger state $i\hbar\partial_t|\psi_S(t)\rangle = (H_0 + V(t))|\psi_S(t)\rangle$, show that the time evolution of the wavefunction in the interaction representation is

$$i\hbar\partial_t|\psi_I(t)\rangle = V_I(t)|\psi_I(t)\rangle = -f(t)\hat{x}(t)|\psi_I(t)\rangle,$$

where $V_I(t) = e^{iH_0 t/\hbar}\hat{V}(t)e^{-iH_0 t/\hbar} = -x(t)f(t)$ is the force term in the interaction representation.

(b) Show that if $|\psi(t)\rangle = |0\rangle$ at $t = -\infty$, then the leading order solution to the above equation of motion is then

$$|\psi_I(t)\rangle = |0\rangle + \frac{i}{\hbar}\int_{-\infty}^{t} dt'\, f(t')\hat{x}(t')|0\rangle + O(f^2),$$

so that

$$\langle\psi_I(t)| = \langle 0| - \frac{i}{\hbar}\int_{-\infty}^{t} dt'\, f(t')\langle 0|\hat{x}(t') + O(f^2).$$

(c) Using the results just derived expand the expectation value $\langle\psi_I(t)|x(t)|\psi_I(t)\rangle$ to linear order in f, obtaining the above cited result.