Corrections:

1. Boosts

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \qquad \text{Det} = 1$$

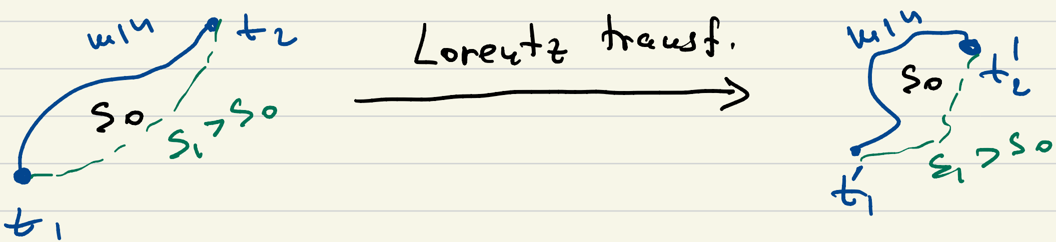
Lorentz group does have transformations with Det = −1, but this means it is not connected rather than noncompact

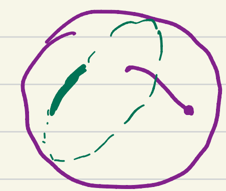
Corrections:

2. If $\mathcal{L}$ is Lorentz scalar, most likely EL eqs are covariant

$$\mathcal{L}(\phi, \partial_\mu \phi) \longrightarrow \mathcal{L}(\phi', \partial'_\mu \phi')$$

$$\partial_\mu F^{\mu\nu} = 0$$





Is there a counterexample?

$$S = \int d^4x \, \mathcal{L}$$

Homework #1 : Peskin & Schröder 2.1, 2.2, 2.3

Due in 3 weeks : Tue, Oct 20 by 11:59pm

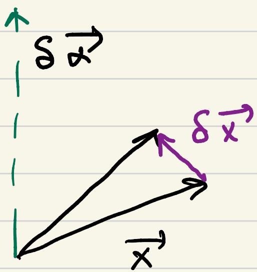
Last time: Momentum opt $\vec{p}$ is the generator of translations

$f(\vec{x}) \to f(\vec{x}+\vec{a})$ on the space of diff. fns

Small $\vec{a}$: $f(\vec{x}+\vec{a}) = (1 + i\vec{a}\cdot\vec{p})\, f(x)$

Arbitrary $\vec{a}$: $f(\vec{x}+\vec{a}) = e^{i\vec{a}\cdot\vec{p}} f(x)$

Rotations: $f(\vec{x}) \longrightarrow f(\vec{x}+\delta\vec{x})$

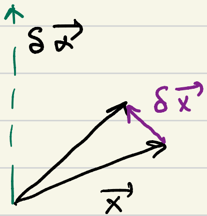


$$\delta\vec{x} = \delta\vec{\alpha}\times\vec{x}$$

$$f(\vec{x}+\delta\vec{x}) = \left[1 + i\left(\delta\vec{\alpha}\times\vec{x}\right)\cdot\vec{p}\right] f(\vec{x})$$

Rotations:

$$f(\vec{x}) \longrightarrow f(\vec{x} + \delta\vec{x})$$



$$\delta\vec{x} = \delta\vec{\alpha} \times \vec{x}$$

$$f(\vec{x} + \delta\vec{x}) = \left[1 + i\left(\delta\vec{\alpha} \times \vec{x}\right)\cdot\vec{p}\right] f(\vec{x})$$

$$\left(\delta\vec{\alpha} \times \vec{x}\right)\cdot\vec{p} = \left(\vec{x} \times \vec{p}\right)\cdot\delta\vec{\alpha}$$

Angular momentum opt: $\vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\nabla)$

$$\delta f = i\,\delta\vec{\alpha}\cdot\vec{J} f$$

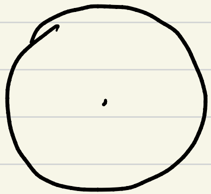
$\delta\vec{\alpha} = \delta\alpha\,\vec{n}$

$$\delta f = i\,\delta\alpha\, J_n f$$

$$\frac{df}{d\alpha} = i J_n f \qquad f_{after} = e^{i\alpha J_n} f_{before}$$

$$e^{i\alpha J_n} = e^{i\vec{\alpha}\cdot\vec{J}} = e^{i\alpha^k J^k}$$

$f(\theta, \varphi)$ Spherical harmonics $Y_{\ell m}$



$$\vec{J}^2 \, Y_{\ell m} = \ell(\ell+1) \, Y_{\ell m} \qquad J^3 \, Y_{\ell m} = m \, Y_{\ell m}$$

$$m = -\ell, -\ell+1, \ldots, \ell \qquad 2\ell+1 \text{ states}$$

In this subspace J_1, J_2, J_3 are $(2\ell+1) \times (2\ell+1)$ matrices

$$\langle \ell m' | J^i | \ell m \rangle \longrightarrow$$

$(2\ell+1)$-dim rep of rotation group

ℓ – weight of the rep

$2\ell+1$ – integer $\Rightarrow$ ℓ – half-integer

Include half-integer spin

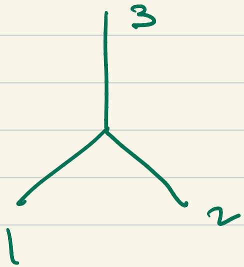
Ex.: spin-1/2 $J^i = \frac{\sigma^i}{2}$ σ^i – Pauli matrices

$\vec{J} = \vec{x} \times (-i \nabla)$

$J^3 \equiv J^{12} = -i(x^1 \nabla^2 - x^2 \nabla^1)$

$\quad\quad\; \| $

$\quad\; -J^{21}$



$J^{ij} = -i(x^i \nabla^j - x^j \nabla^i)$

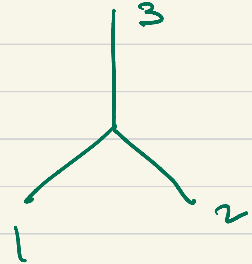
4D Lorentz transform

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

$$J^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)$$

cf. SU(2) $\quad [J^i, J^j] = i\varepsilon^{ijk} J^k$



e.g. $\quad [J^1, J^2] = i J^3$

$$[J^{23}, J^{31}] = i J^{12}$$

$$J^{ij} = - J^{ji}$$

$\mu = 2 \quad \nu = \rho = 3 \quad \sigma = 1$

only $\quad g^{\nu\rho} J^{\mu\sigma} = - J^{21} = J^{12}$ ✓

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}_{\mu\nu}$$

"Rotations" of 4-vectors

$$\tilde{V}^\alpha = R^\alpha{}_\beta V^\beta \qquad R = e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}}$$

$\omega_{\mu\nu}$ – antisymm

$$\frac{\omega_{12}}{2} J^{12} + \frac{\omega_{21}}{2} J^{21} = \omega_{12} J^{12}$$

$$R = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}$$

Consider rotations in 12 plane

$$R = \mathbb{1} - i\,\omega_{12} J^{12}$$

Boosts $\omega_{01} = -\omega_{10} = \beta$

$$V \to \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$R = \mathbb{1} - i\,\omega_{12} J^{12}$ $\varepsilon^{\alpha\beta\partial} \delta_{\alpha\beta} \equiv 0$

Let's find J^{12}

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \simeq \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} \quad -i J^{12}$$

Full

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{0,1,2,3} = \mathbb{1} + \theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{0,1,2,3}$$

Let $\omega_{12} = \theta$

$$(J^{12})_{\alpha\beta} = i\left(\delta_\alpha^1 \delta_\beta^2 - \delta_\alpha^2 \delta_\beta^1\right)$$

$\delta_\alpha^{\alpha_0} \delta_\beta^{\beta_0}$

$$(J^{\mu\nu}) = i\left(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu\right)$$

$(J^{\mu\nu}) = i\,(\delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\,\delta^{\mu}_{\beta})$ – 4-dim rep of Lorentz

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4×4 – dim rep of Lorentz ?

Dirac: Suppose we have 4 4×4 matrices :

$$\{\gamma^{\mu}, \gamma^{\nu}\} \equiv \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\, g^{\mu\nu}\, \mathbb{1}$$

$\Rightarrow \quad S^{\mu\nu} = \frac{i}{4}\,[\gamma^{\mu}, \gamma^{\nu}] \rightarrow$ 4 –dim rep of Lorentz

$$\overset{\shortparallel}{\frac{i}{2}\,\gamma^{\mu}\gamma^{\nu}}$$

$\mu = 1 \, \dots \, d$

γ^{μ} – 4×4

Dirac: Suppose we have 4 $n \times n$ matrices:

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \mathbb{1}$$

$$\Rightarrow \quad S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] \rightarrow n\text{-dim rep of Lorentz}$$

$\mu = 1 \ldots d$

$\gamma^\mu - n \times n$

$d = 1 \quad \gamma^1 - \#$

$d = 2 \quad \gamma^j = i \sigma^j \qquad j = 1, 2$

$d = 3$

$$\{\gamma^i, \gamma^j\} = -2\delta^{ij}$$

$$S^{ij} = -\frac{i}{4} [\sigma^i, \sigma^j] = \frac{1}{2} \varepsilon^{ijk} \sigma^k$$

$$S^3 = S^{12} = \frac{1}{2} \varepsilon^{123} \sigma^3 = \frac{\sigma^3}{2} - n = 2\text{-dim rep of rot. group}$$